Int. J. Math. And Appl., 9(3)(2021), 119–123
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

$(1,2)^{\star}$-$\breve{g}$-Homeomorphisms

A. Ponmalar[1], R. Asokan[1] and O. Nethaji[2,*]

1 Department of Mathematics, School of Mathematics, Madurai Kamaraj University, Madurai, Tamil Nadu, India.

2 PG and Research Department of Mathematics, Kamaraj College, Thoothukudi, Tamil Nadu, India.

Abstract: The purpose of this paper, we introduce the new types of functions called $(1,2)^{\star}$-$\breve{g}$-homeomorphisms and strongly $(1,2)^{\star}$-$\breve{g}$-homeomorphisms by using $(1,2)^{\star}$-$\breve{g}$-closed set prove that the collection of strongly $(1,2)^{\star}$-$\breve{g}$-homeomorphisms forms a group under the operations of composition functions.

MSC: 54A05, 54A10, 54C08, 54C10.

Keywords: $(1,2)^{\star}$-$\breve{g}$-closed functions, strongly $(1,2)^{\star}$-$\breve{g}$-closed, $(1,2)^{\star}$-$\breve{g}$-homeomorphisms and strongly $(1,2)^{\star}$-$\breve{g}$-homeomorphisms.

1. Introduction

J.C.Kelly [1] was uttered the geometrical continuation of bitopological space that is a non empty set X together with two arbitrary topologies defined on X at the stage of significant study the shapes of objects. N. Levine [2] was initiated the study of generalizations of closed sets in topological spaces. In this paper, we introduce the concepts $(1,2)^{\star}$-$\breve{g}$-homeomorphisms and strongly $(1,2)^{\star}$-$\breve{g}$-homeomorphisms. Also basic properties of these two functions are studied and the relation between these types and other existing ones are established.

2. Preliminaries

Throughout this paper (X,τ_1,τ_2) and (Y,σ_1,σ_2) (or simply X and Y) represents the non-empty bitopological spaces on which no separation axiom are assumed, unless otherwise mentioned. For a subset A of X, $\tau_{1,2}$-$cl(A)$ and $\tau_{1,2}$-$int(A)$ represents the closure of A and interior of A respectively.
In the following fundamentals are holds.

Definition 2.1 ([3]). *A subset A of a bitopological space (X,τ_1,τ_2) or X is said to be a $(1,2)^{\star}$-semi open set if $A \subseteq \tau_{1,2}$-$cl(\tau_{1,2}$-$int(A))$.*

The complement of the above mentioned set is called a closed set.

Definition 2.2. *A subset A of a bitopological space (X,τ_1,τ_2) or X is said to be*

(1). a $(1,2)^{\star}$-generalized closed set (briefly, $(1,2)^{\star}$-g-closed) [10] if $\tau_{1,2}$-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-open.

* E-mail: jionetha@yahoo.com

(2). a $(1,2)^{\star}$-sg-closed set [9] if $(1,2)^{\star}$-scl$(A) \subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^{\star}$-semi-open.

(3). a $(1,2)^{\star}$-$\hat{g}$-closed set [7] if $\tau_{1,2}$-cl$(A) \subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^{\star}$-sg-open.

(4). a $(1,2)^{\star}$-$\hat{g}_1$-closed set [6] if $\tau_{1,2}$-cl$(A) \subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^{\star}$-$\hat{g}$-open.

(5). a $(1,2)^{\star}$-$\mathcal{G}$-closed set [6] if $(1,2)^{\star}$-scl$(A) \subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^{\star}$-$\hat{g}_1$-open.

(6). a $(1,2)^{\star}$-$\breve{g}$-closed set [6] if $\tau_{1,2}$-cl$(A) \subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^{\star}$-$\mathcal{G}$-open.

The complements of the above mentioned closed sets are called their respective open sets.

Definition 2.3. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be*

(1). a $(1,2)^{\star}$-continuous [4] if the inverse image of every $\sigma_{1,2}$-closed set of (Y, σ_1, σ_2) is $\tau_{1,2}$-closed set in (X, τ_1, τ_2).

(2). a $(1,2)^{\star}$-$\breve{g}$-continuous [8] if the inverse image of every $\tau_{1,2}$-closed set in (Y, σ_1, σ_2) is $(1,2)^{\star}$-$\breve{g}$-closed set in (X, τ_1, τ_2).

(3). a $(1,2)^{\star}$-$\mathcal{G}$-irresolute [8] if the inverse image of every $(1,2)^{\star}$-$\mathcal{G}$-closed in (Y, σ_1, σ_2) is $(1,2)^{\star}$-$\mathcal{G}$-closed set in (X, τ_1, τ_2).

