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$(p,q)^{th}$ $\psi-$order and $(p,q)^{th}$ $\psi-$type of Entire and Meromorphic Functions and Some of its Estimation

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Abstract: We introduce the concept of $(p,q)^{th}$ $\psi - order$ and $(p,q)^{th}$ $\psi - type$ of entire and meromorphic functions to generalise some results related to the $\varphi - order$ concept introduced by Chyzhykov-Semochko in [7]. In this paper we establish some estimates of the sum, product and the derivative of entire and meromorphic functions in the complex plane.

MSC: 30D35.

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1. Introduction and Definitions

To discuss the growth of functions first we recall the following definitions.

Definition 1.1. *The order $\rho(f)$ of a meromorphic function f is defined as*

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\ T(r,f)}{log\ r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f. Again for $0 < \rho(f) < \infty$, we define the type $\tau(f)$ of a meromorphic function f by

$$\tau(f) = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho(f)}}.$$

Definition 1.2. *The order $\widetilde{\rho}(f)$ of an entire function f is defined as*

$$\widetilde{\rho}(f) = \limsup_{r\to\infty} \frac{\log\log\ M(r,f)}{\log r},$$

where $M(r,f) = \max\{|f(z)|\ :\ |z| = r\}$ is the maximum modulus of f. Again for $0 < \widetilde{\rho}(f) < \infty$, we define the type $\widetilde{\tau}(f)$ of an entire function f by

$$\widetilde{\tau}(f) = \limsup_{r\to\infty} \frac{log\ M(r,f)}{r^{\widetilde{\rho}(f)}}.$$

With this we have two known classical results involving the order and the type of $f_1 + f_2$ and $f_1 f_2$, where f_1 and f_2 are entire or meromorphic functions respectively.

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Theorem 1.3 ([15]). *If f_1 and f_2 be two entire functions, then we have*

$$\rho(f_1+f_2) \le \max\{\rho(f_1), \rho(f_2)\},$$
$$\rho(f_1 f_2) \le \max\{\rho(f_1), \rho(f_2)\}$$

and

$$\widetilde{\tau}(f_1+f_2) \le \max\{\widetilde{\tau}(f_1), \widetilde{\tau}(f_2)\},$$
$$\widetilde{\tau}(f_1 f_2) \le \widetilde{\tau}(f_1) + \widetilde{\tau}(f_2).$$

Theorem 1.4 ([8]). *If f_1 and f_2 be two meromorphic functions and $\rho(f_1) < \rho(f_2)$, then $\rho(f_1+f_2) = \rho(f_1 f_2) = \rho(f_2)$.*

In [14], Latreuch and Belaïdi established new estimates for the order and type of meromorphic functions and they obtained the following results which improved the above two theorems.

Theorem 1.5 ([14]). *Let f_1 and f_2 be two meromorphic functions.*

(i). If $0 < \rho(f_1) < \rho(f_2) < \infty$, then $\tau(f_1+f_2) = \tau(f_1 f_2) = \tau(f_2)$.

(ii). If $0 < \rho(f_1) = \rho(f_2) = \rho(f_1+f_2) = \rho(f_1 f_2) < \infty$, then

$$|\tau(f_1) - \tau(f_2)| \le \tau(f_1+f_2) \le \tau(f_1) + \tau(f_2),$$
$$|\tau(f_1) - \tau(f_2)| \le \tau(f_1 f_2) \le \tau(f_1) + \tau(f_2).$$

Theorem 1.6 ([14]). *If f_1 and f_2 be two meromorphic functions satisfying $0 < \rho(f_1) = \rho(f_2) < \infty$ and $\tau(f_1) \ne \tau(f_2)$, then $\rho(f_1+f_2) = \rho(f_1 f_2) = \rho(f_1) = \rho(f_2)$.*

Theorem 1.7 ([14]). *Let f_1 and f_2 be two entire functions.*

(i). If $0 < \rho(f_1) < \rho(f_2) < \infty$, then $\widetilde{\tau}(f_1+f_2) = \widetilde{\tau}(f_2)$ and $\widetilde{\tau}(f_1 f_2) \le \widetilde{\tau}(f_2)$.

(ii). If $0 < \rho(f_1) = \rho(f_2) = \rho(f_1+f_2) = \rho(f_1 f_2) < \infty$, then

$$\widetilde{\tau}(f_1+f_2) \le \max\{\widetilde{\tau}(f_1), \widetilde{\tau}(f_2)\},$$
$$\widetilde{\tau}(f_1 f_2) \le \widetilde{\tau}(f_1) + \widetilde{\tau}(f_2).$$

Furthermore, if $\widetilde{\tau}(f_1) \ne \widetilde{\tau}(f_2)$, then $\widetilde{\tau}(f_1+f_2) = \max\{\widetilde{\tau}(f_1), \widetilde{\tau}(f_2)\}$.

Theorem 1.8 ([14]). *If f_1 and f_2 be two entire functions and $0 < \rho(f_1) = \rho(f_2) < \infty$ and $\widetilde{\tau}(f_1) \ne \widetilde{\tau}(f_2)$, then $\rho(f_1+f_2) = \rho(f_1) = \rho(f_2)$.*

Analogously $p-order$ and $p-type$ of entire and meromorphic functions are as follows:

Definition 1.9. *Let p be an integer and $p \ge 1$. The iterated $p-order$ $\rho_p(f)$ of a meromorphic function f is defined as*

$$\rho_p(f) = \limsup_{r\to\infty} \frac{log_p\ T(r,f)}{log\ r}.$$

Again if f is an entire function, then

$$\rho_p(f) = \limsup_{r\to\infty} \frac{log_{p+1}\ M(r,f)}{log\ r}.$$

Definition 1.10. *The iterated $p-type$ $\tau_p(f)$ of a meromorphic function f with iterated $p-order$ $(0<\rho_p(f)<\infty)$ is defined as*

$$\tau_p(f)=\limsup_{r\to\infty}\frac{log_{p-1}\ T(r,f)}{r^{\rho_p(f)}}.$$

Again if f is an entire function, then its iterated $p-type$ $\widetilde{\tau}_p(f)$, is defined by

$$\widetilde{\tau}_p(f)=\limsup_{r\to\infty}\frac{log_p\ M(r,f)}{r^{\rho_p(f)}}.$$

From above it is clear that $\rho_1(f)$ and $\tau_1(f)$ coincide with $\rho(f)$ and $\tau(f)$ respectively. Several researchers (see [1, 2, 5, 6, 9, 12]) used the concept of the iterated p-order $\rho_p(f)$ instead of the usual order $\rho(f)$ to study the fast growing solutions. Tu-Zeng-Xu [16] generalized Theorems 1.3-1.6 from the usual order to the iterated $p-order$ as follows.

