Int. J. Math. And Appl., 9(3)(2021), 81–93
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

Generalized Hyers-Ulam Type Stability of the $2k$-Variables Quadratic β-Functional Inequalities And Function in γ-Homogeneous Normed Space

Ly Van An[1,*]

1 Faculty of Mathematics Teacher Education, Tay Ninh University, Ninh Trung, Ninh Son, Tay Ninh Province, Vietnam.

Abstract: In this paper, we study to solve two quadratic β-functional inequalities with $2k$-variables in γ-homogeneous complex Banach spaces and prove the Hyers-Ulam stability of quadratic β-functional equations associated two the quadratic β-functional inequalities in γ-homogeneous complex Banach spaces. We will show that the solutions of the first and second inequalities are quadratic mappings.

MSC: 4610, 4710, 39B62, 39B72, 39B52.

Keywords: Hyers-Ulam stability γ-homogeneous space; quadratic β-functional equation; β-functional inequality.

1. Introduction

Let $\mathbf{X}$ and $\mathbf{Y}$ be a $\gamma-homogeneous$ normed spaces on the same filed $\mathbb{K}$, and $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ be a mapping. We use the notation $\Big\|\cdot\Big\|$ for the norms on both $\mathbf{X}$ and $\mathbf{Y}$. In this paper, we investigate first functional inequalities when $\mathbf{X}$ is a $\gamma-homogeneous$ real or complex Banach space and $\mathbf{Y}$ is a $\gamma-homogeneous$ complex Banach space

$$\begin{aligned}&\left\|f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)+f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f\Big(x_j\Big)\right\|_{\mathbf{Y}}\\&\le\left\|\beta\left(kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f\Big(x_j\Big)\right)\right\|_{\mathbf{Y}}\end{aligned}\tag{1}$$

where β is fixed complex number with $\Big|\beta\Big|<1$, and

when $\mathbf{X}$ is a $\gamma-homogeneous$ real or complex Banach space and $\mathbf{Y}$ is a $\gamma-homogeneous$ complex Banach space

$$\begin{aligned}&\left\|kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f(x_j)\right\|_{\mathbf{Y}}\\&\le\left\|\beta\left(f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)+f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f(x_j)\right)\right\|_{\mathbf{Y}}\end{aligned}\tag{2}$$

where β is a fixed complex number with $\Big|\beta\Big|<\frac{1}{2}$.

* *E-mail: lyvanan145@gmail.com*

The notions of homogeneous real or complex Banach space will remind in the next section. The Hyers-Ulam stability was first investigated for functional equation of Ulam in [31, 32] concerning the stability of group homomorphisms. The functional equation

$$f\left(x+y\right)=f\left(x\right)+f\left(y\right)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping. The Hyers [16] gave firts affirmative partial answer to the equation of Ulam in Banach spaces. After that, Hyers'Theorem was generalized by Aoki [1] additive mappings and by Rassias [27] for linear mappings considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [15] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

$$f\left(\frac{x+y}{2}\right)=\frac{1}{2}f\left(x\right)+\frac{1}{2}f\left(y\right)$$

is called the Jensen equation. The functional equations

$$f\left(x+y\right)+f\left(x-y\right)=2f\left(x\right)+2f\left(y\right)$$

is called the quadratic functional equations. The functional equations

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=2f\left(x\right)+2f\left(y\right)$$

is called the Jensen type quadratic functional equations [17, 20, 21] for more information on functional equations. The Hyers-Ulam stability for functional inequalities have been investigated such as in [13]. Gil*á*nyi showed that is if satisfies the functional inequality

$$\left\|2f\left(x\right)+2f\left(y\right)-f\left(xy^{-1}\right)\right\|\leq\left\|f\left(xy\right)\right\| \tag{3}$$

Then f satisfies the Jordan-von Newman functional equation

$$2f\left(x\right)+2f\left(y\right)=f\left(xy\right)+f\left(xy^{-1}\right)$$

See also [28]. Gil*á*nyi [13, 14] and Fechner [12] proved the Hyers-Ulam stability of the functional inequality. Choonkil $Park^a$ [9] proved the quadratic ρ-functional inequalities. Recently, in [2, 3, 4, 9] the authors studied the Hyers-Ulam stability for the following quadratic functional inequalities.

$$\left\|f\left(x+y\right)\ +f\left(x+y\right)-2f\left(x\right)-2f\left(y\right)\right\|\leq\left\|\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f\left(x\right)-f\left(y\right)\right)\right\|$$

and

$$\left\|2f\left(\frac{x+y}{2}\right)\ +2f\left(\frac{x-y}{2}\right)-f\left(x\right)-f\left(y\right)\right\|\leq\left\|\rho\left(f\left(x+y\right)+f\left(x+y\right)-2f\left(x\right)-2f\left(y\right)\right)\right\|$$

next

$$\left\|f\left(\tfrac{x+y}{2}+z\right)+f\left(\tfrac{x+y}{2}-z\right)-2f\left(\tfrac{x+y}{2}\right)-2f(z)\right\|\leq\left\|\beta\left(2f\left(\tfrac{x+y}{2^2}+\tfrac{z}{2}\right)+2f\left(\tfrac{x+y}{2^2}-\tfrac{z}{2}\right)-f\left(\tfrac{x+y}{2}\right)-f(z)\right)\right\| \tag{4}$$

and

$$\left\|2f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)+2f\left(\frac{x+y}{2^2}-\frac{z}{2}\right)-f\left(\frac{x+y}{2}\right)-f\left(z\right)\right\| \leq \left\|\beta\left(f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)-2f\left(\frac{x+y}{2}\right)-2f\left(z\right)\right)\right\| \quad (5)$$

in homogeneous real and complex Banach spaces.