Definition 2.4 ([5])**.** *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called*

(1). a $(1,2)^{\star}$-$\breve{g}$-closed if the image of every $\tau_{1,2}$-closed set in (X, τ_1, τ_2) is $(1,2)^{\star}$-$\breve{g}$-closed in (Y, σ_1, σ_2).

(2). a $(1,2)^{\star}$-$\breve{g}$-open function if the image $f(A)$ is $(1,2)^{\star}$-$\breve{g}$-open in (Y, σ_1, σ_2) for each $\tau_{1,2}$-open set A in (X, τ_1, τ_2).

Definition 2.5 ([11])**.** *A bijective function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be*

(1). a $(1,2)^{\star}$-homeomorphism if f is both $(1,2)^{\star}$-continuous and $(1,2)^{\star}$-open.

(2). a generalized $(1,2)^{\star}$-g-homeomorphism if f is both $(1,2)^{\star}$-g-continuous and $(1,2)^{\star}$-g-open.

(3). a $(1,2)^{\star}$-gc-homeomorphism if f and f^{-1} are $(1,2)^{\star}$-gc-irresolute function.

3. $(1,2)^{\star}$-$\breve{g}$-Homeomorphisms

Definition 3.1. *A bijection $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called a*

(1). $(1,2)^{\star}$-$\breve{g}$-homeomorphism if f is both $(1,2)^{\star}$-$\breve{g}$-continuous and $(1,2)^{\star}$-$\breve{g}$-open.

(2). strongly $(1,2)^{\star}$-$\breve{g}$-homeomorphism if both f and f^{-1} are $(1,2)^{\star}$-$\breve{g}$-irresolute.

We denote the family of all $(1,2)^{\star}$-$\breve{g}$-homeomorphisms (resp. strongly $(1,2)^{\star}$-$\breve{g}$- homeomorphisms and $(1,2)^{\star}$-homeomorphisms) of a bitopological space (X, τ_1, τ_2) onto itself by $(1,2)^{\star}$-$\breve{g}$-$h(X, \tau_1, \tau_2)$ (resp. strongly $(1,2)^{\star}$-$\breve{g}$-$h(X, \tau_1, \tau_2)$ and $h(X, \tau_1, \tau_2)$)

Theorem 3.2. *Every $(1,2)^{\star}$-homeomorphism is a $(1,2)^{\star}$-$\breve{g}$-homeomorphism.*

Proof. It follows from Definitions 3.1. $\square$

Remark 3.3. *The converse part of Theorem 3.2 is not true as seen from the following Example.*

Example 3.4. *Let $X = Y = \{a, b, c, d\}$, $\tau_1 = \{\phi, \{a, b\}, X\}$ and $\tau_2 = \{\phi, X\}$ then $\tau_{1,2} = \{\phi, \{a, b\}, X\}$ with $\sigma_1 = \{\phi, \{a\}, Y\}$ and $\sigma_2 = \{\phi, \{a, b\}, Y\}$ then $\sigma_{1,2} = \{\phi, \{a\}, \{a, b\}, Y\}$. We have $(1,2)^{\star}$-$\breve{g}$-$C(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$ and $(1,2)^{\star}$-$\breve{g}$-$O(Y) = \{\phi, \{a\}, \{a, b\}, Y\}$. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be the identity function. Then f is $(1,2)^{\star}$-$\breve{g}$-homeomorphism but it is not $(1,2)^{\star}$-homeomorphism because it is not $(1,2)^{\star}$-continuous.*

Theorem 3.5. *Every $(1,2)^\star$-$\breve{g}$-homeomorphism is a $(1,2)^\star$-g-homeomorphism.*

Proof. Since every $(1,2)^\star$-$\breve{g}$-continuous function is $(1,2)^\star$-g-continuous and every $(1,2)^\star$-$\breve{g}$-open function is $(1,2)^\star$-g-open. □

Remark 3.6. *The converse part of Theorem 3.5 is not true as seen from the following Example.*

