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Construction of Weights on the Semigroup $(\mathbb{N}, +)$ Using some Standard Functions

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Abstract: A *weight* on the semigroup $(\mathbb{N}, +)$ of natural numbers is a function $\omega : \mathbb{N} \longrightarrow (0, \infty)$ satisfying the submultiplicativity $\omega(m+n) \leq \omega(m)\omega(n)$ for all $m, n \in \mathbb{N}$. In this simple paper, we exhibit that some standard functions such as $c\cosh(n)$, $c\sinh(n)$, $n^k + c$, $(n+c)^k$, e^{n^c}, e^{-n^c}, $\log(n^k) + c$, $[\log(n) + c]^k$, and much more are weights on $\mathbb{N}$ under certain conditions on the constant c.

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1. Introduction

The weights play crucial role in studying the weighted discrete semigroup algebra $\ell^1(\mathbb{N}, \omega)$ with the convolution product [4, Section 4.6]. Moreover, many Banach algebra properties of $\ell^1(\mathbb{N}, \omega)$ can be characterized in terms of the weight ω. For example, the $\ell^1(\mathbb{N}, \omega)$ is semisimple if and only if $\inf\{\omega(n)^{\frac{1}{n}} : n \in \mathbb{N}\}$ is positive [4, Theorem 4.6.9]. The weight ω greatly influences the Banach algebra structure of $\ell^1(\mathbb{N}, \omega)$. For example, consider $\omega_1(n) = e^{-n^2}$ and $\omega_2(n) = n^2 + 1$. Then, unlike $\ell^1(\mathbb{N}, \omega_2)$, the algebra $\ell^1(\mathbb{N}, \omega_1)$ is a radical, unicellular, ordinary Banach algebra [4, Proposition 4.6.24]. In 1974, G. E. Shilov asked the following question: Does there exist a radical weight ω on $\mathbb{N}$ such that $\ell^1(\mathbb{N}, \omega)$ contains a non-standard closed ideal? [8, Page 189]. In 1984, M. P. Thomas succeeded to construct such weights [10]. There are some open problems also. For example, whether there exists a semisimple weight ω on $\mathbb{N}$ such that $\ell^1(\mathbb{N}, \omega)$ is neither Arens regular nor strongly Arens irregular? [5, Page 56]. There are several types of weights such as radical, semisimple, regular, convex, ordinary, star-shaped, non-quasi-analytic, etc. [2–4, 6, 7]. We also note that the weights on $\mathbb{N}$ have a connection with arithmetical functions in Number Theory [1].

These interesting facts motivated us to find a variety of weights on $\mathbb{N}$. In this paper, we characterize, in terms of the constant c, the standard functions $c\cosh(n)$, $c\sinh(n)$, $n^k + c$, $(n+c)^k$, $n^{-k} + c$, $(n^k + c)^{-n}$, e^{n^c}, e^{-n^c}, $e^{-(n+c)!}$, $\log(n^k) + c$, and $[\log(n) + c]^k$ as weights. General Methods of constructing weights on arbitrary semigroups are given in [9].

2. Main Results

Throughout, we reserve the notation for two constants $k \in \mathbb{N}$ and $c > 0$.

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Theorem 2.1. *Let $c > 0$. Then*

(1). The map $\omega(n) = c\cosh(n)$ is a weight on $\mathbb{N}$ if and only if $c \geq 2$.

(2). The map $\omega(n) = c\sinh(n)$ is a weight on $\mathbb{N}$ if and only if $c \geq \frac{\sinh(2)}{\sinh^2(1)}$.

Proof.

(1). Assume that ω is a weight on $\mathbb{N}$. Then, for any $n \in \mathbb{N}$, we have

$$e^{2n} + e^{-2n} = \frac{2}{c}\omega(2n) \leq \frac{2}{c}\omega(n)^2 = \frac{c}{2}(e^n + e^{-n})^2.$$

This implies that $0 \leq (\frac{c}{2} - 1)e^{2n} + (\frac{c}{2} - 1)e^{-2n} + c$ $(n \in \mathbb{N})$ and hence we have $c \geq 2$. Conversely, assume that $c \geq 2$. Define $\omega_1(n) = e^n$, $\omega_2(n) = e^{-n}$, and $\omega_3(n) = \frac{c}{2}$. Clearly they are weights on $\mathbb{N}$ and hence $\omega = \omega_3(\omega_1 + \omega_2)$ is a weight on $\mathbb{N}$.

