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Generalized Boundary Closed Sets in Fuzzy Topological Spaces

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Abstract: In this paper, fuzzy boundary generalized closed set is introduced and studied. It is proved that every fuzzy closed set and fuzzy boundary closed set is fuzzy Boundary generalized closed set but the converses need not be true. Every fbg-closed set is fspg-closed, every fgs-closed and fpg-closed sets is fbg- closed, but the converses need not be true.

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1. Introduction and Preliminaries

This paper is devoted to the study of another form of generalized fuzzy b-closed set which is known as fuzzy b-generalized closed set. The notion of generalized closed set in general topology is due to Norman Levine. Fuzzy generalized closed set was introduced G.Balasubramanian and P. Sundaram. Similarly other forms of generalized closed sets were defined. For instance, fuzzy pre generalized closed set were defined by Caldas, Navalgi and Saraf in 2006. In 1998 the concept of fuzzy semi-generalized closed sets were given by Maki, Fukutake, Kojima and Harada, fuzzy generalized almost strongly closed set was by defined Bedre, which were also called as fuzzy generalized closed sets by Saraf, Caldas and Mishra. The concept of fuzzy semi-pre-generalized closed sets were given by Saraf, Navalgi and Khanna.

In section 2, fuzzy b-generalized closed set is introduced and studied. It is proved that every fuzzy closed set and fb-elosed set is fuzzy b-generalized closed set but the converses need not be true. Every fbg-closed set is fspg-closed, every fgs-closed and fpg-closed sets is fbg-closed, but the converses need not be true. The interrelationship between these generalized closed sets is shown in form of an implication diagram. A finite union of fbg-open sets is a fbg-open set, but the intersection of any two fbg-open sets need not be fbg-open. Also finite intersection of fbg-closed sets is a fbg-closed set, but the union of two fbg-closed sets need not be a fbg-closed set. It is also proved that a fuzzy set which is fb-open and fbg-closed in (A, τ) is fb-closed. In this section fbg-neighbourhood and fbgq-neighbourhood of a point are also introduced and some properties studied.

Definition 1.1. *A fuzzy subset A in a set X is a function $A : X \to [0, 1]$. A fuzzy subset in X is empty iff its membership function is identically 0 on X and is denoted by 0 or μ_ϕ. The set X can be considered as a fuzzy subset of X whose membership*

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function is identically 1 on X and is denoted by μ_x or I_x. In fact every subset of X is a fuzzy subset of X but not conversely. Hence the concept of a fuzzy subset is a generalization of the concept of a subset.

Definition 1.2. *A fuzzy set on X is 'Crisp' if it takes only the values 0 and 1 on X.*

Definition 1.3. *Let X be a set and be a family of fuzzy subsets of (X,τ) is called a fuzzy topology on X iff τ satisfies the following conditions.*

(i). μ_ϕ, $\mu_X \in \tau$: That is 0 and $1 \in \tau$.

(ii). If $G_i \in \tau$ for $i \in I$, then $\vee G_i \in \tau$, $i \in I$.

(iii). If $G, H \in \tau$ then $G \wedge H \in \tau$.

The pair (X,τ) is called a fuzzy topological space. The members of τ are called fuzzy open sets and a fuzzy set A in X is said to be closed iff $1 - A$ is an fuzzy open set in X.

Remark 1.4. *Every topological space is a fuzzy topological space but not conversely.*

Definition 1.5. *If A and B are any two fuzzy subsets of a set X, then A is said to be included in B or A is contained in B iff $A(x) \leq B(x)$ for all x in X. Equivalently, $A \leq B$ iff $A(x) \leq B(x)$ for all x in X.*

Definition 1.6. *Two fuzzy subsets A and B are said to be equal if $A(x) = B(x)$ for every x in X. Equivalently $A = B$ if $A(x) = B(x)$ for every x in X.*

Definition 1.7. *The complement of a fuzzy subset A in a set X, denoted by $A\prime$ or $1 - A$, is the fuzzy subset of X defined by $A\prime(x) = 1 - A(x)$ for all x in X. Note that $(A\prime)\prime = A$.*

