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Bayesian Analysis of a Renamed Failur Model and its Order Statistics: A Mathematical Study

Shradha Dwivedi[1,*], Sabir Ali Siddiqui[2], Peeyush Dwivedi[3] and Sanjay Jain[4]

1 Department of Information and Technology, University of Technology and Applied Sciences, UTAS, Salalah, Sultanate of Oman.

2 Department of Mathematics and Sciences, CAAS, Dhofar University, Salalah, Sultanate of Oman.

3 Department of Business Studies, University of Technology and Applied Sciences, UTAS, Salalah, Sultanate of Oman.

4 Department of Statistics, St. John's College, Agra, India.

Abstract: The present paper deals in Bayesian analysis of the new distribution which is renamed here as SD-distribution. Bayesian estimates of the expected life, reliability function, hazard rate function is studied mathematically for the distribution, which is already established as a failure model. Similarly Bayesian estimates for the order statistics is also found here. The scale and shape parameters had been considered as random variables separately with certain prior distribution for the purpose of Bayesian estimation.

Keywords: SD Distribution, Reliability function, Hazard rate function, Prior distribution, Posterior distribution.

1. Introduction

The concept of Bayesian inference is not very old in the reliability analysis. Bhattacharya [1] seems to be the initiator in this field. He obtained Bayesian estimates for the exponential model for the uniform, exponential and inverted gamma priors. Sinha [11] carried the same work of Weibull model with Freyja's prior. Bayesian estimation of shape parameter in Pareto type II distribution was done by Sethiya [11] than Siddiqui and Kumar [5] worked on the Bayesian estimation for a finite range model considering the same priors as obtained by Bhattacharya [1]. Siddiqui [5, 8] worked on Bayesian analysis of Mukherjee-Islam distribution and Bayesian estimation of its sizebiased model [9] was done by Dwivedi [4].

In the present paper an attempt has been made to provide Bayesian estimates of reliability function, hazard rate function, expected life etc. for the new distribution studied by Dwivedi [2, 3] which is renamed here in this paper as SD-distribution.So from here-forth we will write all the details regarding new distribution as [2] SD-distribution. The probability density function of the SD-distribution [2] is;

$$f(x) = (1-p)x^{-p}\theta^{p-1}; \quad 0 < x < \theta; 0 < p < 1 \tag{1}$$

Here p is the shape parameter and θ is the scale parameter. Cumulative distribution function,reliability function $R(t)$ and Hazard rate function $H(t)$ are as follows;

$$F(x) = \theta^{p-1}x^{1-p} \tag{2}$$

* *E-mail: k.getshradha@gmail.com*

$$R(t) = 1 - F(t) = 1 - \theta^{p-1} x^{1-p} \tag{3}$$

$$H(t) = \frac{f(x)}{1 - F(t)} = \frac{(1-p)t^{-p}}{\theta^{1-p} - t^{1-p}} \tag{4}$$

Similarly *pdf* of order statistics [3] for n ordered arguments $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ of n; $X_1, X_2, \ldots, X_n$, *iid* observations are given as:

$$f_{n:n} = n(1-p)\theta^{n(p-1)} x^{n(1-p)-1}; \quad 0 < x < \theta; 0 < p < 1 \tag{5}$$

and its reliability function $R_{n:n}(t)$ and hazard rate function $H_{n:n}(t)$ are defined as:

$$R_{n:n}(t) = 1 - \theta^{n(p-1)} t^{n(1-p)} \tag{6}$$

$$H_{n:n}(t) = n(1-p)t^{-1}[\theta^{n(1-p)} t^{n(p-1)}] \tag{7}$$

Further, scale and shape parameters p and θ are considered as random variables with certain priors for the purpose of bayesian estimation. The prior are as follows