In this paper, we solve and proved the Hyers-Ulam stability for two quadratic β-functional inequalities (1)-(2), ie the quadratic β -functional inequalities with $2k-$variables. Under suitable assumptions on spaces $\mathbb{X}$ and $\mathbb{Y}$, we will prove that the mappings satisfying the quadratic β-functional inequatilies (1) or (2). Thus, the results in this paper are generalization of those in [2, 3, 4, 9] for quadratic β-functional inequatilies with $2k$-variables. The paper is organized as followings: In section preliminaries we remind some basic notations in [29] such as $F-norm$ is called γ-homogeneous $\left(\gamma>0\right)$. Section 3 is devoted to prove the Hyers-Ulam stability of the quadratic β- functional inequalities $\left(\left|\beta\right|<1\right)$ (1) and (2) when X is γ_1-homogeneous $\left(\gamma_1\leq 1\right)$ real or complex normed space and Y is γ_2-homogeneous $\left(\gamma_2\leq 1\right)$ complex Bananch space. Section 4 is devoted to prove the Hyers-Ulam stability of the quadratic β- functional inequalities $\left(\left|\beta\right|<\frac{1}{2}\right)$ (1) and (2) when X is γ_1-homogeneous $\left(\gamma_1\leq 1\right)$ real or complex normed space and Y is γ_2-homogeneous $\left(\gamma_2\leq 1\right)$ complex Bananch space.

2. Preliminaries

Definition 2.1 (F^*- spaces)**.** *Let X be a linear space. A nonnegative valued function $\left\|\cdot\right\|$ is an $F-norm$ if it satisfies the following conditions:*

(1). $\left\|x\right\|=0$ *if and only if* $x=0$*;*

(2). $\left\|\lambda x\right\|=\left\|x\right\|$ *for all* $x\in X$ *and all* λ *with* $\left|\lambda\right|=1$*;*

(3). $\left\|x+y\right\|\leq\left\|x\right\|+\left\|y\right\|$ *for all* $x,y\in X$*;*

(4). $\left\|\lambda_n x\right\|\to 0,\ \lambda_n\to 0$*;*

(5). $\left\|\lambda_n x\right\|\to 0,\ x_n\to 0$.

Then $\left(X,\left\|\cdot\right\|\right)$ is called an F^*-space. An F-space is a complete F^*-space. An F-norm is called β-homgeneous $\left(\beta>0\right)$ if $\left\|tx\right\|=\left|t\right|^{\beta}\left\|x\right\|$ for all $x\in X$ and $t\in\mathbb{C}$.

2.1. Solutions of the inequalities

The functional equation

$$f\left(x+y\right)=f\left(x\right)+f\left(y\right)$$

is called the cauchuy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping. The functional equation $f\left(\frac{x+y}{2}\right)=\frac{1}{2}f\left(x\right)+\frac{1}{2}f\left(y\right)$ called the Jensen equation. The functional equation

$$f\left(x+y\right)+f\left(x-y\right)=2f\left(x\right)+2f\left(y\right)$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. The stability of quadratic functional equation was proved by Skof [30] for mappings $f:E_1\to E_2$, where

E_1 is a normed space and E_2 is a Banach space. Cholewe [8] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The functional equation

$$2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)=f\left(x\right)+f\left(y\right)$$

is called a Jensen type quadratic equation [17, 20, 21]. Throughout this paper, let k be a fixed integer with $k\geq 2$.

3. Quadratic γ-functional Inequality

In This section, assume that β is a fixed complex number with $|\beta|<1$. We investigate the quadratic β-fuctional inequality (1) in γ-homogeneous complex Banach space.

Lemma 3.1. *An even mapping $f:\mathbf{X}\to\mathbf{Y}$ satisfies*

$$\begin{aligned}&\left\|f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)+f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f\left(x_j\right)\right\|_{\mathbf{Y}}\\&\leq\left\|\beta\left(kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f\left(x_j\right)\right)\right\|_{\mathbf{Y}}\end{aligned}\tag{6}$$

for all $x_j,x_{k+j}\in\mathbf{X}$ for all $j=1\to k$ if and only if $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ is quadratic mapping.

Proof. Assume that $f:X\to Y$ satisfies (6). Letting $x_j=x_{k+j}=0$, for all $j=1\to k$ in (6), we get

$$\left|2k-1\right|^{\gamma_2}\left\|f\left(0\right)\right\|_{\mathbf{Y}}\leq\left|k\beta\right|^{\gamma_2}\left\|f\left(0\right)\right\|_{\mathbf{Y}}.$$

So $f(0)=0$. Letting $x_{k+j}=0$, $x_j=x$ for all $j=1\to k$ in (6), we get $\left\|f\left(kx\right)-kf\left(x\right)\right\|_{\mathbf{Y}}\leq 0$ and so $f\left(kx\right)=kf\left(x\right)$ for all $x\in X$. Thus

$$f\left(\frac{x}{k}\right)=\frac{1}{k}f(x)\ \ \forall\ \ x\in X\tag{7}$$

It follows from (6) and (7) that:

$$\begin{aligned}&\left\|f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)+f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}-\sum_{j=1}^{k}z_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{k}\right)-2\sum_{j=1}^{k}f\left(z_j\right)\right\|_{\mathbf{Y}}\\&\leq\left\|\beta\left(kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f(x_j)\right)\right\|_{\mathbf{Y}}\\&=\left|\beta\right|^{\gamma_2}\left\|\left(f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)+f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f(x_j)\right)\right\|_{\mathbf{Y}}\end{aligned}$$

and so

$$f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)+f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}-\sum_{j=1}^{k}x_j\right)=2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)+2\sum_{j=1}^{k}f\left(x_j\right)\tag{8}$$

for all $x_j,x_{k+j}\in X$ for all $j=1\to k$. The converse is obviously true. □

Lemma 3.2. *An even mapping $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ satisfies $f\left(0\right)=0$ and*

$$\begin{aligned}&\left\|kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f\left(x_j\right)\right\|_{\mathbf{Y}}\\&\leq\left\|\beta\left(f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)+f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f\left(x_j\right)\right)\right\|_{\mathbf{Y}}\end{aligned}\tag{9}$$

for all $x,y,z\in\mathbf{X}$ if and only if $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ is quadratic.