(2). First we *claim* that, for $k, m, n \in \mathbb{N}$,

$$e^{k+1} + e^{k-1} + e^{-k+1} + e^{-k-1} \leq 2e^{2m+k-1} + 2e^{-2m-k+1}; \tag{1}$$

$$(e^1 - e^{-1})(e^{m+n} - e^{-m-n}) \leq (e^1 + e^{-1})(e^m - e^{-m})(e^n - e^{-n}). \tag{2}$$

The inequality (1) is clear because

$$\begin{aligned} e^{k+1} + e^{k-1} + e^{-k+1} + e^{-k-1} &\leq e^2e^{k-1} + e^2e^{k-1} + e^{-k-1} + e^{-k-1} \\ &\leq e^{2m+k-1} + e^{2m+k-1} + e^{-2m-k+1} + e^{-2m-k+1}. \end{aligned}$$

The inequality (2) clearly holds for $m = n$. Now assume that $m < n$. Then $n = m+k$ for some $k \geq 1$. By inequality (1) above, we have

$$\begin{aligned} & e^{k+1} + e^{k-1} + e^{-k+1} + e^{-k-1} \leq 2e^{2m+k-1} + 2e^{-2m-k+1} \\ \Longrightarrow\ & e^{2m+k+1} + e^{-2m-k-1} + e^{k+1} + e^{k-1} + e^{-k+1} + e^{-k-1} \\ & \leq e^{2m+k+1} + e^{-2m-k-1} + 2e^{2m+k-1} + 2e^{-2m-k+1} \\ \Longrightarrow\ & e^{2m+k+1} + e^{-2m-k-1} - e^{2m+k-1} - e^{-2m-k+1} + e^{k+1} + e^{k-1} \\ & + e^{-k+1} + e^{-k-1} \leq e^{2m+k+1} + e^{-2m-k-1} + e^{2m+k-1} + e^{-2m-k+1} \\ \Longrightarrow\ & e^{2m+k+1} - e^{2m+k-1} - e^{-2m-k+1} + e^{-2m-k-1} \leq e^{2m+k+1} + e^{-2m-k-1} \\ & + e^{2m+k-1} + e^{-2m-k+1} - e^{k+1} - e^{k-1} - e^{-k+1} - e^{-k-1} \\ \Longrightarrow\ & e^{2m+k+1} - e^{2m+k-1} - e^{-2m-k+1} + e^{-2m-k-1} \leq e^{2m+k+1} + e^{2m+k-1} \\ & + e^{-2m-k+1} - e^{k+1} - e^{k-1} - e^{-k+1} - e^{-k-1} + e^{-2m-k-1} \\ \Longrightarrow\ & (e^1 - e^{-1})(e^{2m+k} - e^{-2m-k}) \leq (e^1 + e^{-1})(e^m - e^{-m})(e^{m+k} - e^{-m-k}) \\ \Longrightarrow\ & (e^1 - e^{-1})(e^{m+n} - e^{-m-n}) \leq (e^1 + e^{-1})(e^m - e^{-m})(e^n - e^{-n}) \end{aligned}$$

Thus the inequality (2) is proved.

Now assume that ω is a weight on $\mathbb{N}$. Then $c\sinh(2) = \omega(1+1) \leq \omega(1)\omega(1) = c^2\sinh^2(1)$. Hence $c \geq \frac{\sinh(2)}{\sinh^2(1)}$. Conversely, assume that $c \geq \frac{\sinh(2)}{\sinh^2(1)}$. Without loss of generality, we may assume that $c = \frac{\sinh(2)}{\sinh^2(1)}$. Then

$$\omega(m+n) = \frac{c}{2}(e^{m+n} - e^{-m-n})$$

$$\begin{aligned}
&\leq \frac{c}{2}\frac{(e^1+e^{-1})(e^m+e^{-m})(e^n+e^{-n})}{e^1-e^{-1}} \quad \text{(By inequality (2))}\\
&= \frac{c}{2}\frac{(e^2-e^{-2})}{(e^1-e^{-1})^2}(e^m-e^{-m})(e^n-e^{-n})\\
&= c\frac{\sinh(2)}{\sinh^2(1)}\sinh(m)\sinh(n)\\
&= c^2\sinh(m)\sinh(n) = \omega(m)\omega(n).
\end{aligned}$$

Thus ω is a weight on $\mathbb{N}$. □

Next we prove that the power function $n^k + c$ is a weight on $\mathbb{N}$ under some condition on c. In order to prove this, we need to define two positive numbers; namely,

$$c_k = \frac{\sqrt{2^{k+2}-3}-1}{2} \quad \text{and} \quad d_k = \sqrt{3^k + 2^{2k-2} - 2^k} - 2^{k-1}. \tag{3}$$

Theorem 2.2. *Let $k \in \mathbb{N}$ and $c > 0$. Then*

(1). $k \leq 4$ if and only if $c_k > d_k$.