2. When p is a Variable

When scale parameter p is a variable than following prior density [6] provides the posterior distribution by equation (9) and (10):

$$g(p) = \frac{hp^{h-1}}{q^h - h^h}; h \leq p \leq q \tag{8}$$

The loglikelihood conditional for the failure model [2] and its order statistics [3] on p under the assumption of prior density (8) provides the following posterior distributions:

$$G(p/x_1, x_2, \ldots, x_n) = \frac{\left(\frac{1-p}{\theta^{p-1}}\right)^n \prod_{t=1}^{n} x_i^{-p} \frac{hp^{h-1}}{q^h - h^h}}{\int_h^q \left(\frac{1-p}{\theta^{p-1}}\right)^n \prod_{t=1}^{n} x_i^{-p} \frac{hp^{h-1}}{q^h - h^h} dp} \tag{9}$$

$$G_{n:n}(p/x_1, x_2, \ldots, x_n) = \frac{(n(1-p)\theta^{n(-p)})^n \prod_{t=1}^{n} x_i^{n(1-p)-1} \frac{hp^{h-1}}{q^h - h^h}}{\int_h^q (n(1-p)\theta^{n(-p)})^n \prod_{t=1}^{n} x_i^{n(1-p)-1} \frac{hp^{h-1}}{q^h - h^h} dp} \tag{10}$$

Putting $\prod_{t=1}^{n} t_i^{-1} = A$ in the above equation and with further simplifications we get,

$$G(p/x_1, x_2, \ldots, x_n) = \frac{\left(\frac{1-p}{\theta^{p-1}}\right)^n A^p p^{h-1}}{D_1}, \quad \text{where} \tag{11}$$

$$D_1 = \sum_{i=0}^{n} \sum_{k=0}^{\infty} C_{ik} \frac{(q^d - h^d)}{d} \tag{12}$$

$$C_{ik} = n_{C_i} (-1)^{n-i} \theta^{n(1-p)} \frac{(logA)^k}{k!}$$

$$d = n - i + h + k$$

and for order statistics

$$G_{n:n}(p/x_1, x_2, , \ldots, x_n) = \frac{(n(1-p)\theta^{n(-p)})^n A^{n(1-p)-1} p^{h-1}}{D_{n:n1}} \tag{13}$$

$$D_{n:n1} = \sum_{i=0}^{n}\sum_{k=0}^{\infty}\sum_{s=0}^{k}\sum_{l=0}^{k-s} C_{n:n} \frac{(q^{d_{n:n}} - h^{d_{n:n}})}{d_{n:n}} \tag{14}$$

$$C_{n:n} = n_{C_i}(-1)^{n-i}\theta^{n(1-p)}\frac{(logA)^k}{k!}$$

$$d = n - i + h + k - s - l$$

Where i, h, k, s, l are all integer values. Now,the Bayes estimates of the paraeter p and p^2 and for its order statistics can be obtained with the same above values of $D_1, C_{ik}, d, D_{n:n}, C_{n:n}$ and $d_{n:n}$ as follows:

$$E(p/x_1, x_2, , \ldots, x_n) = \frac{1}{D_1}\int_q^h p.\left(\frac{1-p}{\theta^{p-1}}\right)^n A^p p^{h-1} dp$$

$$= \frac{1}{D_1}\sum_{i=0}^{n}\sum_{k=0}^{\infty} C_{ik}\left(\frac{q^{d+1} - h^{d+1}}{d+1}\right) \tag{15}$$

$$E(p^2/t_1, t_2, \ldots, t_n) = \frac{1}{D_1}\int_q^h p^2.\left(\frac{1-p}{\theta^{p-1}}\right)^n A^p p^{h-1} dp$$

$$= \frac{1}{D_1}\sum_{i=0}^{n}\sum_{k=0}^{\infty} C_{ik}\left(\frac{q^{d+2} - h^{d+2}}{d+2}\right) \tag{16}$$

$$E_{n:n}(p/x_1, x_2, \ldots, x_n) = \frac{1}{D_{n:n1}}\int_q^h p.\left(n(1-p)\theta^{n(1-p)}\right)^n A^{n(1-p)-1} p^{h-1} dp$$