Proof. Assume that $f: X \to Y$ satisfies (9). Letting $x_{j+1} = x_{k+j} = 0$ and $x_1 = x$ for all $j = 1 \to k$ in (9), we get

$$\left\|kf\left(\frac{x}{k}\right) - f\left(x\right)\right\|_{\mathbf{Y}} \leq 0$$

Thus

$$f\left(\frac{x}{k}\right) = \frac{1}{k}f\left(x\right) \tag{10}$$

for all $x \in X$ It follows from (8) and (9) that:

$$\begin{aligned}&\left\|kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} + \frac{1}{k}\sum_{j=1}^{k}x_j\right) + kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} - \frac{1}{k}\sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right) - 2\sum_{j=1}^{k}f\left(x_j\right)\right\|_{\mathbf{Y}}\\
&= \left\|f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} + \sum_{j=1}^{k}x_j\right) + f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} - \sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right) - 2\sum_{j=1}^{k}f\left(x_j\right)\right\|_{\mathbf{Y}}\\
&\leq |\beta|^{\gamma_2}\left\|\left(f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} + \sum_{j=1}^{k}x_j\right) + f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} - \sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right) - 2\sum_{j=1}^{k}f\left(x_j\right)\right)\right\|_{\mathbf{Y}}\end{aligned}$$

and so

$$f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} + \sum_{j=1}^{k}x_j\right) + f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} - \sum_{j=1}^{k}x_j\right) = 2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right) + 2\sum_{j=1}^{k}f\left(x_j\right) \tag{11}$$

for all $x_j, x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$. The converse is obviously true. □

From Lemma 3.1 and Lemma 3.2 we have corollaries.

Corollary 3.3. *An even mapping $f: \mathbf{X} \to \mathbf{Y}$ satisfies*

$$\begin{aligned}&f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} + \sum_{j=1}^{k}x_j\right) + f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} - \sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right) - 2\sum_{j=1}^{k}f\left(x_j\right)\\
&= \beta\left(kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} + \frac{1}{k}\sum_{j=1}^{k}x_j\right) + kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} - \frac{1}{k}\sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right) - 2\sum_{j=1}^{k}f\left(x_j\right)\right)\end{aligned} \tag{12}$$

for all $x_j, x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$ if and only if $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ is quadratic mapping.

Corollary 3.4. *An even mapping $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ satisfies $f(0) = 0$ and*

$$\begin{aligned}&kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} + \frac{1}{k}\sum_{j=1}^{k}x_j\right) + kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} - \frac{1}{k}\sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right) - 2\sum_{j=1}^{k}f\left(x_j\right)\\
&= \beta\left(f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} + \sum_{j=1}^{k}x_j\right) + f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} - \sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right) - 2\sum_{j=1}^{k}f\left(x_j\right)\right)\end{aligned} \tag{13}$$

for all $x, y, z \in \mathbf{X}$ if and only if $\mathbf{f}$:$\mathbf{X} \to \mathbf{Y}$ is quadratic.

The equations (12) and (13) is called a quadratic type β-functional equation.

Theorem 3.5. *Let $r > \frac{2\gamma_2}{\gamma_1}$ and θ be positive real numbers, and let $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ is an even mapping satisfies*

$$\begin{aligned}&\left\|f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} + \sum_{j=1}^{k}x_j\right) + f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} - \sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right) - 2\sum_{j=1}^{k}f\left(z_j\right)\right\|_{\mathbf{Y}}\\
&\leq \left\|\beta\left(kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} + \frac{1}{k}\sum_{j=1}^{k}x_j\right) + kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} - \frac{1}{k}\sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)\right.\right.\end{aligned}$$

$$-2\sum_{j=1}^{k} f\left(x_j\right)\Bigg)\Bigg\|_{\mathbf{Y}} + \theta\left(\sum_{j=1}^{k}\left\|x_j\right\|^r + \sum_{j=1}^{k}\left\|x_{k+j}\right\|^r\right) \tag{14}$$

for all $x_j, x_{k+j} \in X$ for all $j = 1 \to k$. Then there exists a unique quadratic mapping $H : X \to Y$ such that

$$\left\|f(x) - H(x)\right\|_{\mathbf{Y}} \leq \frac{k\theta}{k^{\gamma_1 r} - k^{\gamma_2}} \tag{15}$$

for all $x \in X$.

Proof. Letting $x_j = x_{k+j} = 0$ for all $j = 1 \to k$, in (14) we get

$$\left|2k - 1\right|^{\gamma_2}\left\|f(0)\right\|_{\mathbf{Y}} \leq \left|k\beta\right|^{\gamma_2}\left\|f(0)\right\|_{\mathbf{Y}}.$$

So $f(0) = 0$. Letting $x_{k+j} = 0$ and $x_j = x$ for all $j = 1 \to k$ in (14), we get

$$\left\|f(kx) - kf(x)\right\|_{\mathbf{Y}} \leq k\theta\left\|x\right\|^r \tag{16}$$

for all $x \in X$. So

$$\left\|f(x) - kf\left(\frac{x}{k}\right)\right\|_{\mathbf{Y}} \leq \frac{k}{k^{\gamma_1 r}}\theta\left\|x\right\|^r$$

for all $x \in \mathbf{X}$. Hence

$$\left\|k^l f\left(\frac{x}{k^l}\right) - k^m f\left(\frac{x}{k^m}\right)\right\|_{\mathbf{Y}} \leq \sum_{j=l}^{m-1}\left\|k^j f\left(\frac{x}{k^j}\right) - k^{j+1} f\left(\frac{x}{k^{j+1}}\right)\right\|_{\mathbf{Y}} \leq \frac{k\theta}{k^{\gamma_1 r}}\sum_{j=l}^{m-1}\frac{k^{\gamma_2 j}}{k^{\gamma_1 r j}}\left\|x\right\|^r \tag{17}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbf{X}$. It follows from (17) that the sequence $\left\{k^n f\left(\frac{x}{k^n}\right)\right\}$ is Cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{k^n f\left(\frac{x}{k^n}\right)\right\}$ converges. So one can define the mapping $H : \mathbf{X} \to \mathbf{Y}$ by

$$H(x) := \lim_{n\to\infty} k^n f\left(\frac{x}{k^n}\right)$$

for all $x \in \mathbf{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (17), we get (15) It follows from (14) that