Theorem 1.11 ([16])**.** *Let f_1 and f_2 be two meromorphic functions satisfying $0<\rho_p(f_1)=\rho_p(f_2)<\infty$ and $\tau_p(f_1)<\tau_p(f_2)$. Then*

(i). $\rho_p(f_1+f_2)=\rho_p(f_1f_2)=\rho_p(f_1)=\rho_p(f_2)$.

(ii). If $p>1$, then $\tau_p(f_1+f_2)=\tau_p(f_1f_2)=\tau_p(f_2)$.

(iii). If $p=1$, then $\alpha\leq\tau_p(f_1+f_2)\leq\beta$ and $\alpha\leq\tau_p(f_1f_2)\leq\beta$, where $\alpha=\tau_p(f_2)-\tau_p(f_1)$ and $\beta=\tau_p(f_1)+\tau_p(f_2)$.

Theorem 1.12 ([16])**.** *Let f_1 and f_2 be two entire functions satisfying $0<\rho_p(f_1)=\rho_p(f_2)<\infty$ and $\widetilde{\tau}_p(f_1)<\widetilde{\tau}_p(f_2)$. Then*

(i). If $p\geq1$, then $\rho_p(f_1+f_2)=\rho_p(f_1)=\rho_p(f_2)$ and $\widetilde{\tau}_p(f_1+f_2)=\widetilde{\tau}_p(f_2)$.

(ii). If $p>1$, then $\rho_p(f_1f_2)=\rho_p(f_1)=\rho_p(f_2)$ and $\widetilde{\tau}_p(f_1f_2)=\widetilde{\tau}_p(f_2)$.

Since $\rho_p(f^{'})=\rho_p(f)$, $p\geq1$ and for a meromorphic function f with finite iterated $p-order$, Tu-Zeng-Xu [16] proved the following theorem for the iterated $p-type$.

Theorem 1.13 ([16])**.** *Let $p>1$ and f be meromorphic function satisfying $0<\rho_p(f)<\infty$. Then $\tau_p(f^{'})=\tau_p(f)$.*

In [7], Chyzhykov and Semochko introduced the concept of the $\varphi-order$. After that, Belaïdi ([3, 4]) improved the results in [7] for the lower $\varphi-order$ and the lower $\varphi-type$.

Definition 1.14 ([7])**.** *Let φ be an increasing unbounded function on $[1,\infty)$. The $\varphi-orders$ of a meromorphic function f are defined by*

$$\rho^0_\varphi(f)=\limsup_{r\to\infty}\frac{\varphi(e^{T(r,f)})}{log\ r},$$
$$\rho^1_\varphi(f)=\limsup_{r\to\infty}\frac{\varphi(T(r,f))}{log\ r}.$$

Again if f is an entire function, then the $\varphi-orders$ are defined by

$$\widetilde{\rho}^0_\varphi(f)=\limsup_{r\to\infty}\frac{\varphi(M(r,f))}{log\ r},$$
$$\widetilde{\rho}^1_\varphi(f)=\limsup_{r\to\infty}\frac{\varphi(log\ M(r,f))}{log\ r}.$$

By Φ we define the class of positive unbounded increasing functions on $[1,\infty)$ such that $\varphi(e^t)$ is slowly growing i.e.,

$$\forall c>0:\quad\frac{\varphi(e^{ct})}{\varphi(e^t)}=1,\qquad t\to\infty.$$

Recently, Kara and Belaïdi [11] introduced the following definition.

Definition 1.15 ([11]). *Let φ be an increasing unbounded function on $[1,\infty)$. The $\varphi-types$ of a meromorphic function f with $\varphi-order \in (0,\infty)$ are defined by*

$$\tau_{\varphi}^{0}(f)=\limsup_{r\to\infty}\frac{e^{\varphi(e^{T(r,f)})}}{r^{\rho_{\varphi}^{0}(f)}},$$
$$\tau_{\varphi}^{1}(f)=\limsup_{r\to\infty}\frac{e^{\varphi(T(r,f))}}{r^{\rho_{\varphi}^{1}(f)}}.$$

If f is an entire function, then the $\varphi-types$ are defined by

$$\widetilde{\tau}_{\varphi}^{0}(f)=\limsup_{r\to\infty}\frac{e^{\varphi(M(r,f))}}{r^{\widetilde{\rho}_{\varphi}^{0}(f)}},$$
$$\widetilde{\tau}_{\varphi}^{1}(f)=\limsup_{r\to\infty}\frac{e^{\varphi(log\ M(r,f))}}{r^{\widetilde{\rho}_{\varphi}^{1}(f)}}.$$

In this paper we introduce the definitions of $(p,q)^{th}\psi-orders$ and $(p,q)^{th}\psi-types$ related to $(p,q)^{th}\psi-order$ as follows and generalise all earlier results in our directions where ψ is a positive unbounded increasing function on $[1,\infty)$ satisfying the property $\psi(r_1+r_2)\leq\psi(r_1)+\psi(r_2)$.

Definition 1.16. *Let ψ be an increasing unbounded function on $[1,\infty)$. The $(p,q)^{th}$ $\psi-orders$ of a meromorphic function f are defined by*

$$\rho_{\psi}^{[p,q],0}(f)=\limsup_{r\to\infty}\frac{log\psi(e^{log^{[p-1]}T(r,f)})}{log^{[q]}r},$$
$$\rho_{\psi}^{[p,q],1}(f)=\limsup_{r\to\infty}\frac{log\psi(log^{[p-1]}T(r,f))}{log^{[q]}r},\quad p\geq q\geq 1.$$

If f is an entire function, then the $(p,q)^{th}$ $\psi-orders$ are defined by

$$\widetilde{\rho}_{\psi}^{[p,q],0}(f)=\limsup_{r\to\infty}\frac{log\psi(e^{log^{[p]}M(r,f)})}{log^{[q]}r},$$
$$\widetilde{\rho}_{\psi}^{[p,q],1}(f)=\limsup_{r\to\infty}\frac{log\psi(e^{log^{[p+1]}M(r,f)})}{log^{[q]}r}.$$