$$= \frac{1}{D_{n:n1}}\sum_{i=0}^{n}\sum_{k=0}^{\infty}\sum_{s=0}^{k}\sum_{l=0}^{k-s} C_{n:n}\left(\frac{q^{d_{n:n}+1} - h^{d_{n:n}+1}}{d_{n:n}+1}\right) \tag{17}$$

$$E_{n:n}(p^2/t_1, t_2, \ldots, t_n) = \frac{1}{D_{n:n1}}\int_q^h p^2.\left(n(1-p)\theta^{n(1-p)}\right)^n A^{n(1-p)-1} p^{h-1} dp$$

$$= \frac{1}{D_{n:n1}}\sum_{i=0}^{n}\sum_{k=0}^{\infty}\sum_{s=0}^{k}\sum_{l=0}^{k-s} C_{n:n}\left(\frac{q^{d_{n:n}+2} - h^{d_{n:n}+2}}{d_{n:n}+2}\right) \tag{18}$$

Variance of the parameter p in both cases can be find by using (15),(16) in the following expressions:

$$V(p^2/x_1, x_2, \ldots, x_n) = E(p^2/x_1, x_2, \ldots, x_n) - [E(p/x_1, x_2, \ldots, x_n)]^2$$

and (17), (18) in the following expression:

$$V_{n:n}(p^2/x_1, x_2, \ldots, x_n) = E_{n:n}(p^2/x_1, x_2, \ldots, x_n) - [E_{n:n}(p/x_1, x_2, \ldots, x_n)]^2$$

Now, Bayesian estimate of the reliability function $R(t)$ and $R_{n:n}(t)$ defined by (3), (6) respectively can be obtained as

$$E[R(t)] = \frac{1}{D_1}\int_q^h \left(1 - \theta^{p-1}t^{1-p}\right).\left(\frac{1-p}{\theta^{p-1}}\right)^n A^p p^{h-1} dp$$

$$= 1 - \frac{1}{D1}\left[\sum_{i=0}^{n}\sum_{k=0}^{\infty}\sum_{s=o}^{\infty}\sum_{j=0}^{s} E^R\left(\frac{q^{d^r} - h^{d^r}}{d^r}\right)\right] \tag{19}$$

$$E^R = n_{C_i} s_{C_j}(-1)^{n-i+s-j}\theta^{(1-p)(n-1)}\frac{(logA)^k}{k!}\frac{(logt)^s}{s!} \quad \text{and}$$

$$d^r = n - i + h + k + s - j$$

Similarly,

$$E[R^2(t)] = \frac{1}{D_1}\int_q^h \left(1 - \theta^{p-1}t^{1-p}\right)^2.\left(\frac{1-p}{\theta^{p-1}}\right)^n A^p p^{h-1} dp$$

$$= 1 + \frac{1}{D1}\left[\sum_{i=0}^{n}\sum_{k=0}^{\infty}\sum_{s=o}^{\infty}\sum_{j=0}^{s}\left(E^{2R} - 2E^R\right)\left(\frac{q^{d^r} - h^{d^r}}{d^r}\right)\right], \quad \text{where} \tag{20}$$

$$E^{2R} = n_{C_i} s_{C_j} (-1)^{n-i+s-j} \theta^{(1-p)(n-1)} \frac{(logA)^k}{k!} \frac{(2logt)^s}{s!}$$

Similarly, and for order statistics

$$\begin{aligned} E_{n:n}[R(t)] &= \frac{1}{D_{n:n1}} \int_h^q \left(1{-}\theta^{n(p-1)} t^{n(1-p)}\right) . \left(n(1-p)\theta^{n(1-p)}\right)^n A^{n(1-p)-1} p^{h-1} dp \\ &= 1 - \frac{1}{D_{n:n1}} \left[\sum_{i=0}^{n} \sum_{k=0}^{\infty} \sum_{s=o}^{\infty} \sum_{j=0}^{k+s} E_{n:n}^{R} \left(\frac{q^{d_{n:n}^r} - h^{d_{n:n}^r}}{d_{n:n}^r} \right) \right], \end{aligned} \tag{21}$$