$$\begin{aligned}
&\left\|H\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} + \sum_{j=1}^{k}x_j\right) + H\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} - \sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}H\left(\frac{x_{k+j}}{k}\right) - 2\sum_{j=1}^{k}H\left(x_j\right)\right\|_{\mathbf{Y}}\\
&= \lim_{n\to\infty}\left|k\right|^{n\gamma_2}\left\|f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+1}} + \frac{1}{k^n}\sum_{j=1}^{k}x_j\right) + f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+1}} - \frac{1}{k^n}\sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k^{n+1}}\right) - 2\sum_{j=1}^{k}f\left(\frac{x_j}{k^n}\right)\right\|_{\mathbf{Y}}\\
&\leq \lim_{n\to\infty}\left|k\right|^{n\gamma_2}\left|\beta\right|^{\gamma_2}\Bigg\|\left(kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+2}} + \frac{1}{k^n}\sum_{j=1}^{k}x_j\right) + kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+2}} - \frac{1}{k^{n+1}}\sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k^{n+1}}\right)\right.\\
&\left.-2\sum_{j=1}^{k}f\left(\frac{x_j}{k^n}\right)\right)\Bigg\|_{\mathbf{Y}} + \lim_{n\to\infty}\frac{k^{n\gamma_2}\theta}{k^{nr\gamma_1}}\left(\sum_{j=1}^{k}\left\|x_j\right\|^r + \sum_{j=1}^{k}\left\|x_{k+j}\right\|^r\right)
\end{aligned}$$

for all $x_j, x_{k+j} \in X$ for all $j = 1 \to k$. So

$$\begin{aligned}
&\left\|f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} + \frac{1}{k}\sum_{j=1}^{k}x_j\right) + f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} - \frac{1}{k}\sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right) - 2\sum_{j=1}^{k}f\left(x_j\right)\right\|_{\mathbf{Y}}\\
&\leq \left\|\beta\left(kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} + \sum_{j=1}^{k}x_j\right) + kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} - \sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right) - 2\sum_{j=1}^{k}f\left(x_j\right)\right)\right\|_{\mathbf{Y}}
\end{aligned}$$

for all $x_j, x_{k+j} \in X$ for all $j = 1 \to k$. By Lemma 3.1, the mapping $H : \mathbf{X} \to \mathbf{Y}$ is quadratic. Now, let $T : \mathbf{X} \to \mathbf{Y}$ be another quadratic mapping satisfying (15). Then we have

$$\begin{aligned}\Big\|H(x) - T(x)\Big\|_{\mathbf{Y}} &= k^{\gamma_2 n}\Big\|H\left(\frac{x}{k^n}\right) - T\left(\frac{x}{k^n}\right)\Big\|_{\mathbf{Y}} \\ &\leq k^{\gamma_2 n}\left(\Big\|H\left(\frac{x}{k^n}\right) - f\left(\frac{x}{k^n}\right)\Big\|_{\mathbf{Y}} + \Big\|H\left(\frac{x}{k^n}\right) - f\left(\frac{x}{k^n}\right)\Big\|_{\mathbf{Y}}\right) \\ &\leq \frac{2\theta k^{\gamma_2 n+1}}{(k^{\gamma_1 r} - k^{\gamma_2})\,k^{\gamma_1 n r}}\theta\Big\|x\Big\|^r\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in \mathbf{X}$. So we can conclude that $H(x) = T(x)$ for all $x \in \mathbf{X}$. This prover the uniqueness of H. Thus the mapping $H : \mathbf{X} \to \mathbf{Y}$ be a unique mapping satisfying (15). □

Theorem 3.6. *Let $r < \frac{2\gamma_2}{\gamma_1}$ and θ be positive real numbers, and let $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is an even mapping satisfies*

$$\begin{aligned}&\Big\|f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} + \sum_{j=1}^{k}x_j\right) + f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} - \sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right) - 2\sum_{j=1}^{k}f\left(x_j\right)\Big\|_{\mathbf{Y}} \\ &\leq \Big\|\beta\Big(kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} + \frac{1}{k}\sum_{j=1}^{k}x_j\right) + kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} - \frac{1}{k}\sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right) \\ &- 2\sum_{j=1}^{k}f\left(x_j\right)\Big)\Big\|_{\mathbf{Y}} + \theta\left(\sum_{j=1}^{k}\Big\|x_j\Big\|^r + \sum_{j=1}^{k}\Big\|x_{k+j}\Big\|^r\right)\end{aligned} \tag{18}$$

for all $x_j, x_{k+j} \in X$ for all $j = 1 \to k$. Then there exists a unique quadratic mapping $H : X \to Y$ such that

$$\Big\|f\left(x\right) - H\left(x\right)\Big\| \leq \frac{k\theta}{k^{\gamma_2} - k^{\gamma_1 r}} \tag{19}$$

for all $x \in \mathbf{X}$

Proof. Letting $x_j = x_{k+j} = 0$ for all $j = 1 \to k$, in (18) we get

$$\Big|2k - 1\Big|^{\gamma_2}\Big\|f\left(0\right)\Big\|_{\mathbf{Y}} \leq \Big|k\beta\Big|^{\gamma_2}\Big\|f\left(0\right)\Big\|_{\mathbf{Y}}$$

So $f\left(0\right) = 0$. Letting $x_{k+j} = 0$, and $x_j = x$ for all $j = 1 \to k$ in (18), we get

$$\Big\|f\left(kx\right) - kf\left(x\right)\Big\|_{\mathbf{Y}} \leq k\theta\Big\|x\Big\|^r \tag{20}$$

for all $x \in \mathbf{X}$.So

$$\Big\|f\left(x\right) - \frac{1}{k}f\left(kx\right)\Big\|_{\mathbf{Y}} \leq \frac{k}{k^{\gamma_2}}\theta\Big\|x\Big\|^r$$