Definition 1.8. *The union of two fuzzy subsets A and B in X, denoted by $A \vee B$, is a fuzzy subset in X defined by $(A \vee B)(x) = \max\{\mu_A(x), \mu_B(X)\}$ for all x in X.*

Definition 1.9. *The intersection of two fuzzy subsets A and B in X, denoted by $A \wedge B$, is a fuzzy subset in X defined by $(A \wedge B)(x) = \min\{A(x), B(x)\}$ for all x in X.*

2. Fuzzy Boundary Generalized Closed Sets

Definition 2.1. *A fuzzy set A in a Fuzzy topological Space (X,τ) is called fuzzy boundary generalized closed (briefly fbg-closed) iff $bCl(A) < B$, whenever $A < B$ and B is fb-open in X.*

Remark 2.2. *A fuzzy set A in a Fuzzy topological Space (X,τ) is called fuzzy boundary generalized open (briefly fbg-open) if its compliment $1 - A$ is fbg-closed.*

Theorem 2.3. *Every fuzzy closed set in (X,τ) is fbg-closed.*

Proof. Let A be a fuzzy closed in Fuzzy topological Space X. Let $A < B$, where B is fb-open in X. Since A is closed it is fb-closed and so $Cl(A) = bCl(A) = A < B$. Thus $bCl(A) < B$. Hence A is fbg-closed. □

Theorem 2.4. *Every fb-closed set in (X,τ) is fbg-closed.*

Proof. Let A be a fb-closed set in Fuzzy topological Space X. Let $A < B$. Where B is fb open in X. Since A is fb-closed, $bCl(A) = A < B$. Thus $bCl(A) < B$. Hence A is fbg-closed. □

Remark 2.5.

(i). Every fbg-closed set is fgb-closed.

(ii). Every fbg-closed set is fspg-closed.

(iii). Every fsg-closed, fpg-closed sets are fbg-closed.

The following example shows that every fgb-closed set is not fbg-closed.

Example 2.6. *Let* $X = \{a, b\}$, $\tau = \{0, A, 1\}$, *where* $A = \{(a, 0.3), (b, 0.6)\}$. *Let* $B = \{(a, 0.5), (b, 0.3)\}$, $C = \{(a, 0.6), (b, 0.4)\}$ *for* $B < C$, $bCl(B) < C$. *C is fuzzy open but not fb-open in X. Hence B is fgb-closed but not fbg-closed.*

The following example shows that every fbg-closed is not fsg-closed.

Example 2.7. *Let* $X = \{a, b\}$, $\tau = \{0, A, 1\}$, *where* $A = \{(a, 0.4), (b, 0.7)\}$. *Let* $B = \{(a, 0.5), (b, 0.2)\}$, $C = \{(a, 0.7), (b, 0.5)\}$ *for* $B < C$, $bCl(B) < C$. *But* $sCl(B) < C$.

The following example shows that every fspg-closed is not fbg-closed.

Example 2.8. *Let* $X = \{x, y, z\}$, $\tau = \{0, A, 1\}$, *where* $A = \{(x, 0.1), (y, 0.2), (z, 0.7)\}$. *Let* $B = \{(x, 0.9), (y, 0.2), (z, 0.5)\}$, $C = \{(x, 0.9), (y, 0.2), (z, 0.7)\}$ *for* $B < C$, $spCl(B) < C$, *C is fuzzy semi-open in* (X, τ) *but* $bCl(B) < C$.

Theorem 2.9. *A fuzzy set A of a Fuzzy topological Space* (X, τ) *is called fbg-open iff* $B < bint(A)$, *whenever B is fb-closed and* $B < A$.