$$\text{where} \quad E_{n:n}^{R} = ^n C_i \ ^{(k+s)} C_j (-1)^{n-i+k+s-j} n^{n+k} \theta^{[n(1-p)+(n(1-p))^n]} \frac{(nlogA)^k}{k!} \frac{(logt)^s}{s!}, \quad \text{and}$$

$$d_{n:n}^r = n - i + h + k + s - j$$

$$\begin{aligned} E_{n:n}[R^2(t)] &= \frac{1}{D_{n:n1}} \int_h^q \left(1{-}\theta^{n(p-1)} t^{n(1-p)}\right)^2 . \left(n(1-p)\theta^{n(1-p)}\right)^n A^{n(1-p)-1} p^{h-1} dp \\ &= 1 + \frac{1}{D_{n:n1}} \left[\sum_{i=0}^{n} \sum_{k=0}^{\infty} \sum_{s=o}^{\infty} \sum_{j=0}^{k+s} \left(E_{n:n}^{2R} - 2E_{n:n}^{R}\right) \left(\frac{q^{d_{n:n}^r} - h^{d_{n:n}^r}}{d_{n:n}^r} \right) \right], \end{aligned} \tag{22}$$

$$\text{where} \quad E_{n:n}^{2R} = ^n C_i \ ^{(k+s)} C_j (-1^{n-i+k+s-j}) n^{n+k} \theta^{[n(1-p)+(n(1-p))^n]} \frac{(nlogA)^k}{k!} \frac{(2logt)^s}{s!}$$

Using values from equations from (19) and (20) in (23) and (21),similarly (22) in (24),we can obtain the Bayes' estimates of variance of the Reliability function $R(t)$ and $R_{n:n}(t)$ as:

$$V\left[R(t)\right] = E\left[R^2(t)\right] - \left[E(R(t))\right]^2 \tag{23}$$

$$V_{n:n}\left[R(t)\right] = E_{n:n}\left[R^2(t)\right] - \left[E_{n:n}(R(t))\right]^2 \tag{24}$$

Now the Bayes' estimates of the Hazard rate function $H(t)$ in the equation (4) can be given as:

$$\begin{aligned} E\left[H(t)\right] &= \frac{1}{D_1} \int_h^q (1-p) t^{-1} \left(\theta^{1-p} t^{p-1} - 1\right)^{-1} . \left(\frac{1-p}{\theta^{p-1}}\right)^n A^p p^{h-1} dp \\ &= \frac{1}{D_1} \left[\sum_{i=0}^{n+1} \sum_{k=0}^{\infty} \sum_{m'=0}^{\infty} \sum_{k'=0}^{\infty} \sum_{s=0}^{k'} E^H \frac{\left(q^{d^h+1} - h^{d^h+1}\right)}{d^h+1} \right] \end{aligned} \tag{25}$$

Similarly,

$$\begin{aligned} E\left[H^2(t)\right] &= \frac{1}{D_1} \int_h^q (1-p)^2 t^{-2} \left(\theta^{1-p} t^{p-1} - 1\right)^{-2} . \left(\frac{1-p}{\theta^{p-1}}\right)^n A^p p^{h-1} dp \\ &= \frac{1}{D_1} \left[\sum_{i=0}^{n+1} \sum_{k=0}^{\infty} \sum_{m'=0}^{\infty} \sum_{k'=0}^{\infty} \sum_{s'=0}^{k'} E^{2H} \frac{\left(q^{d^h+2} - h^{d^h+2}\right)}{d^h+2} \right] \end{aligned} \tag{26}$$