(2). If $\omega(n) = n^k + c$ is a weight on $\mathbb{N}$, then $c \geq \max\{c_k, d_k\}$.

(3). If $k \leq 4$, then $\omega(n) = n^k + c_k$ is a weight on $\mathbb{N}$.

(4). If $k \geq 5$, then $\omega(n) = n^k + d_k$ is a weight on $\mathbb{N}$.

(5). The map $\omega(n) = n^k + c$ is a weight on $\mathbb{N}$ if and only if $c \geq \max\{c_k, d_k\}$.

Proof.

(1). If $k \leq 4$, then one can check $c_k > d_k$ manually. Conversely, assume that $k \geq 5$. Clearly $c_5 = \frac{\sqrt{125}-1}{2} < \sqrt{467} - 16 = d_5$. If $k \geq 6$, then

$$\begin{aligned}
&2 + 2^{\frac{k}{2}} < (\frac{3}{2})^k\\
\implies &2^k + 2^{\frac{3k}{2}} + 2^{2k-2} < 3^k + 2^{2k-2} - 2^k\\
\implies &(2^{\frac{k}{2}} + 2^{k-1})^2 < 3^k + 2^{2k-2} - 2^k\\
\implies &2^{\frac{k}{2}} < \sqrt{3^k + 2^{2k-2} - 2^k} - 2^{k-1} = d_k
\end{aligned}$$

On the other hand, $c_k = \frac{\sqrt{2^{k+2}-3}-1}{2} < 2^{\frac{k}{2}}$. Thus $c_k < d_k$ for all $k \geq 5$.

(2). Assume that ω is a weight on $\mathbb{N}$. Then

$$\begin{aligned}
&2^k + c = \omega(1+1) \leq \omega(1)\omega(1) = 1 + 2c + c^2\\
\iff &2^k \leq 1 + c + c^2 = \frac{3}{4} + (\frac{1}{2} + c)^2\\
\iff &\frac{\sqrt{2^{k+2}-3}}{2} \leq \frac{1}{2} + c\\
\iff &\frac{\sqrt{2^{k+2}-3}-1}{2} \leq c.
\end{aligned}$$

So that $c \geq c_k$. Also ω must satisfy

$$3^k + c = \omega(1+2) \leq \omega(1)\omega(2) = 2^k + c + 2^k c + c^2$$

$$\begin{aligned} &\iff 3^k - 2^k \le c^2 + 2^k c \\ &\iff 3^k - 2^k + 2^{2k-2} \le (2^{k-1} + c)^2 \\ &\iff \sqrt{3^k + 2^{2k-2} - 2^k} - 2^{k-1} \le c. \end{aligned}$$

So that $c \ge d_k$. Thus $c \ge \max\{c_k, d_k\}$.

(3). We shall prove this result in four cases.

Case-(i): $k = 1$. In this case, $c_1 > \frac{1}{2}$. The inequality $\omega(1+1) \le \omega(1)\omega(1)$ follows from the proof of Statement (2) above. For $n \ge 2$,

$$\omega(1+n) < n + c_1 + nc_1 < (1+c_1)(n+c_1) = \omega(1)\omega(n).$$

For $m, n \ge 2$, we have

$$\omega(m+n) = m + n + c_1 \le mn + c_1 < \omega(m)\omega(n).$$

Case-(ii): $k = 2$. In this case, $c_2 > 1$. Again $\omega(1+1) \le \omega(1)\omega(1)$ follows from the proof of Statement (2) above. For $n \ge 2$,

$$\omega(1+n) = 1 + 2n + n^2 + c_2 < n^2 + c_2 + n^2 c_2 + c_2^2 = \omega(1)\omega(n).$$

Finally, for $m, n \ge 2$, we have

$$(\frac{1}{n} + \frac{1}{m})^k + \frac{c_2}{m^k n^k} \le 1 + \frac{c_2}{m^k n^k} \le 1 + \frac{c_2^2}{m^k n^k} < (1 + \frac{c_2}{m^k})(1 + \frac{c_2}{n^k}).$$

Multiplying both sides by $m^k n^k$, we get $\omega(m+n) \le \omega(m)\omega(n)$.