Definition 1.17. *Let ψ be an increasing unbounded function on $[1,\infty)$. The $(p,q)^{th}$ $\psi-types$ of a meromorphic function f with $(p,q)^{th}$ $\psi-order \in$ $(0,\infty)$ are defined by*

$$\tau_{\psi}^{[p,q],0}(f)=\limsup_{r\to\infty}\frac{\psi(e^{log^{[p-1]}T(r,f)})}{[log^{[q-1]}r]^{\rho_{\psi}^{[p,q],0}(f)}},$$
$$\tau_{\psi}^{[p,q],1}(f)=\limsup_{r\to\infty}\frac{\psi(log^{[p-1]}T(r,f))}{[log^{[q-1]}r]^{\rho_{\psi}^{[p,q],1}(f)}}.$$

If f is an entire function, then the $(p,q)^{th}$ $\psi-types$ are defined as

$$\widetilde{\tau}_{\psi}^{[p,q],0}(f)=\limsup_{r\to\infty}\frac{\psi(e^{log^{[p]}M(r,f)})}{[log^{[q-1]}r]^{\widetilde{\rho}_{\psi}^{[p,q],0}(f)}},$$
$$\widetilde{\tau}_{\psi}^{[p,q],1}(f)=\limsup_{r\to\infty}\frac{\psi(e^{log^{[p+1]}M(r,f)})}{[log^{[q-1]}r]^{\widetilde{\rho}_{\psi}^{[p,q],1}(f)}}.$$

Through out this paper, we assume the standard notations of Nevanlinna value distribution theory of meromorphic functions (see [8, 10, 13, 17]), also we mean by a meromorphic function a function which is meromorphic in the whole complex plane. Also we assume ψ be a positive unbounded increasing function on $[1,\infty)$ satisfying the property $\psi(r_1+r_2)\leq\psi(r_1)+\psi(r_2)$ for large r_1,r_2.

2. Basic Theorems

Theorem 2.1. *Let f, f_1, f_2 be three meromorphic functions. Then*

1.

$$\rho_\psi^{[p,q],j}(f_1 \pm f_2) \leq \ max\{\rho_\psi^{[p,q],j}(f_1), \rho_\psi^{[p,q],j}(f_2)\}, \quad j = 0, 1. \tag{1}$$

2.

$$\rho_\psi^{[p,q],j}(f_1 f_2) \leq \ max\{\rho_\psi^{[p,q],j}(f_1), \rho_\psi^{[p,q],j}(f_2)\}, \quad j = 0, 1. \tag{2}$$

3.

$$\rho_\psi^{[p,q],j}(\frac{1}{f}) = \rho_\psi^{[p,q],j}(f), \quad j = 0, 1 \ and \ f \not\equiv 0. \tag{3}$$

Proof. Let $\alpha = \rho_\psi^{[p,q],1}(f_1)$ and $\beta = \rho_\psi^{[p,q],1}(f_2)$. Without loss of generality, we may suppose that $\alpha \leq \beta$. Now from the definition of $(p,q)^{th}$ $\psi - order$, for any $\epsilon > 0$ and for all large r

$$\begin{aligned} \frac{log\psi(log^{[p-1]}T(r, f_k))}{log^{[q]}r} &\leq (\rho_\psi^{[p,q],1}(f_k) + \epsilon), \ k = 1, 2 \\ or, log\psi(log^{[p-1]}T(r, f_k)) &\leq (\beta + \epsilon) log^{[q]}r \\ or, log^{[p-1]}T(r, f_k) &\leq \psi^{-1}(e^{(\beta+\epsilon)log^{[q]}r}) \\ or, \ T(r, f_k) &\leq exp^{[p-1]}(\psi^{-1}(e^{(\beta+\epsilon)log^{[q]}r})). \end{aligned}$$

Now from the properties of Nevanlinna characteristic functions, we have

$$\begin{aligned} T(r, f_1 \pm f_2) &\leq T(r, f_1) + T(r, f_2) + O(1) \\ &\leq 3[exp^{[p-1]}(\psi^{-1}(e^{(\beta+\epsilon)log^{[q]}r}))] \\ &< exp^{[p-1]}(\psi^{-1}(e^{(\beta+3\epsilon)log^{[q]}r})). \end{aligned}$$

Hence, $\frac{log\psi(log^{[p-1]}T(r, f_1 \pm f_2))}{log^{[q]}r} \leq (\beta + 3\epsilon)$ or, $\rho_\psi^{[p,q],1}(f_1 \pm f_2) \leq \max\{\rho_\psi^{[p,q],1}(f_1), \rho_\psi^{[p,q],1}(f_2)\}$. Properties 2 and 3 can be proved similarly and proofs for $\rho_\psi^{[p,q],0}$ are analogous. □

Theorem 2.2. *Let f_1, f_2 be two meromorphic functions. If $\rho_\psi^{[p,q],j}(f_1) < \rho_\psi^{[p,q],j}(f_2)$, $(j = 0, 1)$, then $\rho_\psi^{[p,q],j}(f_1 + f_2) = \rho_\psi^{[p,q],j}(f_1 f_2) = \rho_\psi^{[p,q],j}(f_2)$ for $j = 0, 1$.*

Proof. Assume that $\rho_\psi^{[p,q],j}(f_1) < \rho_\psi^{[p,q],j}(f_2)$. So by (1), we have

$$\rho_\psi^{[p,q],j}(f_1 + f_2) \leq \rho_\psi^{[p,q],j}(f_2). \tag{4}$$

Again from (1), we get

$$\rho_\psi^{[p,q],j}(f_2) = \rho_\psi^{[p,q],j}(f_1 + f_2 - f_1) \leq max\{\rho_\psi^{[p,q],j}(f_1 + f_2), \rho_\psi^{[p,q],j}(f_1)\}.$$