Where,

$$E^H = ^{(n+1)} C_i \ ^{k'} C_s \ ^s C_j (-1)^{n+1-i+m'+s+j} \theta^{(n+m')(1-p)} (m')^s \frac{(logA)^k}{k!} \frac{(logt)^{k'}}{k'!}$$

$$E^{2H} = ^{(n+2)} C_i \ ^{k'} C_s \ ^s C_j (-1)^{n+2-i+m'+s+j} \theta^{(n+m')(1-p)} (m')^s (m'+1) \frac{(logA)^k}{k!} \frac{(logt)^{k'}}{k'!}, \quad \text{and}$$

$$d^h = n - i + h + k + j$$

For order statistics:

$$E_{n:n}\left[H(t)\right] = \frac{1}{D_{n:n1}} \int_h^q n(1-p) t^{-1} \left(\theta^{n(1-p)} t^{n(p-1)} - 1\right)^{-1} . \left(n(1-p)\theta^{n(1-p)}\right)^n A^{n(1-p)-1} p^{h-1} dp$$

$$= \frac{1}{D_{n:n1}}\left[\sum_{i=0}^{n+1}\sum_{k=0}^{\infty}\sum_{m'=0}^{\infty}\sum_{k'=0}^{\infty}\sum_{s=0}^{k'}\sum_{j=0}^{s}\sum_{s'=0}^{k}\sum_{l=0}^{k-s'} E_{n:n}^{H}\frac{\left(q^{d_{n:n}^{h}+1}-h^{d_{n:n}^{h}+1}\right)}{d_{n:n}^{h}+1}\right] \tag{27}$$

$$E_{n:n}\left[H^2(t)\right] = \frac{1}{D_{n:n1}}\int_h^q n(1-p)^2 t^{-2}\left(\theta^{n(1-p)}t^{n(p-1)}-1\right)^{-2}.\left(n(1-p)\theta^{n(1-p)}\right)^n A^{n(1-p)-1}p^{h-1}dp$$

$$= \frac{1}{D_{n:n1}}\left[\sum_{i=0}^{n+2}\sum_{k=0}^{\infty}\sum_{m'=0}^{\infty}\sum_{k'=0}^{\infty}\sum_{s=0}^{k'}\sum_{j=0}^{s}\sum_{s'=0}^{k}\sum_{l=0}^{k-s'} E_{n:n}^{2H}\frac{\left(q^{d_{n:n}^{h}+2}-h^{d_{n:n}^{h}+2}\right)}{d_{n:n}^{h}+2}\right] \tag{28}$$

$$E_{n:n}^{H} = {}^{(n+1)}C_i\;{}^{(k'-s)}C_s\;{}^{s}C_j(-1)^{n+1-i+m+k'-j+k-s'-l}\,\theta^{(n+m')(1-p)}(m')^{(k'-s)}n^{n+k-s'}\frac{(nlogA)^k}{k!}\frac{(logt)^{k'}}{k'!}$$

$$E_{n:n}^{2H} = {}^{(n+2)}C_i\;{}^{(k'-s)}C_s\;{}^{s}C_j(-1)^{n+2-i+m+k'-j+k-s'-l}\,\theta^{(n+m')(1-p)}(m')^{(k'-s)}n^{n+k-s'}(m'+1)\frac{(nlogA)^k}{k!}\frac{(logt)^{k'}}{k'!}$$

$$\text{and}\quad d_{n:n}^{h} = n-i+h+m'+k'-j+k-s'-l$$

Using the values from (25) and (26) in the following expressions(29), we can obtain the Bayes' estimates of the variance of the Hazard Rate function $H(t)$ as:

$$V[H(t)] = E[H^2(t)] - E[H(t)]^2 \tag{29}$$

Similarly using (27) and (28) in (30) as:

$$V_{n:n}[H(t)] = E_{n:n}[H^2(t)] - E_{n:n}[H(t)]^2 \tag{30}$$

will give Bayesian estimation of order statistics of failure model.