Case-(iii): $k = 3$. In this case, $c_3 > 2$. Now this can be proved as per the arguments given in Case-(ii) above.

Case-(iv): $k = 4$. In this case, $3.4 < c_4 < 3.5$. The inequality $\omega(1+1) \le \omega(1)\omega(1)$ is clear. Let $m = 1$ and $n = 2$. Now

$$\omega(1+2) < 84.5 < 16 + 17(3.4) + (3.4)^2 < 16 + 17c_4 + c_4^2 = \omega(1)\omega(2),$$

and, for $n \ge 3$, we have

$$\omega(1+n) \le 4n^4 + c_4 \le (1+c_4)n^4 + (1+c_4)c_4 = \omega(1)\omega(n).$$

Finally, the $m, n \ge 2$ can be proved as in Case-(ii) above.

(4). By Statement (1) above, we have

$$\frac{\sqrt{2^{k+2} - 3} - 1}{2} = c_k < d_k \Longrightarrow 2^{k+2} - 3 < (1 + 2d_k)^2 \Longrightarrow 2^k + d_k < (1 + d_k)^2.$$

Thus $\omega(1+1) < \omega(1)\omega(1)$. Next we have

$$\begin{aligned} d_k &= \sqrt{3^k + 2^{2k-2} - 2^k} - 2^{k-1} \\ \Longrightarrow & (c_2 + 2^{k-1})^2 = 3^k + 2^{2k-2} - 2^k \\ \Longrightarrow & 2^k + c_2 2^k + c_2^2 = 3^k \\ \Longrightarrow & (1^k + c_2)(2^k + c_2) = 3^k + c_2 \end{aligned}$$

Hence $\omega(1)\omega(2) = \omega(1+2)$. The remaining two cases can be proved as in Case-(ii) of Statement (2) above.

(5). The necessary condition is proved in Statement (2) above. For the sufficient condition, we first note that if ω is a weight on $\mathbb{N}$ and if $\omega(n) \geq 1$ $(n \in \mathbb{N})$, then $\widetilde{\omega}(n) = \omega(n) + d$ is also a weight on $\mathbb{N}$ for any $d > 0$. So, we can assume that $c = \max\{c_k, d_k\}$. Now the result follows from Statements (1), (3), and (4). □

Theorem 2.3. *Let $k \in \mathbb{N}$ and $c > 0$. Then*

(1). $\omega(n) = (n+c)^k$ is a weight on $\mathbb{N}$ if and only if $c \geq c_1$.

(2). $\omega(n) = \frac{1}{n^k} + c$ is a weight on $\mathbb{N}$ if and only if $c \geq 1$.

(3). $\omega(n) = \frac{1}{(n^k+c)^n}$ is always a weight on $\mathbb{N}$.

(4). $\omega(n) = \frac{1}{n^k+c}$ is never a weight on $\mathbb{N}$.

Proof.

(1). Define $\omega_1(n) = n + c$ $(n \in \mathbb{N})$. Then $\omega(n) = \omega_1(n)^k$ for all n. Clearly, ω is a weight if and only if ω_1 is a weight. Now the result follows from Theorem 2.2(5).

(2). Assume that ω is a weight on $\mathbb{N}$. Then, for all $m, n \in \mathbb{N}$,

$$\frac{1}{(m+n)^k} + c = \omega(m+n) \leq \omega(m)\omega(n) = (\frac{1}{m^k} + c)(\frac{1}{n^k} + c).$$

Taking $m, n \longrightarrow \infty$, we get $c \leq c^2$. Hence $c \geq 1$. Conversely, assume that $c \geq 1$. Then, for $m, n \in \mathbb{N}$,

$$\begin{aligned} m^k n^k \{1 + c(m+n)^k\} &= m^k n^k + c m^k n^k (m+n)^k \\ &\leq (cm^k)(m+n)^k + c^2 m^k n^k (m+n)^k \\ &\leq (cm^k + cn^k + 1)(m+n)^k + c^2 m^k n^k (m+n)^k \\ &= (cm^k + cn^k + 1 + c^2 m^k n^k)(m+n)^k \\ &= (cm^k + 1)(cn^k + 1)(m+n)^k. \end{aligned}$$

Dividing both sides by $m^k n^k (m+n)^k$, we get $\omega(m+n) \leq \omega(m)\omega(n)$.