So if we suppose that $\rho_\psi^{[p,q],j}(f_1) > \rho_\psi^{[p,q],j}(f_1 + f_2)$, then

$$\rho_\psi^{[p,q],j}(f_2) = \rho_\psi^{[p,q],j}(f_1 + f_2 - f_1) \leq max\{\rho_\psi^{[p,q],j}(f_1 + f_2), \rho_\psi^{[p,q],j}(f_1)\} = \rho_\psi^{[p,q],j}(f_1).$$

which contradicts the assumption $\rho_{\psi}^{[p,q],j}(f_1) < \rho_{\psi}^{[p,q],j}(f_2)$. Hence

$$\rho_{\psi}^{[p,q],j}(f_2) \leq \rho_{\psi}^{[p,q],j}(f_1 + f_2). \tag{5}$$

So, from (4) and (5) we get,

$$\rho_{\psi}^{[p,q],j}(f_1 + f_2) = \rho_{\psi}^{[p,q],j}(f_2). \tag{6}$$

Again from (2), it follows that

$$\rho_{\psi}^{[p,q],j}(f_1 f_2) \leq \rho_{\psi}^{[p,q],j}(f_2). \tag{7}$$

Now by (3), we have

$$\rho_{\psi}^{[p,q],j}(f_2) = \rho_{\psi}^{[p,q],j}(f_1 f_2 \frac{1}{f_1}) \leq \max\{\rho_{\psi}^{[p,q],j}(f_1 f_2), \rho_{\psi}^{[p,q],j}(\frac{1}{f_1})\} = \max\{\rho_{\psi}^{[p,q],j}(f_1 f_2), \rho_{\psi}^{[p,q],j}(f_1)\}.$$

So if we suppose $\rho_{\psi}^{[p,q],j}(f_1) > \rho_{\psi}^{[p,q],j}(f_1 f_2)$, then

$$\rho_{\psi}^{[p,q],j}(f_2) = \rho_{\psi}^{[p,q],j}(f_1 f_2 \frac{1}{f_1}) \leq \max\{\rho_{\psi}^{[p,q],j}(f_1 f_2), \rho_{\psi}^{[p,q],j}(f_1)\} = \rho_{\psi}^{[p,q],j}(f_1).$$

which is a contradiction. Hence

$$\rho_{\psi}^{[p,q],j}(f_2) \leq \rho_{\psi}^{[p,q],j}(f_1 f_2). \tag{8}$$

So from (7) and (8) we get,

$$\rho_{\psi}^{[p,q],j}(f_1 f_2) = \rho_{\psi}^{[p,q],j}(f_2). \tag{9}$$

Hence the theorem follows from (6) and (9). □

3. Main Theorems

Theorem 3.1. *Let f_1, f_2 be two meromorphic functions.*

(i) If $0 < \rho_{\psi}^{[p,q],j}(f_1) < \rho_{\psi}^{[p,q],j}(f_2) < \infty$ and $0 = \tau_{\psi}^{[p,q],j}(f_1) < \tau_{\psi}^{[p,q],j}(f_2)$, $(j = 0, 1)$, then

$$\tau_{\psi}^{[p,q],j}(f_1 + f_2) = \tau_{\psi}^{[p,q],j}(f_1 f_2) = \tau_{\psi}^{[p,q],j}(f_2).$$

(ii) If $0 < \rho_{\psi}^{[p,q],j}(f_1) = \rho_{\psi}^{[p,q],j}(f_2) = \rho_{\psi}^{[p,q],j}(f_1 + f_2) < \infty$, $(j = 0, 1)$, then

$$\tau_{\psi}^{[p,q],j}(f_1 + f_2) \leq \tau_{\psi}^{[p,q],j}(f_1) + \tau_{\psi}^{[p,q],j}(f_2).$$

(iii) If $0 < \rho_{\psi}^{[p,q],j}(f_1) = \rho_{\psi}^{[p,q],j}(f_2) = \rho_{\psi}^{[p,q],j}(f_1 f_2) < \infty$, $(j = 0, 1)$, then

$$\tau_{\psi}^{[p,q],j}(f_1 f_2) \leq \tau_{\psi}^{[p,q],j}(f_1) + \tau_{\psi}^{[p,q],j}(f_2).$$

Proof. We will prove the theorem for $j = 1$ and the proofs for $j = 0$ are analogous.

(i) From the definition of the $\tau_{\psi}^{[p,q],1}-type$ for any given $\epsilon > 0$, there exists a sequence $\{r_n, n \geq 1\}$ tending to infinity such that

$$\tau_{\psi}^{[p,q],1}(f_2) - \epsilon \leq \frac{\psi(log^{[p-1]}T(r_n, f_2))}{[log^{[q-1]}r_n]^{\rho_{\psi}^{[p,q],1}(f_2)}}$$

or, $\psi(log^{[p-1]}T(r_n, f_2)) \geq (\tau_{\psi}^{[p,q],1}(f_2) - \epsilon)[log^{[q-1]}r_n]^{\rho_{\psi}^{[p,q],1}(f_2)}$ and for all sufficiently large values of r,

$$\psi(log^{[p-1]}T(r, f_1)) \leq (\tau_{\psi}^{[p,q],1}(f_1) + \epsilon)[log^{[q-1]}r]^{\rho_{\psi}^{[p,q],1}(f_1)}.$$

We know that $T(r, f_1 + f_2) \geq T(r, f_2) - T(r, f_1) - log2$
or, $log^{[p-1]}T(r_n, f_1 + f_2) \geq log^{[p-1]}T(r_n, f_2) - log^{[p-1]}T(r_n, f_1) + O(1)$
or, $\psi(log^{[p-1]}T(r_n, f_1 + f_2)) \geq \psi(log^{[p-1]}T(r_n, f_2)) - \psi(log^{[p-1]}T(r_n, f_1)) + O(1)$
or, $\psi(log^{[p-1]}T(r_n, f_1 + f_2)) \geq (\tau_{\psi}^{[p,q],1}(f_2) - \epsilon)[log^{[q-1]}r_n]^{\rho_{\psi}^{[p,q],1}(f_2)} - (\tau_{\psi}^{[p,q],1}(f_1) + \epsilon)[log^{[q-1]}r_n]^{\rho_{\psi}^{[p,q],1}(f_1)} + O(1)$
or, $\psi(log^{[p-1]}T(r_n, f_1 + f_2)) \geq (\tau_{\psi}^{[p,q],1}(f_2) - 2\epsilon)[log^{[q-1]}r_n]^{\rho_{\psi}^{[p,q],1}(f_2)} + O(1)$
provided ϵ such that $0 < 2\epsilon < \tau_{\psi}^{[p,q],1}(f_2)$. Again we get from Theorem 2.2, $\rho_{\psi}^{[p,q],1}(f_1 + f_2) = \rho_{\psi}^{[p,q],1}(f_2)$ and hence from above

$$\frac{\psi[log^{[p-1]}T(r_n, f_1 + f_2)]}{[log^{[q-1]}r_n]^{\rho_{\psi}^{[p,q],1}(f_1+f_2)}} \geq \tau_{\psi}^{[p,q],1}(f_2) - 2\epsilon + o(1).$$