3. When θ is a Variable

When shape parameter θ is a variable than following prior density [7] provides the posterior distribution by equations (32) and (33).

$$g(\theta) = \frac{a}{b^a}\theta^{a-1};\;\; 0 \le \theta \le b \tag{31}$$

The likelihood conditional under the assumption of prior density (31) provides the following posterior distribution for SD distribution [2] and its order statistics [3].

$$G(\theta/x_1, x_2, \ldots, x_n) = \frac{\left(\frac{1-p}{\theta^{p-1}}\right)^n \prod_{i=1}^{n} x_i^{-p}\frac{a}{b^a}\theta^{a-1}}{\int_0^b \left(\frac{1-p}{\theta^{p-1}}\right)^n \prod_{i=1}^{n} x_i^{-p}\frac{a}{b^a}\theta^{a-1}d\theta} \tag{32}$$

$$G_{n:n}(\theta/x_1, x_2 \ldots . x_n) = \frac{\left(n(1-p)\theta^{n(1-p)}\right)^n \prod_{i=1}^{n} x_i^{(1-p)-1}\frac{a}{b^a}\theta^{a-1}}{\int_0^b \left(n(1-p)\theta^{n(1-p)}\right)^n \prod_{i=1}^{n} x_i^{-p}\frac{a}{b^a}\theta^{a-1}d\theta} \tag{33}$$

$$D_2 = \int_0^b \left(\frac{1-p}{\theta^{p-1}}\right)^n \prod_{i=1}^{n} x_i^{-p}\frac{a}{b^a}\theta^{a-1}d\theta \tag{34}$$

$$= \frac{b^{a-n(p-1)-1}}{a+n(1-p)}$$

$$D_{n:n2} = \int_0^b \left(n(1-p)\theta^{n(1-p)}\right)^n \prod_{i=1}^{n} x_i^{n(1-p)-1}\frac{a}{b^a}\theta^{a-1}d\theta \tag{35}$$

$$= \frac{b^{a-n(p-1)-1}}{(a+n(1-p))^n}$$

Now Bayes' estimates of the parameter θ and θ^2 can be obtained as:

$$E\left(\theta/x_1, x_2, \ldots, x_n\right) = \frac{1}{D_2}\int_0^b \theta.\left(\frac{1-p}{\theta^{p-1}}\right)^n \prod_{i=1}^{n} x_i^{-p}\theta^{a-1}d\theta$$

$$= \frac{1}{D_2} B_i \left[\frac{b^{a+n(1-p)+1}}{a+n(1-p)+1} \right], \quad \text{and} \tag{36}$$

$$E\left(\theta^2 / x_1, x_2, \ldots, x_n\right) = \frac{1}{D_2} \int_0^b \theta^2 . \left(\frac{1-p}{\theta^{p-1}}\right)^n \prod_{t=1}^{n} x_i^{-p} \theta^{a-1} d\theta$$

$$= \frac{1}{D_2} B_i \left[\frac{b^{a+n(1-p)+2}}{a+n(1-p)+2} \right] \tag{37}$$

$$E_{n:n}\left(\theta / x_1, x_2, \ldots, x_n\right) = \frac{1}{D_{n:n2}} \int_0^b \theta . \left(n(1-p)\theta^{n(1-p)}\right)^n \prod_{i=1}^{n} x_i^{n(1-p)-1} \theta^{a-1} d\theta$$

$$= \frac{1}{D_{n:n2}} B_{n:n} \left[\frac{b^{a+n(1-p)^n+1}}{a+n(1-p)^n+1} \right] \tag{38}$$

$$E_{n:n}\left(\theta^2 / x_1, x_2, \ldots, x_n\right) = \frac{1}{D_{n:n2}} \int_0^b \theta^2 . \left(n(1-p)\theta^{n(1-p)}\right)^n \prod_{i=1}^{n} x_i^{n(1-p)-1} \theta^{a-1} d\theta$$

$$= \frac{1}{D_{n:n2}} B_{n:n} \left[\frac{b^{a+n(1-p)^n+2}}{a+n(1-p)^n+2} \right] \tag{39}$$