So for all $x \in \mathbf{X}$. Hence

$$\begin{aligned}\Big\|\frac{1}{k^l}f\left(k^l x\right) - \frac{1}{k^m}f\left(k^m x\right)\Big\|_{\mathbf{Y}} &\leq \sum_{j=l1}^{m-1}\Big\|\frac{1}{k^j}f\left(k^j x\right) - \frac{1}{k^{j+1}}f\left(k^{j+1}x\right)\Big\| \\ &\leq \frac{k\theta}{k^{\gamma_2}}\sum_{j=l}^{m-1}\frac{k^{\gamma_1 r j}}{k^{\gamma_2 j}}\Big\|x\Big\|^r\end{aligned} \tag{21}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbf{X}$. It follows from (21) that the sequence $\left\{\frac{1}{k^n}f\left(k^n x\right)\right\}$ is Cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{\frac{1}{k^n}f\left(k^n x\right)\right\}$ coverges. So one can define te mapping $H : \mathbf{X} \to \mathbf{Y}$ by

$$H\left(x\right) := \lim_{n\to\infty}\frac{1}{k^n}f\left(k^n x\right)$$

for all $x \in \mathbf{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (21), we get (19) The rest of the proof is similar to the proof of the Theorem 3.5 □

From Theorem 3.5 and Theorem 3.6 we have corollaries.

Corollary 3.7. *Let $r > \frac{2\gamma_2}{\gamma_1}$ and θ be positive real numbers, and let $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ is an even mapping satifies*

$$\begin{aligned}&\left\| f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)+f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f\left(z_j\right)\right\|_{\mathbf{Y}}\\&\leq\left\|\beta\left(kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)\right.\right.\\&\left.\left.-2\sum_{j=1}^{k}f\left(x_j\right)\right)\right\|_{\mathbf{Y}}+\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|^r+\sum_{j=1}^{k}\left\|x_{k+j}\right\|^r\right)\end{aligned}\tag{22}$$

for all $x_j, x_{k+j} \in X$ for all $j = 1 \to k$. Then there exists a unique quadratic mapping $H : X \to Y$ such that

$$\left\|f\left(x\right)-H\left(x\right)\right\|_{\mathbf{Y}}\leq\frac{k\theta}{k^{\gamma_1 r}-k^{\gamma_2}},\tag{23}$$

for all $x \in X$.

Corollary 3.8. *Let $r < \frac{2\gamma_2}{\gamma_1}$ and θ be positive real numbers, and let $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is an even mapping satisfies*

$$\begin{aligned}&\left\| f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)+f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f\left(x_j\right)\right\|_{\mathbf{Y}}\\&\leq\left\|\beta\left(kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)\right.\right.\\&\left.\left.-2\sum_{j=1}^{k}f\left(x_j\right)\right)\right\|_{\mathbf{Y}}+\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|^r+\sum_{j=1}^{k}\left\|x_{k+j}\right\|^r\right)\end{aligned}\tag{24}$$

for all $x_j, x_{k+j} \in X$ for all $j = 1 \to k$. Then there exists a unique quadratic mapping $H : X \to Y$ such that

$$\left\|f\left(x\right)-H\left(x\right)\right\|\leq\frac{k\theta}{k^{\gamma_2}-k^{\gamma_1 r}}\tag{25}$$

for all $x \in \mathbf{X}$.

4. Quadratic γ-functional Inequality

In This section, assume that β is a fixed complex number with $\left|\beta\right| < \frac{1}{2}$. We investigate the quadratic β-functional inequality (2) in γ-homogeneous complex Banach space.

Theorem 4.1. *Let $r > \frac{2\gamma_2}{\gamma_1}$ and θ be positive real numbers, and let $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is an even mapping with $f\left(0\right) = 0$ such that*

$$\begin{aligned}&\left\| kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f(x_j)\right\|_{\mathbf{Y}}\\&\leq\left\|\beta\left(f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)+f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f(x_j)\right)\right\|_{\mathbf{Y}}\\&+\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|^r+\sum_{j=1}^{k}\left\|x_{k+j}\right\|^r\right)\end{aligned}\tag{26}$$

for all $x_j, x_{k+j} \in X$ for all $j = 1 \to k$. Then there exists a unique quadratic mapping $H : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|f\left(x\right)-H\left(x\right)\right\|_{\mathbf{Y}}\leq\frac{\theta}{2^{\gamma_2}}\cdot\frac{k^{\gamma_1 r}}{k^{\gamma_1 r}-k^{\gamma_2}}\tag{27}$$

for all $x \in X$

Proof. Letting $x_{j+1} = x_{k+j} = 0$ and $x_1 = x$ for all $j = 1 \to k$ in (26), we get

$$\left\|kf\left(\frac{x}{k}\right) - f\left(x\right)\right\|_{\mathbf{Y}} \leq \frac{\theta}{2^{\gamma_2}}\left\|x\right\|^r \tag{28}$$

for all $x \in \mathbf{X}$. Hence

$$\begin{aligned}\left\|k^l f\left(\frac{x}{k^l}\right) - k^m f\left(\frac{x}{k^m}\right)\right\|_{\mathbf{Y}} &\leq \sum_{j=1}^{m-1}\left\|k^j f\left(\frac{x}{k^j}\right) - k^{j+1} f\left(\frac{x}{k^{j+1}}\right)\right\|_{\mathbf{Y}} \\ &\leq \frac{\theta}{2^{\gamma_2}}\sum_{j=l}^{m-1}\frac{k^{\gamma_2 j}}{k^{\gamma_1 r j}}\left\|x\right\|^r \end{aligned}\tag{29}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbf{X}$. It follows from (29) that the sequence $\left\{k^n f\left(\frac{x}{k^n}\right)\right\}$ is Cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{k^n f\left(\frac{x}{k^n}\right)\right\}$ coverges. So one can define the mapping $H : \mathbf{X} \to \mathbf{Y}$ by

$$H\left(x\right) := \lim_{n\to\infty} k^n f\left(\frac{x}{k^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (29), we get (27) It follows from (26) that