(3). Let $m, n \in \mathbb{N}$. Then

$$\begin{aligned} (m^k + c)^m (n^k + c)^n &\leq [(m+n)^k + c]^m (n^k + c)^n \\ &\leq [(m+n)^k + c]^m [(m+n)^k + c]^n \\ &= [(m+n)^k + c]^{m+n}. \end{aligned}$$

Thus $\frac{1}{[(m+n)^k+c]^{m+n}} \leq \frac{1}{m^k+c^m} \frac{1}{n^k+c^n}$ which implies $\omega(m+n) \leq \omega(m)\omega(n)$.

(4). Suppose, if possible, $\omega(n) = \frac{1}{n^k+c}$ is a weight. Then, for $m, n \in \mathbb{N}$,

$$\begin{aligned} &\omega(m+n) \leq \omega(m)\omega(n) \\ \Longrightarrow\ &\frac{1}{(m+n)^k + c} \leq (\frac{1}{m^k + c})(\frac{1}{n^k + c}) \\ \Longrightarrow\ &(m^k + c)(n^k + c) \leq (m+n)^k + c \\ \Longrightarrow\ &(1 + \frac{c}{m^k})(1 + \frac{c}{n^k}) \leq (\frac{1}{n} + \frac{1}{m})^k + \frac{c}{m^k n^k} \end{aligned}$$

Taking $m, n \longrightarrow \infty$, we get $1 \leq 0$. It is a contradiction. So ω is not a weight on $\mathbb{N}$. □

Theorem 2.4. *Let $k \in \mathbb{N}$ and $c \geq 0$. Then*

(1). $\omega(n) = e^{n^c}$ is a weight on $\mathbb{N}$ if and only if $0 \leq c \leq 1$.

(2). $\omega(n) = e^{-n^c}$ is a weight on $\mathbb{N}$ if and only if $c \geq 1$.

(3). If $c \in \mathbb{N} \cup \{0\}$, then $\omega(n) = e^{-(n+c)!}$ is a weight on $\mathbb{N}$.

Proof.

(1). If ω is a weight, then $2^c = \ln \omega(2) = \ln \omega(1+1) \leq 2 \ln \omega(1) = 2$. Hence $c \leq 1$. Conversely, if $c \leq 1$, then $(m+n)^c \leq m^c + n^c$, and so $\omega(m+n) \leq \omega(m)\omega(n)$.

(2). If ω is a weight, then $-2^c = \ln \omega(1+1) \leq 2 \ln \omega(1) = -2$. Hence $c \geq 1$. Conversely, if $c \geq 1$, then $(m+n)^c \geq m^c + n^c$. Hence $\omega(n) = e^{-n^c}$ is a weight on $\mathbb{N}$.

(3). Clearly, $[(m+n)^k + c]! \geq 2\{[(m+n-1)^k + c]!\} \geq (m^k + c)! + (n^k + c)!$. Thus ω is a weight on $\mathbb{N}$. □

Theorem 2.5. *Let $k \in \mathbb{N}$ and $c > 0$. Then*

(1). $\omega(n) = \log(n^k) + c$ is a weight on $\mathbb{N}$ if and only if $\log(2^k) \leq c(c-1)$.

(2). $\omega(n) = [\log(n) + c]^k$ is a weight on $\mathbb{N}$ if and only if $\log(2) \leq c(c-1)$.

(3). If $c \geq c_1$ and $\log[(1+c)^k] \geq 2$, then $\omega(n) = \log[(n+c)^k]$ is a weight on $\mathbb{N}$.

(4). If $c \geq c_1$ and $\log(1+c) \geq 2$, then $\omega(n) = [\log(n+c)]^k$ is a weight on $\mathbb{N}$.

Proof.

(1). Assume that ω is a weight. Then $\log(2^k) + c = \omega(1+1) \leq \omega(1)^2 = c^2$. Conversely, assume that $\log(2^k) \leq c(c-1)$. Since $\left\{\left(\frac{n+1}{n}\right)\right\}$ is a decreasing sequence and $c > 1$, we have

$$\begin{aligned} & \log\left[\left(\frac{n+1}{n^c}\right)^k\right] < \log\left[\left(\frac{n+1}{n}\right)^k\right] \leq \log(2^k) \leq c(c-1) \\ \Longrightarrow\ & \log[(n+1)^k] - \log[n^{kc}] \leq c^2 - c \\ \Longrightarrow\ & \log[(n+1)^k] + c \leq c\log(n^k) + c^2 \\ \Longrightarrow\ & \log[(n+1)^k] + c \leq c[\log(n^k) + c]. \end{aligned}$$

Thus $\omega(1+n) \leq \omega(1)\omega(n)$. Now let $m \geq 2$, and $n \geq 2$. Since $c > 1$, we have

$$\log[(m+n)^k] + c \leq \log[(mn)^k] + c < c[\log(m^k) + \log(n^k)] + c^2.$$

Hence $\omega(m+n) \leq \omega(m)\omega(n)$. Thus ω is a weight on $\mathbb{N}$.