Since $\epsilon > 0$ is arbitrary so

$$\tau_{\psi}^{[p,q],1}(f_1 + f_2) \geq \tau_{\psi}^{[p,q],1}(f_2). \tag{10}$$

For reverse inequality since

$$\rho_{\psi}^{[p,q],1}(f_1 + f_2) = \rho_{\psi}^{[p,q],1}(f_2) > \rho_{\psi}^{[p,q],1}(f_1) = \rho_{\psi}^{[p,q],1}(-f_1),$$

so applying (10) we obtain

$$\tau_{\psi}^{[p,q],1}(f_2) = \tau_{\psi}^{[p,q],1}(f_1 + f_2 - f_1) \geq \tau_{\psi}^{[p,q],1}(f_1 + f_2). \tag{11}$$

Hence from (10) and (11) we get $\tau_{\psi}^{[p,q],1}(f_1 + f_2) = \tau_{\psi}^{[p,q],1}(f_2)$. Now we have to show that $\tau_{\psi}^{[p,q],1}(f_1 f_2) = \tau_{\psi}^{[p,q],1}(f_2)$. By the property

$$T(r, f_1 f_2) \geq T(r, f_2) - T(r, f_1) + O(1). \tag{12}$$

and a similar discussion as in the above proof, one can easily show that

$$\tau_{\psi}^{[p,q],1}(f_1 f_2) \geq \tau_{\psi}^{[p,q],1}(f_2). \tag{13}$$

Since $\rho_{\psi}^{[p,q],1}(f_1 f_2) = \rho_{\psi}^{[p,q],1}(f_2) > \rho_{\psi}^{[p,q],1}(f_1) = \rho_{\psi}^{[p,q],1}(\frac{1}{f_1})$. So, from (13), we get

$$\tau_{\psi}^{[p,q],1}(f_2) = \tau_{\psi}^{[p,q],1}(f_1 f_2 \frac{1}{f_1}) \geq \tau_{\psi}^{[p,q],1}(f_1 f_2)$$

and therefore we get from above

$$\tau_{\psi}^{[p,q],1}(f_1 f_2) = \tau_{\psi}^{[p,q],1}(f_2).$$

This proves the first part of the theorem.

(ii) From the definition of the $\tau_{\psi}^{[p,q],1}-type$ for any given $\epsilon>0$ and for all sufficiently large values of r we have

$$\psi(log^{[p-1]}T(r,f_i))\leq(\tau_{\psi}^{[p,q],1}(f_i)+\epsilon)[log^{[q-1]}r]^{\rho_{\psi}^{[p,q],1}(f_i)}\quad i=1,2.$$

Now $T(r,f_1+f_2)\leq T(r,f_1)+T(r,f_2)+O(1)$

or, $log^{[p-1]}T(r,f_1+f_2)\leq log^{[p-1]}T(r,f_1)+log^{[p-1]}T(r,f_2)+O(1)$

or, $\psi(log^{[p-1]}T(r,f_1+f_2))\leq\psi(log^{[p-1]}T(r,f_1))+\psi(log^{[p-1]}T(r,f_2))+O(1)$

or, $\psi(log^{[p-1]}T(r,f_1+f_2))\leq(\tau_{\psi}^{[p,q],1}(f_1)+\epsilon)[log^{[q-1]}r]^{\rho_{\psi}^{[p,q],1}(f_1+f_2)}+(\tau_{\psi}^{[p,q],1}(f_2)+\epsilon)[log^{[q-1]}r]^{\rho_{\psi}^{[p,q],1}(f_1+f_2)}+O(1)$

or, $\psi(log^{[p-1]}T(r,f_1+f_2))\leq(\tau_{\psi}^{[p,q],1}(f_1)+\tau_{\psi}^{[p,q],1}(f_2))+2\epsilon)[log^{[q-1]}r]^{\rho_{\psi}^{[p,q],1}(f_1+f_2)}+O(1)$.

Hence,

$$\frac{\psi(log^{[p-1]}T(r,f_1+f_2))}{[log^{[q-1]}r]^{\rho_{\psi}^{[p,q],1}(f_1+f_2)}}\leq\tau_{\psi}^{[p,q],1}(f_1)+\tau_{\psi}^{[p,q],1}(f_2)+2\epsilon+o(1).$$

Since $\epsilon>0$ is arbitrary, so we get

$$\tau_{\psi}^{[p,q],1}(f_1+f_2)\leq\tau_{\psi}^{[p,q],1}(f_1)+\tau_{\psi}^{[p,q],1}(f_2).$$

This proves the second part of the theorem.

(iii) From the definition of the $\tau_{\psi}^{[p,q],1}-type$ for any given $\epsilon>0$ and for all sufficiently large values of r we have

$$\psi(log^{[p-1]}T(r,f_i))\leq(\tau_{\psi}^{[p,q],1}(f_i)+\epsilon)[log^{[q-1]}r]^{\rho_{\psi}^{[p,q],1}(f_i)}\quad i=1,2.$$

Now $T(r,f_1f_2)\leq T(r,f_1)+T(r,f_2)$

or, $log^{[p-1]}T(r,f_1f_2)\leq log^{[p-1]}T(r,f_1)+log^{[p-1]}T(r,f_2)$

or, $\psi(log^{[p-1]}T(r,f_1f_2))\leq\psi(log^{[p-1]}T(r,f_1))+\psi(log^{[p-1]}T(r,f_2))$

or, $\psi(log^{[p-1]}T(r,f_1f_2))\leq(\tau_{\psi}^{[p,q],1}(f_1)+\epsilon)[log^{[q-1]}r]^{\rho_{\psi}^{[p,q],1}(f_1f_2)}+(\tau_{\psi}^{[p,q],1}(f_2)+\epsilon)[log^{[q-1]}r]^{\rho_{\psi}^{[p,q],1}(f_1f_2)}$

or, $\psi(log^{[p-1]}T(r,f_1f_2))\leq(\tau_{\psi}^{[p,q],1}(f_1)+\tau_{\psi}^{[p,q],1}(f_2))+2\epsilon)[log^{[q-1]}r]^{\rho_{\psi}^{[p,q],1}(f_1f_2)}$.

