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Matrix Representations of Group Algebras of General Metacyclic Groups

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Abstract: In [1], we have computed matrix representations of group algebras of split metacyclic groups. In this paper we extend the computation to include the case of non-split metacyclic group.

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1. Preliminaries

Let F be a field. A ring A is an algebra over F (briefly F-algebra) if A is a vector space over F and the following compatibility condition holds $(sa)\,.b = s\,(a.b) = a.(sb)$ for any $a, b \in A$ and any $s \in F$. A is also called associative algebra (over F). The dimension of the algebra A is the dimension of A as a vector space over F.

Theorem 1.1 ([2])**.** *Let A be a n-dimensional algebra over a field F. Then there is a one to one algebra homomorphism from A into $M_n(F)$, the algebra of n-matrices over F.*

Let $G = \{g_1 = 1, g_2, \dots, g_n\}$ be a finite group of order n and F a field. Define $FG = \{a_1g_1 + a_2g_2 + \cdots + a_ng_n :\ a_i \in F\}$. FG is n-dimensional vector space over F with basis G. Multiplication of G can be extended linearly to FG by using group operation of G. Thus FG becomes an algebra over F of dimension n. FG is called group algebra. The following identifications should be realized.

(i). $0_F g_G = 0_{FG} = 0$.

(ii). $1_F g_G = g_{FG} = g$.

(iii). $a_F 1_F = a_{FG}$.

A group G is metacyclic if it has a cyclic normal subgroup N such that G/N is cyclic. Equivalently, G has cyclic subgroups H and K such that $H \lhd G$ and $G = HK$ [3]. If $H \cap K = \{1\}$ also, then G is called a split metacyclic group. Otherwise, it is called non-split. If G is a non-split metacyclic group, then G has a presentation of the following form [4]. $G = \langle \alpha, \beta : \alpha^n = 1, \beta^m = \alpha^t, \beta\alpha = \alpha^r\beta \rangle$, where $r^m \equiv 1 (mod\ n)$ and $tr \equiv t(mod\ n)$. $|G| = nm$. The general element of G is of

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the form $\alpha^i\beta^j$, where $0 \le i \le n-1$ and $0 \le j \le m-1$. If $t = 0$, we get the presentation of a split metacyclic group $G = \langle \alpha, \beta : \alpha^n = \beta^m = 1, \beta\alpha = \alpha^r\beta \rangle$; $r^m \equiv 1(mod\ n)$. By direct substitution we have the following in G.

Lemma 1.2. *$\beta^v\alpha^u = \alpha^{ur^v}\beta^v$, where u, v are integers.*

A circulant matrix M on parameters $a_0, a_1, \ldots, a_{n-1}$ is defined as follows

$$M(a_0, a_1, \ldots, a_{n-1}) = \begin{bmatrix} a_0 & a_{n-1} & \cdots & a_1 \\ a_1 & a_0 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 \end{bmatrix}$$

M is said to be circulant block matrix if it is of the form $M(M_1, M_2, \ldots, M_n)$. i.e. it is circulant blockwise on the blocks $M_1, M_2, \ldots, M_n$. Thus

$$M = \begin{bmatrix} M_1 & M_n & \cdots & M_2 \\ M_2 & M_1 & \cdots & M_3 \\ \vdots & \vdots & \ddots & \vdots \\ M_n & M_{n-1} & \cdots & M_1 \end{bmatrix}$$

2. Main Result

The general element of the group algebra FG is given by $w = w_0 + w_1 + \cdots + w_{m-1}$, where $w_i = (a_{0i} + a_{1i}\alpha + \cdots + a_{n-1i}\alpha^{n-1})\beta^i$ for $i = 0, 1, \ldots, m-1$. We take the following natural basis of the group algebra FG. $B = \{1, \alpha, \ldots, \alpha^{n-1}; \beta, \alpha\beta, \ldots, \alpha^{n-1}\beta; \ldots; \beta^{m-1}, \alpha\beta^{m-1}, \ldots, \alpha^{n-1}\beta^{m-1}\}$. This can be written as follows: $B = \{1, \alpha, \ldots, \alpha^{n-1}\}\beta^0 \cup \{1, \alpha, \ldots, \alpha^{n-1}\}\beta \cup \cdots \cup \{1, \alpha, \ldots, \alpha^{n-1}\}\beta^{m-1}$. Briefly, $= B_0 \cup B_1 \cup \cdots \cup B_{m-1}$, where $B_j = \{1, \alpha, \ldots, \alpha^{n-1}\}\beta^j$. Let $T_B : FG \to M_n(F)$ be the linear transformation of our matrix representation relative to the basis B. Let $T_{B_j} = T_B \mid_{B_j}$. By [1, Theorem 3] we have the following.