Example 3.7. *Let $X = Y = \{a,b,c,d,e\}, \tau_1 = \{\phi,\{a\},X\}$ and $\tau_2 = \{\phi,X\}$ then $\tau_{1,2} = \{\phi,\{a\},X\}$ with $\sigma_1 = \{\phi,\{b\},Y\}$ and $\sigma_2 = \{\phi,Y\}$ then $\sigma_{1,2} = \{\phi,\{b\},Y\}$. We have $(1,2)^\star$-$\breve{g}$-$C(X) = \{\phi,\{b,c\},X\}$ and $(1,2)^\star$-g-$C(X) = \{\phi,\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},X\}, (1,2)^\star$-$\breve{g}$-$O(Y) = \{\phi,\{b\},Y\}$ and $(1,2)^\star$-g-$O(Y) = \{\phi,\{a\},\{b\},\{c\},\{a,b\},\{b,c\},Y\}$. Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be defined by $f(\{a\}) = \{c\}, f(\{b\}) = \{a\}$ and $f(\{c\}) = \{b\}$. Then f is $(1,2)^\star$-g-homeomorphism but it is not $(1,2)^\star$-$\breve{g}$-homeomorphism.*

Theorem 3.8. *Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be a bijective $(1,2)^\star$-$\breve{g}$-continuous function. Then the following are equivalent:*

(1). f is an $(1,2)^\star$-$\breve{g}$-open function.

(2). f is $(1,2)^\star$-$\breve{g}$-homeomorphism.

(3). f is $(1,2)^\star$-$\breve{g}$-closed function.

Remark 3.9. *The composition of two $(1,2)^\star$-$\breve{g}$-homeomorphism functions need not be a $(1,2)^\star$-$\breve{g}$-homeomorphism function as shown in the following Example.*

Example 3.10. *Let $X = Y = Z = \{a,b,c\}$, $\tau_1 = \{\phi,\{a\},\{b\},\{a,b\},X\}$ and $\tau_2 = \{\phi,\{a\},X\}$ then $\tau_{1,2} = \{\phi,\{a\},\{b\},\{a,b\},X\}$, $\sigma_1 = \{\phi,\{a,b\},Y\}$ and $\sigma_2 = \{\phi,Y\}$ then $\sigma_{1,2} = \{\phi,\{a,b\},Y\}$ with $\eta_1 = \{\phi,\{a\},Z\}$ and $\eta_2 = \{\phi,\{a,b\},Z\}$ then $\eta_{1,2} = \{\phi,\{a\},\{a,b\},Z\}$. Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ and $g : (Y,\sigma_1,\sigma_2) \to (Z,\eta_1,\eta_2)$ be the identity functions. Then f and g are $(1,2)^\star$-$\breve{g}$-homeomorphisms but their $g \circ f : (X,\tau_1,\tau_2) \to (Z,\eta_1,\eta_2)$ is not $(1,2)^\star$-$\breve{g}$-homeomorphism, since for the set $S = \{b\}$ is $(1,2)^\star$-open in $(X,\tau_1,\tau_2), (g \circ f)(S) = f(g(S)) = f(g(\{b\}) = f(\{b\}) = \{b\}$ is not $(1,2)^\star$-$\breve{g}$-open in (Z,η_1,η_2). Hence $g \circ f$ is not $(1,2)^\star$-$\breve{g}$-open function and $g \circ f$ is not a $(1,2)^\star$-$\breve{g}$-homeomorphism.*

Theorem 3.11. *Every strongly $(1,2)^\star$-$\breve{g}$-homeomorphism is a $(1,2)^\star$-$\breve{g}$-homeomorphism. i.e., for any space (X,τ_1,τ_2), strongly $(1,2)^\star$-$\breve{g}$-$h(X,\tau_1,\tau_2) \subseteq (1,2)^\star$-$\breve{g}$-$h(X,\tau_1,\tau_2)$.*

Proof. Since every strongly $(1,2)^\star$-$\breve{g}$-open function is $(1,2)^\star$-$\breve{g}$-open. □

Remark 3.12. *The converse part of Theorem 3.11 is not true as seen from the following Example.*

Example 3.13. *The function g in Example 3.10 is a $(1,2)^\star$-$\breve{g}$-homeomorphism but not a strongly $(1,2)^\star$-$\breve{g}$-homeomorphism, since for the $(1,2)^\star$-$\breve{g}$-closed set $\{a,c\}$ in $(Y,\sigma_1,\sigma_2), (g^{-1})^{-1}(\{a,c\}) = g(\{a,c\}) = \{a,c\}$ which is not $(1,2)^\star$-$\breve{g}$-closed in (Z,η_1,η_2). Therefore, g^{-1} is not $(1,2)^\star$-$\breve{g}$-irresolute and so g is not a strongly $(1,2)^\star$-$\breve{g}$-homeomorphism.*

Theorem 3.14. *Every strongly $(1,2)^\star$-$\breve{g}$-homeomorphism is a $(1,2)^\star$-g-homeomorphism.*