Hence,

$$\frac{\psi(log^{[p-1]}T(r,f_1f_2))}{[log^{[q-1]}r]^{\rho_{\psi}^{[p,q],1}(f_1f_2)}}\leq\tau_{\psi}^{[p,q],1}(f_1)+\tau_{\psi}^{[p,q],1}(f_2)+2\epsilon.$$

Since $\epsilon>0$ is arbitrary, so we get

$$\tau_{\psi}^{[p,q],1}(f_1f_2)\leq\tau_{\psi}^{[p,q],1}(f_1)+\tau_{\psi}^{[p,q],1}(f_2).$$

This completes the proof. □

Theorem 3.2. *Let f_1,f_2 be two meromorphic functions.*

(i) If $0<\rho_{\psi}^{[p,q],j}(f_1)=\rho_{\psi}^{[p,q],j}(f_2)=\rho_{\psi}^{[p,q],j}(f_1+f_2)<\infty$, $(j=0,1)$, then

$$\tau_{\psi}^{[p,q],j}(f_1)\leq\tau_{\psi}^{[p,q],j}(f_1+f_2)+\tau_{\psi}^{[p,q],j}(f_2),\ \ for\ j=0,1.$$

(ii) If $0<\rho_{\psi}^{[p,q],j}(f_1)=\rho_{\psi}^{[p,q],j}(f_2)=\rho_{\psi}^{[p,q],j}(f_1f_2)<\infty$, $(j=0,1)$ then

$$\tau_{\psi}^{[p,q],j}(f_1)\leq\tau_{\psi}^{[p,q],j}(f_1f_2)+\tau_{\psi}^{[p,q],j}(f_2),\ \ for\ j=0,1.$$

Proof. The proofs of the theorem are follows immediately from Theorem 3.1 (ii). Since $\rho_{\psi}^{[p,q],j}(f_1) = \rho_{\psi}^{[p,q],j}(f_1+f_2) = \rho_{\psi}^{[p,q],j}(f_2) = \rho_{\psi}^{[p,q],j}(-f_2)$, then we get

$$\tau_{\psi}^{[p,q],j}(f_1) = \tau_{\psi}^{[p,q],j}(f_1+f_2-f_2) \leq \tau_{\psi}^{[p,q],j}(f_1+f_2) + \tau_{\psi}^{[p,q],j}(f_2).$$

Similarly using Theorem 3.1 (iii) and since $\rho_{\psi}^{[p,q],j}(f_1) = \rho_{\psi}^{[p,q],j}(f_1 f_2) = \rho_{\psi}^{[p,q],j}(f_2) = \rho_{\psi}^{[p,q],j}(\frac{1}{f_2})$, so we have

$$\tau_{\psi}^{[p,q],j}(f_1) = \tau_{\psi}^{[p,q],j}(f_1 f_2 \frac{1}{f_2}) \leq \tau_{\psi}^{[p,q],j}(f_1 f_2) + \tau_{\psi}^{[p,q],j}(f_2).$$

This completes the proof. □

Theorem 3.3. *Let f_1, f_2 be two entire functions.*

(i) If $0 < \widetilde{\rho}_{\psi}^{[p,q],j}(f_1) < \widetilde{\rho}_{\psi}^{[p,q],j}(f_2) < \infty$ and $0 = \widetilde{\tau}_{\psi}^{[p,q],j}(f_1) < \widetilde{\tau}_{\psi}^{[p,q],j}(f_2)$, $(j = 0, 1)$, then

$$\widetilde{\tau}_{\psi}^{[p,q],j}(f_1+f_2) = \widetilde{\tau}_{\psi}^{[p,q],j}(f_2),$$
$$\widetilde{\tau}_{\psi}^{[p,q],j}(f_1 f_2) \leq \widetilde{\tau}_{\psi}^{[p,q],j}(f_2).$$

(ii) If $0 < \widetilde{\rho}_{\psi}^{[p,q],j}(f_1) = \widetilde{\rho}_{\psi}^{[p,q],j}(f_2) = \widetilde{\rho}_{\psi}^{[p,q],j}(f_1+f_2) < \infty$, $(j = 0, 1)$, then

$$\widetilde{\tau}_{\psi}^{[p,q],j}(f_1+f_2) \leq \widetilde{\tau}_{\psi}^{[p,q],j}(f_1) + \widetilde{\tau}_{\psi}^{[p,q],j}(f_2).$$

(iii) If $0 < \widetilde{\rho}_{\psi}^{[p,q],j}(f_1) = \widetilde{\rho}_{\psi}^{[p,q],j}(f_2) = \widetilde{\rho}_{\psi}^{[p,q],j}(f_1 f_2) < \infty$, $(j = 0, 1)$, then

$$\widetilde{\tau}_{\psi}^{[p,q],j}(f_1 f_2) \leq \widetilde{\tau}_{\psi}^{[p,q],j}(f_1) + \widetilde{\tau}_{\psi}^{[p,q],j}(f_2).$$

Proof. We will prove the theorem for $j = 1$ and the proofs for $j = 0$ are analogous.

(i) From the definition of $\widetilde{\tau}_{\psi}^{[p,q],1} - type$ for any given $\epsilon > 0$, there exists a sequence $\{r_n, n \geq 1\}$ tending to infinity such that

$$\psi(log^{[p]}M(r_n, f_2)) \geq (\widetilde{\tau}_{\psi}^{[p,q],1}(f_2) - \epsilon)[log^{[q-1]}r_n]^{\widetilde{\rho}_{\psi}^{[p,q],1}(f_2)}$$

and for all sufficiently large values of r we obtain,

$$\psi(log^{[p]}M(r, f_1) \leq (\widetilde{\tau}_{\psi}^{[p,q],1}(f_1) + \epsilon)[log^{[q-1]}r]^{\widetilde{\rho}_{\psi}^{[p,q],1}(f_1)}.$$