$$\begin{aligned}&\left\|kH\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} + \frac{1}{k}\sum_{j=1}^{k}x_j\right) + kH\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} - \frac{1}{k}\sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}H\left(\frac{x_{k+j}}{k}\right) - 2\sum_{j=1}^{k}H\left(x_j\right)\right\|_{\mathbf{Y}} \\ &= \lim_{n\to\infty}\left|k\right|^{n\gamma_2}\left\|kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+1}} + \frac{1}{k^n}\sum_{j=1}^{k}x_j\right) + kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+1}} - \frac{1}{k^n}\sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k^{n+1}}\right) - 2\sum_{j=1}^{k}f\left(\frac{x_j}{k^n}\right)\right\|_{\mathbf{Y}} \\ &\leq \lim_{n\to\infty}\left|k\right|^{n\gamma_2}\left|\beta\right|^{\gamma_2}\left\|\left(f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+1}} + \frac{1}{k^n}\sum_{j=1}^{k}x_j\right) + f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+1}} - \frac{1}{k^n}\sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k^{n+1}}\right) - \right.\right. \\ &\left.\left. 2\sum_{j=1}^{k}f\left(\frac{x_j}{k^n}\right)\right)\right\|_{\mathbf{Y}} + \lim_{n\to\infty}\frac{k^{n\gamma_2}\theta}{k^{nr\gamma_1}}\left(\sum_{j=1}^{k}\left\|x_j\right\|^r + \sum_{j=1}^{k}\left\|x_{k+j}\right\|^r\right)\end{aligned}$$

for all $x_j, x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$. So

$$\begin{aligned}&\left\|kH\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} + \frac{1}{k}\sum_{j=1}^{k}x_j\right) + kH\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} - \frac{1}{k}\sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}H\left(\frac{x_{k+j}}{k}\right) - 2\sum_{j=1}^{k}H\left(x_j\right)\right\|_{\mathbf{Y}} \\ &\leq \left\|\beta\left(H\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} + \sum_{j=1}^{k}x_j\right) + H\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} - \sum_{j=1}^{k}x_j\right) - 2\sum_{j=1}^{k}H\left(\frac{x_{k+j}}{k}\right) - 2\sum_{j=1}^{k}H\left(x_j\right)\right)\right\|_{\mathbf{Y}}\end{aligned}$$

for all $x_j, x_{k+j} \in X$ for all $j = 1 \to k$. By Lemma 3.2, the mapping $H : \mathbf{X} \to \mathbf{Y}$ is quadratic. Now, let $T : \mathbf{X} \to \mathbf{Y}$ be another quadratic mapping satisfying (27). Then we have

$$\begin{aligned}\left\|H\left(x\right) - T\left(x\right)\right\|_{\mathbf{Y}} &= k^{\gamma_2 n}\left\|H\left(\frac{x}{k^n}\right) - T\left(\frac{x}{k^n}\right)\right\|_{\mathbf{Y}} \\ &\leq k^{\gamma_2 n}\left(\left\|H\left(\frac{x}{k^n}\right) - f\left(\frac{x}{k^n}\right)\right\|_{\mathbf{Y}} + \left\|H\left(\frac{x}{k^n}\right) - f\left(\frac{x}{k^n}\right)\right\|_{\mathbf{Y}}\right) \\ &\leq \frac{2\theta}{2^{\gamma_2}}\cdot\frac{1}{\left(k^{\gamma_1 r} - k^{\gamma_2}\right)k^{\gamma_1 r}}\cdot\frac{k^{\gamma_2 n}}{k^{\gamma_1 n r}}\left\|x\right\|^r\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in \mathbf{X}$. So we can conclude that $H\left(x\right) = T\left(x\right)$ for all $x \in \mathbf{X}$. This prover the uniqueness of H. Thus the mapping $H : \mathbf{X} \to \mathbf{Y}$ be a unique mapping satisfying (27). □

Theorem 4.2. *Let $r < \frac{2\gamma_2}{\gamma_1}$ and θ be positive real numbers, and let $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is an even mapping with $f\Big(0\Big) = 0$ such that*

$$\begin{aligned}&\left\|kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f\left(x_j\right)\right\|_{\mathbf{Y}}\\&\leq\left\|\beta\left(f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)+f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}-\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}f\left(x_j\right)\right)\right\|_{\mathbf{Y}}\\&+\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|^r+\sum_{j=1}^{k}\left\|x_{k+j}\right\|^r\right)\end{aligned}\tag{30}$$

for all $x_j, x_{k+j} \in X$ for all $j = 1 \to k$. Then there exists a unique quadratic mapping $H : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|f\Big(x\Big)-H\Big(x\Big)\right\|_{\mathbf{Y}}\leq 2^{\gamma_2}\theta\cdot\frac{k^{\gamma_1 r}}{k^{\gamma_2}-k^{\gamma_1 r}}\tag{31}$$

for all $x \in \mathbf{X}$

Proof. Letting $x_{j+1} = x_{k+j} = 0$ and $x_1 = x$ for all $j = 1 \to k$ in (30), we get

$$\left\|f\Big(x\Big)-\frac{1}{k}f\Big(kx\Big)\right\|_{\mathbf{Y}}\leq 2^{\gamma_2}\theta\frac{k^{\gamma_1 r}}{k^{\gamma_2}}\left\|x\right\|^r\tag{32}$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{k^l}f\Big(k^lx\Big)-\frac{1}{k^m}f\Big(k^mx\Big)\right\|_{\mathbf{Y}}&\leq\sum_{j=l}^{m-1}\left\|\frac{1}{k^j}f\Big(k^jx\Big)-\frac{1}{k^{j+1}}f\Big(k^{j+1}x\Big)\right\|_{\mathbf{Y}}\\&\leq 2^{\gamma_2}\theta\sum_{j=l}^{m-1}\frac{k^{\gamma_1 j}}{k^{\gamma_2 rj}}\left\|x\right\|^r\end{aligned}\tag{33}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbf{X}$. It follows from (33) that the sequence $\left\{\frac{1}{k^n}f\Big(k^nx\Big)\right\}$ is Cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{\frac{1}{k^n}f\Big(k^nx\Big)\right\}$ coverges. So one can define te mapping $H : \mathbf{X} \to \mathbf{Y}$ by

$$H\Big(x\Big):=\lim_{n\to\infty}\frac{1}{k^n}f\Big(k^nx\Big)$$

for all $x \in \mathbf{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (33), we get (31) It follows from (30) that