Lemma 2.1. $T_{B_0}(w_0) = M(a_{00}, a_{10}, \ldots, a_{n-1,0})$.

By Lemma 1.2, we have $B_{j,i} = \beta^i\{1, \alpha, \ldots, \alpha^{n-1}\}\beta^j = \left\{1, \alpha^{r^i}, \ldots, \alpha^{(n-1)r^i}\right\}\beta^{i+j}$. By division algorithm $i + j = qm + v$; $0 \le v \le m-1$; $\beta^{i+j} = \beta^{qm+v} = (\beta^m)^q\beta^v = \alpha^{qt}\beta^v$. Then $B_{j,i} = \left\{\alpha^{qt}, \alpha^{qt+r^i}, \ldots, \alpha^{qt+(n-1)r^i}\right\}\beta^v$. By construction of the linear transformation we get:

Lemma 2.2. *$T_{B_j}(w_i)$ is obtained by columns interchange of $M(a_{0i}, a_{1i}, \ldots, a_{n-1,i})$ according to the order of the elements $\alpha^{qt}, \alpha^{qt+r^i}, \ldots, \alpha^{qt+(n-1)r^i}$.*

With notations as above and by the construction of the linear transformation T, we have the following main result.

Theorem 2.3. *The matrix representation of $w = w_0 + w_1 + \cdots + w_{m-1}$ in FG relative to the basis $B = B_0 \cup B_1 \cup \cdots \cup B_{m-1}$ is given by*

$$T_B(w) = \begin{bmatrix} T_{B_0}(w_0) & T_{B_1}(w_{m-1}) & \cdots & T_{B_{m-1}}(w_1) \\ T_{B_0}(w_1) & T_{B_1}(w_0) & \cdots & T_{B_{m-1}}(w_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_{B_0}(w_{m-1}) & T_{B_1}(w_{m-2}) & \cdots & T_{B_{m-1}}(w_0) \end{bmatrix}$$

Now, we discuss the special case of G when $t = 0$. Thus $G = \langle \alpha, \beta : \alpha^n = \beta^m = 1, \beta\alpha = \alpha^r \beta \rangle$, where $r^m \equiv 1 (mod\ n)$. It is the split case.

Let $\left\{1, \alpha, \ldots, \alpha^{n-1}\right\}^{\beta^j} \equiv \left\{\beta^j 1 \beta^{-j}, \beta^j \alpha \beta^{-j}, \ldots, \beta^j \alpha^{n-1} \beta^{-j}\right\}$. Call the matrix obtained from the basis $\left\{1, \alpha, \ldots, \alpha^{n-1}\right\}^{\beta^j}$ by M^{β^j}.

By Lemma 2.2, $T_{B_j}(w_i) = M^{\beta^j}(a_{0j},\ a_{1j}, \ldots, a_{n-1,j})$. Thus we have

Corollary 2.4. *Let F be a field and G a split metacyclic group as above. The representation of the general element $\sum_{j=0}^{m-1}\sum_{i=0}^{n-1} a_{ij}\alpha^i\beta^j$ in FG is given by the circulant block matrix $M\left(M(a_{i0}), M^{\beta}(a_{i1}), \ldots, M^{\beta^{m-1}}(a_{im-1})\right)$; $i = 0, 1, \ldots, n-1$. It is the same result of [1, Theorem 4].*

3. Application

We compute matrix representations of FD_4 and FQ_4, where $D_4 = \left\langle \alpha, \beta : \alpha^4 = \beta^2 = 1, \beta\alpha = \alpha^3\beta \right\rangle$, the dihedral group and $Q_4 = \left\langle \alpha, \beta : \alpha^4 = 1, \alpha^2 = \beta^2, \beta\alpha = \alpha^3\beta \right\rangle$, the quaternion group. Q_4 is a non-split metacyclic group.