Proof. It follows from Theorem 3.5. □

Proposition 3.15. *The function f in Example 3.7 is a $(1,2)^\star$-g-homeomorphism but not a strongly $(1,2)^\star$-$\breve{g}$- homeomorphism.*

Theorem 3.16. *Every $(1,2)^\star$-$\breve{g}$-homeomorphism is a $(1,2)^\star$-gc-homeomorphism.*

Proof. It follows from Definitions 2.5(2) and 3.1(1). □

Remark 3.17. *The converse part of Theorem 3.16 is not true as seen from the following Example.*

Example 3.18. *Let $X = Y = \{a, b, c, d\}, \tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{a, b\}, X\}$ then $\tau_{1,2} = \{\phi, \{a\}, \{a, b\}, X\}$ with $\sigma_1 = \{\phi, \{b\}, Y\}$ and $\sigma_2 = \{\phi, \{b, c\}, Y\}$ then $\sigma_{1,2} = \{\phi, \{b\}, \{b, c\}, Y\}$. We have $(1,2)^{\star}$-$\breve{g}$-$C(X) = \{\phi, \{c\}, \{b, c\}, X\}$ and $(1,2)^{\star}$-g-$C(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}, (1,2)^{\star}$-$\breve{g}$-$O(Y) = \{\phi, \{b\}, \{b, c\}, Y\}$ and $(1,2)^{\star}$-g-$C(Y) = \{\phi, \{a\}, \{a, b\}, \{a, c\}, Y\}$. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be defined by $f(\{a\}) = \{c\}, f(\{b\}) = \{b\}$ and $f(\{c\}) = \{a\}$. Then f is $(1,2)^{\star}$-gc-homeomorphism but it is not $(1,2)^{\star}$-$\breve{g}$-homeomorphism.*

Theorem 3.19. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ are strongly $(1,2)^{\star}$-$\breve{g}$-homeomorphisms, then their composition $g \circ f : (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2)$ is also strongly $(1,2)^{\star}$-$\breve{g}$-homeomorphism.*

Proof. Let F be $(1,2)^{\star}$-$\breve{g}$-open set in (Z, η_1, η_2). Now, $(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F)) = f^{-1}(G)$, where $G = g^{-1}(F)$. By hypothesis, G is $(1,2)^{\star}$-$\breve{g}$-open in (Y, σ_1, σ_2) and so again by hypothesis, $f^{-1}(G)$ is $(1,2)^{\star}$-$\breve{g}$-open in(X, τ_1, τ_2). Therefore, $g \circ f$ is $(1,2)^{\star}$-$\breve{g}$-irresolute. Also for an $(1,2)^{\star}$-$\breve{g}$-open set F in (X, τ_1, τ_2), we have $(g \circ f)(F) = g(f(F)) = g(H)$, where $H = f(F)$. By hypothesis $f(F)$ is $(1,2)^{\star}$-$\breve{g}$-open in (Y, σ_1, σ_2) and so again by hypothesis, $g(f(F))$ is $(1,2)^{\star}$-$\breve{g}$-open in (Z, η_1, η_2) i.e., $(g \circ f)(F)$ is $(1,2)^{\star}$-$\breve{g}$-open in (Z, η_1, η_2) and therefore $(g \circ f)^{-1}$ is $(1,2)^{\star}$-$\breve{g}$-irresolute. Hence $g \circ f$ is a strongly $(1,2)^{\star}$-$\breve{g}$-homeomorphism. □

Theorem 3.20. *The set strongly $(1,2)^{\star}$-$\breve{g}$-$h(X, \tau_1, \tau_2)$ is a group under the composition of functions.*

Proof. Define a binary operation $*$: strongly $(1,2)^{\star}$-$\breve{g}$-$h(X, \tau_1, \tau_2)\times$ strongly $(1,2)^{\star}$-$\breve{g}$-$h(X, \tau_1, \tau_2) \to$ strongly $(1,2)^{\star}$-$\breve{g}$-$h(X, \tau_1, \tau_2)$ by $f * g = g \circ f$ for all $f, g \in$ strongly $(1,2)^{\star}$-$\breve{g}$-$h(X, \tau_1, \tau_2)$ and $\circ$ is the usual operation of composition of functions. Then by Theorem 3.19, $g \circ f \in$ strongly $(1,2)^{\star}$-$\breve{g}$-$h(X, \tau_1, \tau_2)$. We know that the composition of functions is associative and the identity function $I : (X, \tau_1, \tau_2) \to (X, \tau_1, \tau_2)$ belonging to strongly $(1,2)^{\star}$-$\breve{g}$-$h(X, \tau_1, \tau_2)$ serves as the identity element. If $f \in$ strongly $(1,2)^{\star}$-$\breve{g}$-$h(X, \tau_1, \tau_2)$, then $f^{-1} \in$ strongly $(1,2)^{\star}$-$\breve{g}$-$h(X, \tau_1, \tau_2)$ such that $f \circ f^{-1} = f^{-1} \circ f = I$ and so inverse exists for each element of strongly $(1,2)^{\star}$-$\breve{g}$-$h(X, \tau_1, \tau_2)$. Therefore, (strongly $(1,2)^{\star}$-$\breve{g}$-$h(X, \tau_1, \tau_2), \circ)$ is a group under the operation of composition of functions. □