(2). Define $\omega_1(n) = \log(n) + c$. Then $\omega(n) = \omega_1(n)^k$. Clearly, ω is a weight iff ω_1 is a weight. By Statement (1) above, ω_1 is a weight if and only if $\log(2) \leq c(c-1)$.

(3). Define $\omega_1(n) = (n+c)^k$. By Theorem 2.3(1), ω_1 is a weight. Then, by the hypothesis, $\omega(n) = \log \omega_1(n) \geq 2$ ($n \in \mathbb{N}$). Hence $\omega(m+n) = \log \omega_1(m+n) \leq \log[\omega_1(m)\omega_1(n)] = \omega(m) + \omega(n) \leq \omega(m)\omega(n)$ because $a \geq 2$ and $b \geq 2$ implies $a + b \leq ab$. Thus ω is a weight.

(4). Define $\omega_1 = n+c$ and $\omega_2(n) = \log \omega_1(n)$. By Theorem 2.3(1), ω_1 is a weight on $\mathbb{N}$. By the hypothesis, $\omega_2(n) \geq 2$ $(n \in \mathbb{N})$. So $\omega_2(m+n) = \log \omega_1(m+n) \leq \log[\omega_1(m)\omega_1(n)] = \omega_2(m) + \omega_2(n) \leq \omega_2(m)\omega_2(n)$. Thus ω_2 is a weight. Hence $\omega(n) = \omega_2(n)^k$ is a weight on $\mathbb{N}$. □

Remark 2.6. *Note that the maps $e^{|\sin(n)|}$ and $e^{|\sin(n)|+|\cos(n)|}$ are weights. But their ranges are contained in a bounded subset of $[1,\infty)$. Such weights are not interesting in studying the Banach algebra $\ell^1(\mathbb{N}, \omega)$. Moreover, $\mathbb{N}$ is the smallest subsemigroup of $\mathbb{R}$. If ω is a weight on a subsemigroup S of $\mathbb{R}$, then we can get a weight on $\mathbb{N}$ simply by $\widetilde{\omega}(n) = \omega(ns_0)$ with some fixed $s_0 \in S$. On the other hand, we can extend weights on $\mathbb{N}$ to a subsemigroup S of $\mathbb{R}$ containing $\mathbb{N}$ under some conditions on ω [2, Section 3.4]. The reader could refer to [2, 5, 9] and references their in for more weights and for methods of constructing them.*

References

[1] Tom M. Apostol, *Introduction to Analytic Number Theory*, Springer International Student Edition, Narosa Publishing House, New Delhi, (1989).

[2] Shreema S. Bhatt, *Weights on Groups and Semigroups, and their relevance in Banach Algebras*, M.Phil. Dissertation, Sardar Patel University, (2019).

[3] T. D. Blackmore, *Weak Amenability of Discrete Semigroup Algebras*, Semigroup Forum, 55(1997), 196-205.

[4] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Math. Soc. Monographs, Vol 24, Clarendon Press, Oxford, (2000).

[5] H. G. Dales and H. V. Dedania, *Weighted Convolution Algebras on Subsemigroups of the Real Line*, Dissertationes Mathematicae (Rosprawy Matematicze), 459(2009), 1-60.

[6] H. R. Ebrahimi Vishki, B. Khodsiani and A. Rejali, *Arens regularity of certain weighted semigroup algebras and countability*, Semigroup Forum, 92(2016), 304–310.

[7] N. Grønbæk, *Amenability of Weighted Discrete Semigroup Algebras on Cancellative Semigroups*, Proc. Roy. Soc. Edinburgh Series-A, 110(1988), 351-360.

[8] N. K. Nikolskii, *Selected Problems of Weighted Approximations and Spectral Analysis*, Proc. Steklov Inst. Math., 120(1974), 1-278.

[9] J. H. Shah, *Vector-valued Weighted Discrete Semigroup Algebras*, Ph.D. Thesis, Sardar Patel University, (2008).

[10] M. P. Thomas, *A Non-standard Ideal of a Radical Banach Algebra of Power Series*, Acta Math., 152(1984), 199-217.