Let $a1 + b\alpha + c\alpha^2 + d\alpha^3 + e\beta + f\alpha\beta + g\alpha^2\beta + h\alpha^3\beta$ be the general element of FD_4 and $B_0 = \left\{1, \alpha, \alpha^2, \alpha^3\right\}$ the natural basis of FD_4. $B_1 = \left\{1, \alpha, \alpha^2, \alpha^3\right\}\beta$, $B_0^{\beta} = \left\{1, \alpha^3, \alpha^2, \alpha\right\}$, $B_1^{\beta} = \left\{1, \alpha^3, \alpha^2, \alpha\right\}\beta$.

$$T_{B_0}(w_0) = M(a,b,c,d) = \begin{bmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{bmatrix}, \quad T_{B_0}(w_1) = M^{\beta}(e,f,g,h) = \begin{bmatrix} e & f & g & h \\ f & g & h & e \\ g & h & e & f \\ h & e & f & g \end{bmatrix},$$

$$T_{B_1}(w_0) = M(a,b,c,d) = \begin{bmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{bmatrix}, \quad T_{B_1}(w_1) = M^{\beta}(e,f,g,h) = \begin{bmatrix} e & f & g & h \\ f & g & h & e \\ g & h & e & f \\ h & e & f & g \end{bmatrix}.$$

Then

$$T_B(w) = \begin{bmatrix} M(a,b,c,d) & \vdots & M^{\beta}(e,f,g,h) \\ \ldots\ldots & \vdots & \ldots\ldots \\ M^{\beta}(e,f,g,h) & \vdots & M(a,b,c,d) \end{bmatrix} \text{ by corollary 2.4.}$$

Thus $T_B(w)$ is given by the following 8- square matrix.

$$\begin{bmatrix} a & d & c & b & \vdots & e & f & g & h \\ b & a & d & c & \vdots & f & g & h & e \\ c & b & a & d & \vdots & g & h & e & f \\ d & c & b & a & \vdots & h & e & f & g \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e & f & g & h & \vdots & a & d & c & b \\ f & g & h & e & \vdots & b & a & d & c \\ g & h & e & f & \vdots & c & b & a & d \\ h & e & f & g & \vdots & d & c & b & a \end{bmatrix}$$

Now for FQ_4, let $w = w_0 + w_1 \in FQ_4$, where $w_0 = a1 + b\alpha + c\alpha^2 + d\alpha^3$ and $w_1 = e\beta + f\alpha\beta + g\alpha^2\beta + h\alpha^3\beta$. $B = B_0 \cup B_1$, where $B_0 = \{1, \alpha, \alpha^2, \alpha^3\}$ and $B_1 = \{1, \alpha, \alpha^2, \alpha^3\}\beta$. $r = 3, t = 2$. Then $B_{0,0} = \{1, \alpha, \alpha^2, \alpha^3\}, B_{0,1} = \{1, \alpha^3, \alpha^2, \alpha\}\beta, B_{1,0} = \{1, \alpha, \alpha^2, \alpha^3\}\beta, B_{1,1} = \{\alpha^2, \alpha, 1, \alpha^3\}$. By Theorem 2.3

$$T_B(w) = \begin{bmatrix} T_{B_0}(w_0) & \vdots & T_{B_1}(w_1) \\ \ldots\ldots & \vdots & \ldots\ldots \\ T_{B_0}(w_1) & \vdots & T_{B_1}(w_0) \end{bmatrix} = \begin{bmatrix} M^{B_{0,0}}(a,b,c,d) & \vdots & M^{B_{1,1}}(e,f,g,h) \\ \ldots\ldots & \vdots & \ldots\ldots \\ M^{B_{0,1}}(e,f,g,h) & \vdots & M^{B_{1,0}}(a,b,c,d) \end{bmatrix}$$

which is the following 8-square matrix

$$\begin{bmatrix} a & d & c & b & \vdots & g & h & e & f \\ b & a & d & c & \vdots & h & e & f & g \\ c & b & a & d & \vdots & e & f & g & h \\ d & c & b & a & \vdots & f & g & h & e \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e & f & g & h & \vdots & a & d & c & b \\ f & g & h & e & \vdots & b & a & d & c \\ g & h & e & f & \vdots & c & b & a & d \\ h & e & f & g & \vdots & d & c & b & a \end{bmatrix}$$

Note that the two matrix representations of FD_4 and FQ_4 are distinct.

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