$$\begin{aligned}&\left\|kH\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+kH\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}H\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}H\Big(x_j\Big)\right\|_{\mathbf{Y}}\\&=\lim_{n\to\infty}\frac{1}{\left|k\right|^{n\gamma_2}}\left\|kf\left(\sum_{j=1}^{k}k^{n-2}x_{k+j}+\sum_{j=1}^{k}k^{n-1}x_j\right)+kf\left(\sum_{j=1}^{k}k^{n-2}x_{k+j}-\sum_{j=1}^{k}k^{n-1}x_j\right)\right.\\&\left.-2\sum_{j=1}^{k}f\Big(k^{n-1}x_{k+j}\Big)-2\sum_{j=1}^{k}f\Big(k^nx_j\Big)\right\|_{\mathbf{Y}}\\&\leq\lim_{n\to\infty}\frac{1}{\left|k\right|^{n\gamma_2}}\left|\beta\right|^{\gamma_2}\left\|\left(f\left(\sum_{j=1}^{k}k^{n-1}x_{k+j}+\sum_{j=1}^{k}k^nx_j\right)+f\left(\sum_{j=1}^{k}k^{n-1}x_{k+j}-\sum_{j=1}^{k}k^nx_j\right)\right.\right.\\&\left.\left.-2\sum_{j=1}^{k}f\Big(k^{n-1}x_{k+j}\Big)-2\sum_{j=1}^{k}f\Big(k^nx_j\Big)\right)\right\|_{\mathbf{Y}}+\lim_{n\to\infty}\frac{k^{n\gamma_1}\theta}{k^{nr\gamma_2}}\left(\sum_{j=1}^{k}\left\|x_j\right\|^r+\sum_{j=1}^{k}\left\|x_{k+j}\right\|^r\right)\end{aligned}$$

for all $x_j, x_{k+j} \in \mathbf{X}$for all $j = 1 \to k$. So

$$\left\|kH\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)+kH\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}-\frac{1}{k}\sum_{j=1}^{k}x_j\right)-2\sum_{j=1}^{k}H\left(\frac{x_{k+j}}{k}\right)-2\sum_{j=1}^{k}H(x_j)\right\|_{\mathbf{Y}}$$

$$\leq \left\| \beta\left(H\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k} + \sum_{j=1}^{k} x_j \right) + H\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k} - \sum_{j=1}^{k} x_j \right) - 2\sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k} \right) - 2\sum_{j=1}^{k} H\left(x_j \right) \right) \right\|_{\mathbf{Y}}$$

for all $x_j, x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$. By *Lemma* 3.2, the mapping $H : \mathbf{X} \to \mathbf{Y}$ is quadratic. Now, let $T : \mathbf{X} \to \mathbf{Y}$ be another quadratic mapping satisfying (31). Then we have

$$\begin{aligned} \|H(x) - T(x)\|_{\mathbf{Y}} &= \frac{1}{k^{\gamma_2 n}} \left\| H\left(k^n x\right) - T\left(k^n x\right) \right\|_{\mathbf{Y}} \\ &\leq \frac{1}{k^{\gamma_2 n}} \left(\left\| H\left(k^n x\right) - f\left(k^n x\right) \right\|_{\mathbf{Y}} + \left\| H\left(k^n x\right) - f\left(k^n x\right) \right\|_{\mathbf{Y}} \right) \\ &\leq \frac{2 \cdot 2^{\gamma_2} \theta}{\left(k^{\gamma_2} - k^{\gamma_1 r}\right) k^{\gamma_1 r}} \cdot \frac{k^{\gamma_1 n r}}{k^{\gamma_2 n}} \left\|x\right\|^r \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in \mathbf{X}$. So we can conclude that $H(x) = T(x)$ for all $x \in \mathbf{X}$. This prover the uniqueness of H. Thus the mapping $H : \mathbf{X} \to \mathbf{Y}$ be a unique mapping satisfying (31). □

From Theorem 4.1 and Theorem 4.2 we have corollaries.

Corollary 4.3. *Let $r > \frac{2\gamma_2}{\gamma_1}$ and θ be positive real numbers, and let $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is an even mapping with $f\left(0\right) = 0$ such that*

$$\begin{aligned} &\left\| kf\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k^2} + \frac{1}{k}\sum_{j=1}^{k} x_j \right) + kf\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k^2} - \frac{1}{k}\sum_{j=1}^{k} x_j \right) - 2\sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k} \right) - 2\sum_{j=1}^{k} f\left(x_j \right) \right\|_{\mathbf{Y}} \\ &\leq \left\| \beta\left(f\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k} + \sum_{j=1}^{k} x_j \right) + f\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k} - \sum_{j=1}^{k} x_j \right) - 2\sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k} \right) - 2\sum_{j=1}^{k} f\left(x_j \right) \right) \right\|_{\mathbf{Y}} \\ &+ \theta\left(\sum_{j=1}^{k} \left\|x_j\right\|^r + \sum_{j=1}^{k} \left\|x_{k+j}\right\|^r \right) \end{aligned} \tag{34}$$

for all $x_j, x_{k+j} \in X$ for all $j = 1 \to k$. Then there exists a unique quadratic mapping $H : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\| f\left(x\right) - H\left(x\right) \right\|_{\mathbf{Y}} \leq \frac{\theta}{2^{\gamma_2}} \cdot \frac{k^{\gamma_1 r}}{k^{\gamma_1 r} - k^{\gamma_2}} \tag{35}$$

for all $x \in X$

Corollary 4.4. *Let $r < \frac{2\gamma_2}{\gamma_1}$ and θ be positive real numbers, and let $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is an even mapping with $f\left(0\right) = 0$ such that*

$$\begin{aligned} &\left\| kf\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k^2} + \frac{1}{k}\sum_{j=1}^{k} x_j \right) + kf\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k^2} - \frac{1}{k}\sum_{j=1}^{k} x_j \right) - 2\sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k} \right) - 2\sum_{j=1}^{k} f\left(x_j \right) \right\|_{\mathbf{Y}} \\ &\leq \left\| \beta\left(f\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k} + \sum_{j=1}^{k} x_j \right) + f\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k} - \sum_{j=1}^{k} x_j \right) - 2\sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k} \right) - 2\sum_{j=1}^{k} f\left(x_j \right) \right) \right\|_{\mathbf{Y}} \\ &+ \theta\left(\sum_{j=1}^{k} \left\|x_j\right\|^r + \sum_{j=1}^{k} \left\|x_{k+j}\right\|^r \right) \end{aligned} \tag{36}$$