Now from of the each circle $|z| = r_n$ we choose a sequence $\{z_n,\ n \geq 1\}$ with $|z_n| = r_n$ and satisfying $|f_2(z_n)| = M(r_n, f_2)$, we get, $M(r_n, f_1+f_2) \geq |f_1(z_n) + f_2(z_n)|$

or, $M(r_n, f_1+f_2) \geq |f_2(z_n)| - |f_1(z_n)|$

or, $M(r_n, f_1+f_2) \geq M(r_n, f_2) - M(r_n, f_1)$

or, $log^{[p]}M(r_n, f_1+f_2) \geq log^{[p]}M(r_n, f_2) - log^{[p]}M(r_n, f_1)$

$$\begin{aligned} or,\quad \psi(log^{[p]}M(r_n, f_1+f_2)) &\geq \psi(log^{[p]}M(r_n, f_2)) - \psi(log^{[p]}M(r_n, f_1)) \\ &\geq [(\widetilde{\tau}_{\psi}^{[p,q],1}(f_2) - \epsilon) - (\widetilde{\tau}_{\psi}^{[p,q],1}(f_1) + \epsilon)][log^{[q-1]}r_n]^{\widetilde{\rho}_{\psi}^{[p,q],1}(f_2)} \\ &= (\widetilde{\tau}_{\psi}^{[p,q],1}(f_2) - 2\epsilon)[log^{[q-1]}r_n]^{\widetilde{\rho}_{\psi}^{[p,q],1}(f_2)} \end{aligned}$$

provided ϵ such that $0 < 2\epsilon < \widetilde{\tau}_{\psi}^{[p,q],1}(f_2)$ and $r_n \to \infty$. It follows from Theorem 2.2, we get, $\widetilde{\rho}_{\psi}^{[p,q],1}(f_1+f_2) = \widetilde{\rho}_{\psi}^{[p,q],1}(f_2)$. So we get from above

$$\frac{\psi(log^{[p]}M(r_n, f_1+f_2))}{[log^{[q-1]}r_n]^{\widetilde{\rho}_{\psi}^{[p,q],1}(f_1+f_2)}} \geq \widetilde{\tau}_{\psi}^{[p,q],1}(f_2) - 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary so

$$\widetilde{\tau}_{\psi}^{[p,q],1}(f_1+f_2) \geq \widetilde{\tau}_{\psi}^{[p,q],1}(f_2). \tag{14}$$

For reverse inequality since

$$\widetilde{\rho}_{\psi}^{[p,q],1}(f_1+f_2) = \widetilde{\rho}_{\psi}^{[p,q],1}(f_2) > \widetilde{\rho}_{\psi}^{[p,q],1}(f_1) = \widetilde{\rho}_{\psi}^{[p,q],1}(-f_1),$$

so applying (14) we obtain

$$\widetilde{\tau}_{\psi}^{[p,q],1}(f_2) = \widetilde{\tau}_{\psi}^{[p,q],1}(f_1+f_2-f_1) \geq \widetilde{\tau}_{\psi}^{[p,q],1}(f_1+f_2).$$

So finally we get from above $\widetilde{\tau}_{\psi}^{[p,q],1}(f_1+f_2) = \widetilde{\tau}_{\psi}^{[p,q],1}(f_2)$. Now we have to show that $\widetilde{\tau}_{\psi}^{[p,q],1}(f_1f_2) \leq \widetilde{\tau}_{\psi}^{[p,q],1}(f_2)$. By the property $M(r, f_1f_2) \leq M(r, f_1)M(r, f_2)$

or,

$$\begin{aligned} logM(r, f_1f_2) &\leq log(M(r, f_1)M(r, f_2)) \\ &= logM(r, f_1) + logM(r, f_2) \end{aligned}$$

or, $log^{[p]}M(r, f_1f_2) \leq log^{[p]}M(r, f_1) + log^{[p]}M(r, f_2)$

or,

$$\begin{aligned} \psi(log^{[p]}M(r, f_1f_2)) &\leq \psi(log^{[p]}M(r, f_1)) + \psi(log^{[p]}M(r, f_2)) \\ &\leq [(\widetilde{\tau}_{\psi}^{[p,q],1}(f_1)+\epsilon) + (\widetilde{\tau}_{\psi}^{[p,q],1}(f_2)+\epsilon)][log^{[q-1]}r]^{\widetilde{\rho}_{\psi}^{[p,q],1}(f_2)} \\ &= (\widetilde{\tau}_{\psi}^{[p,q],1}(f_2)+2\epsilon)[log^{[q-1]}r]^{\widetilde{\rho}_{\psi}^{[p,q],1}(f_2)}. \end{aligned}$$

It follows from Theorem 2.2 we get, $\widetilde{\rho}_{\psi}^{[p,q],1}(f_2) = \widetilde{\rho}_{\psi}^{[p,q],1}(f_1f_2)$. So we get from above

$$\frac{\psi(log^{[p]}M(r, f_1f_2))}{[log^{[q-1]}r]^{\widetilde{\rho}_{\psi}^{[p,q],1}(f_1f_2)}} \leq \widetilde{\tau}_{\psi}^{[p,q],1}(f_2) + 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we get

$$\widetilde{\tau}_{\psi}^{[p,q],1}(f_1f_2) \leq \widetilde{\tau}_{\psi}^{[p,q],1}(f_2).$$

This proves the first part of the theorem.

(ii) From the definition of $\widetilde{\tau}_{\psi}^{[p,q],1} - type$ for any given $\epsilon > 0$ and for all sufficiently large values of r

$$\psi(log^{[p]}M(r, f_i)) \leq (\widetilde{\tau}_{\psi}^{[p,q],1}(f_i)+\epsilon)[log^{[q-1]}r]^{\widetilde{\rho}_{\psi}^{[p,q],1}(f_i)} \quad i = 1, 2.$$

Now, $M(r, f_1+f_2) \leq M(r, f_1) + M(r, f_2)$

or, $log^{[p]}M(r, f_1+f_2) \leq log^{[p]}M(r, f_1) + log^{[p]}M(r, f_2)$

or,

$$\begin{aligned} \psi(log^{[p]}M(r, f_1+f_2)) &\leq \psi(log^{[p]}M(r, f_1)) + \psi(log^{[p]}M(r, f_2)) \\ &\leq [(\widetilde{\tau}_{\psi}^{[p,q],1}(f_1)+\epsilon) + (\widetilde{\tau}_{\psi}^{[p,q],1}(f_2)+\epsilon)][log^{[q-1]}r]^{\widetilde{\rho}_{\psi}^{[p,q],1}(f_1+f_2)}. \end{aligned}$$

Hence,
$$\frac{\psi(log^{[p]}M(r,f_1+f_2))}{[log^{[q-1]}r]^{\widetilde{\rho}_\psi^{[p,q],1}(f_1+f_2)}} \le \widetilde{\tau}_\psi^{[p,q],1}(f_1)+\widetilde{\tau}_\psi^{[p,q],1}(f_2)+2\epsilon.$$
Since $\epsilon>0$ is arbitrary, so we get
$$\widetilde{\tau}_\psi^{[p,q],1}(f_1+f_2) \le \ \widetilde{\tau}_\psi^{[p,q],1}(f_1)+\widetilde{\tau}_\psi^{[p,q],1}(f_2).$$

This proves the second part of the theorem.