Theorem 3.21. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be strongly $(1,2)^{\star}$-$\breve{g}$-homeomorphism. Then f induces an isomorphism from the group strongly $(1,2)^{\star}$-$\breve{g}$-$h(X, \tau_1, \tau_2)$ on to the group strongly $(1,2)^{\star}$-$\breve{g}$-$h(Y, \sigma_1, \sigma_2)$.*

Proof. Using the map f, we define a map θ_f : strongly $(1,2)^{\star}$-$\breve{g}$-$h(X, \tau_1, \tau_2) \to$ strongly $(1,2)^{\star}$-$\breve{g}$-$h(Y, \sigma_1, \sigma_2)$ by $\theta_f(h) = f \circ h \circ f^{-1}$ for every $h \in$ strongly $(1,2)^{\star}$-$\breve{g}$-$h(X, \tau_1, \tau_2)$. Then θ_f is a bijection. Further, for all $h_1, h_2 \in$ strongly $(1,2)^{\star}$-$\breve{g}$-$h(X, \tau_1, \tau_2)$, $\theta_f(h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1} = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) = \theta_f(h_1) \circ \theta_f(h_2)$. Therefore, θ_f is a $(1,2)^{\star}$-homomorphism and so it is $(1,2)^{\star}$-isomorphism induced by f. □

Theorem 3.22. *Strongly $(1,2)^{\star}$-$\breve{g}$-homeomorphism is an equivalence relation in the collection of all topological spaces.*

Proof. Clearly it satisfied reflexivity, symmetry and transitivity. □

References

[1] J. C. Kelly, *Bitopological spaces*, Proc. London. Math. Soc., 3(13)(1963), 71-89.

[2] N. Levine, *Generalized closed sets in topology*, Rend. Circ. Math. Palermo, 17(2)(1970), 89-96.

[3] M. Lellis Thivagar and O. Ravi, *On stronger forms of $(1,2)^{\star}$-quotient function in bitopological spaces*, Int. J. Math. Game theory and Algebra, 14(6)(2004), 481-492.

[4] M. Lellis Thivagar, O. Ravi and E. Hatir, *Decomposition of* $(1,2)^{\star}$*-continuity and* $(1,2)^{\star}$*-*α*-continuity*, Miskolc Mathematical Notes, 10(2)(2009), 163-71.

[5] A. Ponmalar, R. Asokan and O. Nethaji, *On* $(1,2)^{\star}$*-*$\breve{g}$*-closed and open functions*, (to Appear).

[6] M. Ramaboopathi and K. M. Dharmalingam, *On* $(1,2)^{\star}$*-*$\breve{g}$*-closed sets in bitopological spaces*, Malaya Journal of Matematik, 7(3)(2019), 463-467.

[7] M. Ramaboopathi and K. M. Dharmalingam, *New forms of generalized closed sets in bitopological spaces*, Journal of Applied Science and Computations, VI(III)(2019), 712-718.

[8] M. Ramaboopathi and K. M. Dharmalingam, $(1,2)^{\star}$*-*$\breve{g}$*-Continuity and it's decompositions*, The International Journal of Analytical and Experimental Modal Analysis, XIII(V)(2021), 927-941.

[9] O. Ravi and M. Lellis Thivagar, *A bitopological* $(1,2)^{\star}$*-semi-generalized continuous maps*, Bull. Malaysian Math. Sci. Soc., (2)(29)(1)(2006), 76-88.

[10] O. Ravi, M. L. Thivagar and Jinjinli, *Remarks on extensions of* $(1,2)^{\star}$*-*g*-closed functionpings in bitopology*, Archimedes J. Math., 1(2)(2011), 177-187.

[11] O. Ravi, S. Pious Missier and T. Salai Parkunan, *On bitopological* $(1,2)^{\star}$*-generalized homeomorphisms*, Int J. Contemp. Math. Sciences., 5(11)(2010), 543-557.