for all $x_j, x_{k+j} \in X$ for all $j = 1 \to k$. Then there exists a unique quadratic mapping $H : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\| f\left(x\right) - H\left(x\right) \right\|_{\mathbf{Y}} \leq 2^{\gamma_2}\theta \cdot \frac{k^{\gamma_1 r}}{k^{\gamma_2} - k^{\gamma_1 r}} \tag{37}$$

for all $x \in \mathbf{X}$

Remark 4.5. *If β is a real number such that $-1 < \beta < 1$ and $\mathbf{Y}$ is a γ_2-homogeneous real Banach space, then all the assertions in this sections remain valid.*

5. Conclusion

In this paper, I have shown that the solutions of the first and second quadratic β-functional inequalities are quadratic mappings. The Hyers-Ulam stability for these given from theorems. These are the main results of the paper , which are the generalization of the results [2, 3, 4, 9].

References

[1] T. Aoki, *On the stability of the linear transformation in Banach space*, J. Math. Soc. Japan, 2(1995), 64-66.

[2] Ly Van An, *Hyers-Ulam stability of functional inequalities with three variable in Banach spaces and Non-Archemdean Banach spaces*, International Journal of Mathematical Analysis, 13(11)(2019), 296-310.

[3] Ly Van An, *Hyers-Ulam stability additive β-functional inequalities with three variable in Banach spaces and Non-Archemdean Banach spaces*, International Journal of Mathematical Analysis, 14(5-8)(2020), 296-310.

[4] Ly Van An, *Hyers-Ulam stability quadratic β-functional inequalities with three variable in γ-homogeneous normed spaces*, International Journal of Mathematic Research, 12(1)(2020), 47-64.

[5] J. Bae and W. Park, *Approximate bi-homomorphisms and bi-derivations in C^*-ternary algebras*, Bull. Korean Math. Soc, 47(2010), 195-209.

[6] A. Bahyrycz and M. Piszczek, *Hyers stability of the Jensen function equation*, Acta Math. Hungar., 142(2014), 353-365.

[7] M. Balcerowski, *On the functional equations related to a problem of z Boros and Z. Dróczy*, Acta Math. Hungar., 138(2013), 329-340.

[8] P. W. Cholewa, *Remarks on the stability of functional eqution*, Aequationes Math., 27(1984), 76-86

[9] Chookil, Sang Og Kim, Jung Rye Lee and Dong Yun Shin, *Quadraic ρfunctional inequalities in β-homogeueous normed Space*, Int. J. Nonlinear Anal. Appl., 6(2)(2015), 21-26.

[10] Z. Daróczy and Gy. Mackasa, *Afunction equation involving comparable weighted quasi-arthmetric means*, Acta Math Hungar., 138(2013), 329-340.

[11] I. -i. EL-Fassi, *Solution and approximation of radical quintic functional equation related to quintic mapping in quasi-β-Banach spaces*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat., 2018(2018), 1-13.

[12] W. Fechner, *Stability of a functional inequlities associated with the Jordan-von Newmann functional equation*, Aequtiones Math., 71(2006), 149-161.

[13] A. Gil*ányi*, *Eine zur Parallelogrammaleichung ä Ungleichung*, Aeq. Math., 62(2001), 303-309.

[14] A. Gil*ányi*, *On u problemby K*, Nikodem Math. Inequal. Appl., 5(2002), 707-710.

[15] P. Găvruta, *A generalization of the Hyers-Ulam -Rassias stability, Locally bounded linear topological spaces*, Proc. Imp. Acad. Tokyo, 18(10)(1942), 588-594.

[16] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA., 27(1941), 222-224.

[17] S. Jung, *On the quadrtic functional equation modul modulo a subgroup*, Indian J. Pure Appl. Math., 36(2005), 441-450.

[18] P. Kaskasem, C. Klin-eam and Y. J. Cho, *On the stability of the generalized Cauchy-Jensen set-valued functional equations*, J. Fixed Point Theory Appl., 20(2018), 1-14.

[19] C. Park, *Quadrtic β-functional inequalites and equation*, J. Nonlinear Anal. Appl., 2014(2014).

[20] C. Park, *Functional equation in Banach modules*, Indian J. Pure Anal. Appl Math., 33(2002), 1077-1086.

[21] C. Park, *Multilinear mappings in Banach modules over a C^*-algebra*, Indian J. Pure Appl Math., 35(2004), 183-192.

[22] C. Park, Y. Cho and M. Han, *Functional inequalities associiated with Jordan -von Neuman- type additive functional*

equational, J. Inequal. Appl., 2007(2007).

[23] A. Najati and G. Z. Eskandani, *Stability of a mixed additive and cubic functional equation in quasi-Banach spaces*, J. Math. Anal. Appl., 342(2)(2008), 1318-1331.

[24] A. Najati and M. B. Moghimi, *Stability of a functional equation deriving from quadratic and additive functions in quasi-Banach spaces*, J. Math. Anal. Appl., 337(1)(2008), 399-415.

[25] Th. M. Rassias, *On the stability of the linear mappings in Banach space*, Proc. Amer. Math. So., 72(1978), 297-300.

[26] J. Rätz, *On inqualities associated with the Jordan Neumann functional equation*, Aequationes Math., 66(2003), 191-200.

[27] S. Rolewicz, *Metric linear space*, PWN-Polish Scientific Publishersm, Warsaw, (1972).

[28] S. M. Ulam, *Collection of the mathematical Problems*, Interscience Publ New York, (1960).

[29] S. M. Ulam, *Problems in modern mathematics*, Wiley, (1964).

[30] T. Z. Xu, J. M. Rassias and W. X. Xu, *Generalized Hyers-Ulam stability of a general mixed additive-cubic functional equation in quasi-Banach spaces*, Acta Math. Sin. (Engl. Ser.), 28(3)(2012), 529-560.