Where, D_2, $D_{n:n2}$ is defined by equations (34) and (35) respectively. Here,

$$B_i = (1-p)^n \prod_{i=1}^{n} x_i^{-p} \quad \text{and}$$

$$B_{n:n} = (n(1-p))^n \prod_{i=1}^{n} x^{n(1-p)-1}$$

Variance in both cases can be find by (40) and (41) by substituting values from (36), (37) and (38), (39) respectively.

$$V(\theta/x_1, x_2, \ldots, x_n) = E(\theta^2/x_1, x_2, \ldots, x_n) - [E(\theta/x_1, x_2, \ldots, x_n)]^2 \tag{40}$$

$$V_{n:n}(\theta/x_1, x_2, \ldots, x_n) = E_{n:n}(\theta^2/x_1, x_2, \ldots, x_n) - [E_{n:n}(\theta/x_1, x_2, \ldots, x_n)]^2 \tag{41}$$

Now, the Bayes' estimates of the reliability function $R(t)$ in equation (3) will be obtained as:

$$E[R(t)] = \frac{1}{D_2} \int_0^b (1-\theta^{p-1} t^{1-p}) \left(\frac{1-p}{\theta^{p-1}}\right)^n A^p \theta^{a-1} d\theta$$

$$= 1 - \frac{1}{D_2} B_i^R \left[\frac{b^{a+(n-1)(1-p)-1}}{a+(n-1)(1-p)-1} \right] \tag{42}$$

$$E[R^2(t)] = \frac{1}{D_2} \int_0^b (1-\theta^{p-1} t^{1-p})^2 \left(\frac{1-p}{\theta^{p-1}}\right)^n A^p \theta^{a-1} d\theta$$

$$= 1 - \frac{1}{D_2} \left[2B_i^R \left(\frac{b^{a(n-1)(1-p)}}{a+(n-1)(1-p)} \right) + B_i^{2R} \left(\frac{b^{a+(n-2)(1-p)}}{a+(n-2)(1-p)} \right) \right] \tag{43}$$

Where, $B_i^R = (1-p)^n A^p t^{1-p}$; $B_i^{2R} = (1-p)^n A^p t^{2(1-p)}$. Using the values from (42) and (43) in the following expressions we can obtain the Bayes'estimates of the Variance of the reliability function $R(t)$ as:

$$V[R(t)] = E\left[R^2(t)\right] - E\left[R(t)\right]^2 \tag{44}$$

$$E_{n:n}[R(t)] = \frac{1}{D_{n:n2}} \int_0^b (1-\theta^{n(p-1)} t^{n(1-p)}) \left(n(1-p)\theta^{n(p-1)}\right)^n A^{n(1-p)-1} \theta^{a-1} d\theta$$

$$= 1 - \frac{1}{D_{n:n2}} B_{n:n}^R \left[\frac{b^{a+p+(n(1-p))^n-1}}{a+p+(n(1-p))^n-1} \right] \tag{45}$$

$$E_{n:n}[R^2(t)] = \frac{1}{D_{n:n2}} \int_0^b (1-\theta^{n(p-1)} t^{n(1-p)})^2 \left(n(1-p)\theta^{n(p-1)}\right)^n A^{n(1-p)-1} \theta^{a-1} d\theta$$

$$= 1 - \frac{1}{D_{n:n2}} \left[2B_{n:n}^R \left(\frac{b^{a+p+(n(1-p))^n-1}}{a+p+(n(1-p))^n-1} \right) + B_{n:n}^{2R} \left(\frac{b^{a+p+(n(1-p))^n-2}}{a+p+(n(1-p))^n-2} \right) \right] \tag{46}$$

Where,

$$B_{n:n}^{R} = (n(1-p))^{n} A^{n(1-p)-1} t^{n(1-p)}$$
$$B_{n:n}^{2R} = (n(1-p))^{n} A^{n(1-p)-1} t^{2n(1-p)}$$

Variance for order statistics for reliability function when θ is a variable can be found by using (45) and (46) in

$$V_{n:n}[R(t)] = E_{n:n}\left[R^{2}(t)\right] - E_{n:n}\left[R(t)\right]^{2} \tag{47}$$