(iii) From the definition of $\widetilde{\tau}_\psi^{[p,q],1}-type$ for any given $\epsilon>0$ and for all sufficiently large values of r
$$\psi(log^{[p]}M(r,f_i)) \le (\widetilde{\tau}_\psi^{[p,q],1}(f_i)+\epsilon)[log^{[q-1]}r]^{\widetilde{\rho}_\psi^{[p,q],1}(f_i)} \quad i=1,2.$$

Now $M(r,f_1f_2)\le M(r,f_1)M(r,f_2)$

or,
$$\begin{aligned} logM(r,f_1f_2) &\le log(M(r,f_1)M(r,f_2)) \\ &= logM(r,f_1)+logM(r,f_2) \end{aligned}$$
or, $log^{[p]}M(r,f_1f_2)\le log^{[p]}M(r,f_1)+log^{[p]}M(r,f_2)$

or,
$$\begin{aligned} \psi(log^{[p]}M(r,f_1f_2)) &\le \psi(log^{[p]}M(r,f_1))+\psi(log^{[p]}M(r,f_2)) \\ &\le [(\widetilde{\tau}_\psi^{[p,q],1}(f_1)+\epsilon)+(\widetilde{\tau}_\psi^{[p,q],1}(f_2)+\epsilon)][log^{[q-1]}r]^{\widetilde{\rho}_\psi^{[p,q],1}(f_1f_2)} \\ &= (\widetilde{\tau}_\psi^{[p,q],1}(f_1)+\widetilde{\tau}_\psi^{[p,q],1}(f_2)+2\epsilon)[log^{[q-1]}r]^{\widetilde{\rho}_\psi^{[p,q],1}(f_1f_2)}. \end{aligned}$$
Hence,
$$\frac{\psi(log^{[p]}M(r,f_1f_2))}{[log^{[q-1]}r]^{\widetilde{\rho}_\psi^{[p,q],1}(f_1f_2)}} \le \widetilde{\tau}_\psi^{[p,q],1}(f_1)+\widetilde{\tau}_\psi^{[p,q],1}(f_2)+2\epsilon.$$
Since $\epsilon>0$ is arbitrary, we get
$$\widetilde{\tau}_\psi^{[p,q],1}(f_1f_2) \le \widetilde{\tau}_\psi^{[p,q],1}(f_1)+\widetilde{\tau}_\psi^{[p,q],1}(f_2).$$

This completes the proof. □

Theorem 3.4. *Let f be a meromorphic function. Then*
$$\rho_\psi^{[p,q],j}(f^{'}) \le \rho_\psi^{[p,q],j}(f) \ for \ j=0,1.$$

Proof. Take $\rho_\psi^{[p,q],1}(f)=\alpha$. So from the definition of $\rho_\psi^{[p,q],1}-order$ for any $\epsilon>0$ and for all $r>r_0$, we have
$$T(r,f)=O(exp^{[p-1]}(\psi^{-1}(e^{log^{[q]}r(\alpha+\epsilon)}))).$$

Now by the lemma of logarithmic derivative ([10, 13]), we get
$$\begin{aligned} T(r,f^{'}) &= m(r,f^{'})+N(r,f^{'}) \\ &\le m(r,\frac{f^{'}}{f})+m(r,f)+2N(r,f) \\ &\le m(r,\frac{f^{'}}{f})+2T(r,f) \\ &= O(logT(r,f)+logr)+2T(r,f) \\ &\le 3T(r,f)+O(1) \\ &= O(exp^{[p-1]}(\psi^{-1}(e^{(\alpha+3\epsilon)log^{[q]}r}))), \quad r\notin E. \end{aligned}$$

where $E \subset [0,\infty)$ is a set of finite linear measure. So from above for all sufficiently large values of r

$$log\psi(log^{[p-1]}T(r,f^{'})) \leq (\alpha+3\epsilon)log^{[q]}r.$$

By the arbitrariness of ϵ, we finally get

$$\rho_{\psi}^{[p,q],1}(f^{'}) \leq \alpha = \rho_{\psi}^{[p,q],1}(f).$$

This proves the theorem. □

Theorem 3.5. *Let f be a meromorphic function. Then*

$$\tau_{\psi}^{[p,q],j}(f^{'}) \leq \tau_{\psi}^{[p,q],j}(f) \ for \ j=0,1.$$

Proof. Take $\rho_{\psi}^{[p,q],1}(f)=\alpha$. So from the definition of $\tau_{\psi}^{[p,q],1}-type$ for any $\epsilon>0$ and for all $r>r_0$, we have

$$T(r,f) = O[exp^{[p-1]}[\psi^{-1}((\tau_{\psi}^{[p,q],1}+\epsilon)[log^{[q-1]}r]^{\alpha})]].$$

Now by the lemma of logarithmic derivative ([10, 13]), we get

$$\begin{aligned}
T(r,f^{'}) &= m(r,f^{'})+N(r,f^{'})\\
&\leq m(r,\frac{f^{'}}{f})+m(r,f)+2N(r,f)\\
&\leq m(r,\frac{f^{'}}{f})+2T(r,f)\\
&= O(logT(r,f)+logr)+2T(r,f)\\
&\leq 3T(r,f)+O(1)\\
&= O[exp^{[p-1]}[\psi^{-1}((\tau_{\psi}^{[p,q],1}+3\epsilon)[log^{[q-1]}r]^{\alpha})]], \quad r\notin E.
\end{aligned}$$

where $E \subset [0,\infty)$ is a set of finite linear measure. So from above for all sufficiently large values of r

$$\psi(log^{[p-1]}T(r,f^{'})) \leq ((\tau_{\psi}^{[p,q],1}+3\epsilon)[log^{[q-1]}r]^{\alpha}).$$

By the arbitrariness of ϵ, we finally get

$$\tau_{\psi}^{[p,q],1}(f^{'}) \leq \tau_{\psi}^{[p,q],1}(f).$$

This proves the theorem. □

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