Similarly, Bayes estimates of the Hazard rate function $H(t)$ will be obtained as

$$\begin{aligned} E[H(t)] &= \frac{1}{D_2}\int_0^b (1-p)t^{-1}\left(\theta^{1-p}t^{p-1}-1\right)^{-1}\left(\frac{1-p}{\theta^{p-1}}\right)^{n} A^{p}\theta^{a-1}d\theta \\ &= \frac{1}{D_2}\sum_{m'=0}^{\infty} C_i^{H}\left(\frac{b^{a+(1-p)(n+m')}}{a+(1-p)(n+m')}\right) \end{aligned} \tag{48}$$

$$\begin{aligned} E[H^2(t)] &= \frac{1}{D_2}\int_0^b (1-p)^2t^{-2}\left(\theta^{1-p}t^{p-1}-1\right)^{-2}\left(\frac{1-p}{\theta^{p-1}}\right)^{n} A^{p}\theta^{a-1}d\theta \\ &= \frac{1}{D_2}\sum_{m''=0}^{\infty} C_i^{2H}\left(\frac{b^{a+(1-p)(n+2m'')}}{a+(1-p)(n+2m'')}\right) \end{aligned} \tag{49}$$

Where,

$$C_i^{H} = (1-p)^{n+1}(-1)^{m'} A^{p} t^{m'(p-1)-1}$$
$$C_i^{2H} = (1-p)^{n+2}(-1)^{m''} A^{p} t^{2m''(p-1)-2}$$

$$\begin{aligned} E_{n:n}[H(t)] &= \frac{1}{D_{n:n2}}\int_0^b (1-p)t^{-1}\left(\theta^{1-p}t^{p-1}-1\right)^{-1}\left(n(1-p)\theta^{n(1-p)}\right)^{n} A^{n(1-p)-1}\theta^{a-1}d\theta \\ &= \frac{1}{D_{n:n2}}\sum_{m'=0}^{\infty} C_{n:n}^{H}\left(\frac{b^{a+(1-p)m'+(n(1-p))^n}}{a+(1-p)m'+(n(1-p))^n}\right) \end{aligned} \tag{50}$$

$$\begin{aligned} E_{n:n}[H^2(t)] &= \frac{1}{D_{n:n2}}\int_0^b (1-p)^2t^{-2}\left(\theta^{1-p}t^{p-1}-1\right)^{-2}\left(n(1-p)\theta^{n(1-p)}\right)^{n} A^{n(1-p)-1}\theta^{a-1}d\theta \\ &= \frac{1}{D_{n:n2}}\sum_{m''=0}^{\infty} C_{n:n}^{2H}\left(\frac{b^{a+(1-p)m''+(n(1-p))^n}}{a+(1-p)m''+(n(1-p))^n}\right) \end{aligned} \tag{51}$$

Where

$$C_{n:n}^{H} = (1-p)^{n+1}(-1)^{m'} A^{n(1-p)-1} t^{m'(p-1)-1}$$
$$C_{n:n}^{2H} = (1-p)^{n+2}(-1)^{m''} A^{n(1-p)-1} t^{2m''(p-1)-2}$$

Using the values from (48) and (49) in (52) and (50), (51) in (53) respectively, we can obtain the Bayes'estimates of the Variance of the Hazard Rate function $H(t)$ and its order statistics $H_{n:n}(t)$ as:

$$V[H(t)] = E[H(t)] - E[H^2(t)] \tag{52}$$
$$V_{n:n}[H(t)] = E_{n:n}[H(t)] - E_{n:n}[H^2(t)] \tag{53}$$

4. Concluding Remark

In the given mathematical literature here, it is observed that the results presented on Bayesian estimates were complicated, but a comprehensive programming in different computer software can tackle these complexities. The priors are successfully tried in this paper in order to reduce the complexity of the results. In addition, the newly introduced SD distribution is much more flexible and useful for modelling failure time data which is already established.

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