Proof. Suppose A is fbg-open in X. Then $1 - A$ is fbg-closed in X. Let B be a fb-closed set in X such that $B < A$. Then $1 - A < 1 - B$, $1 - B$ is fb-open set in X. Since $1 - A$ is fbg-closed, $bCl(1 - A) < 1 - B$, which implies $1 - blnt(A) < 1 - B$. Thus $B < blnt(A)$.
Conversely, assume that $B < b - int(A)$, whenever $B < A$ and B is fb-closed set in X. Then $1 - blnt(A) < 1 - B = C$, where C is fb-open set in X. Hence $bCl(1 - A) < C$, which implies $1 - A$ is fbg-closed. Therefore A is fbg-open. □

Theorem 2.10. *If A is fbg-closed set in* (X, τ) *and* $A < B < bCl(A)$, *then B is fbg-closed set in* (X, τ).

Proof. Let C be fb-open in X such that $B < C$, then $A < C$. Since A is a fbg-closed in X, it follows that $bCl(A) < C$. Now $B < bCl(A)$ implies $bCl(B) < bCl(bCl\{A)) = bCl(A)$. Thus $bCl\{B) < C$. Hence B is fbg-closed in X. □

Theorem 2.11. *If A is fbg-open set in* (X, τ) *and* $blnt(A) < B < A$ *then B is fbg-open set in* (X, τ).

Proof. Let A be fbg-open and B be any fuzzy set in X such that $blnt(A) < B < A$. Then $1 - A$ is fbg-closed and $1 - A < 1 - B < bCl(l - A)$. Then $1 - B$ is fbg-closed set. Hence B is fbg-open set of X. □

Theorem 2.12. *Finite intersection of fbg-closed sets is a fbg-closed set.*

Proof. Let A and B be fbg-closed sets in X. Let $F < A \wedge B$, where F is fb-closed set in X. Then $F < A$ and $F < B$. Since A and B are fbg-closed sets, $F < A = blnt(A)$ and $F < B = bInt(B)$, which implies $F < (blnt(A) \wedge blnt(B))$. Hence $F < bInt(A \wedge B)$. Therefore $A \wedge B$ fbg-closed set in X. □

However, intersection of any two fbg-open sets need not be fbg-open, as shown in the following example.

Example 2.13. *Let $X = \{a, b, c\}$ and $\tau = \{0, 1, A\}$, where $A = \{(a, 1), (b, 0.5), (c, 0.3)\}$. Let $C = \{(a, 0.1), (b, 0.6), (c, 0.3)\}$ and $B = \{(a, 0), (b, 0.6), (c, 0)\}$ be fuzzy sets in X. Then C and B are fb-open in X and hence C and B are fbg-open in X. But $C \wedge B = \{(a, 0), (b, 0.4), (c, 0)\}$ is not fbg-open in X, since $bCl(C \wedge B) = 1 < C \vee B$. Since every fbg-open set is fgb-open the same applies to fgb-open sets.*

Theorem 2.14. *A finite union of fbg-open sets is a fbg-open set.*

Proof. Let A and B be fbg-open in X. Let $A \vee B < F$, where F is fb-open in X. Then $A < F$ or $B < F$. Since A and B are fbg-open, $bCl\{A) = A < F$ or $bCl(B) = B < F$, which implies $bCl\{A)VbCl\{B\} < F$. Hence $bCl(A \vee B) < F$. Therefore $A \vee B$ fbg-open in X. □

However, union of two fbg-closed sets need not be a fbg-closed set as shown in the following example.

Example 2.15. *Let $X = \{a, b, c\}$ and $\tau = \{0, 1, A\}$, where $A = \{(a, 1), (b, 0.5), (c, 0)\}$. Let $C = \{(a, 0), (b, 0.4), (c, 1)\}$ and $B = \{(a, 1), (b, 0.4), (c, 1)\}$ be fuzzy sets in X. Then C and B are fb-closed set in X and hence C and B are fbg-closed in X. But $C \vee B = \{(a, 1), (b, 0.4), (c, 1)\}$ is not fbg-closed set in X, since $bCl(C \vee B) = 1 < C \vee B$. Since every fbg-open set is fgb-open the same applies to fgb-open sets.*

Theorem 2.16. *If A is fb-closed set in (X, τ) and fbg-closed, then A is fb-closed in (X, τ).*

Proof. Let A be fb-open and fbg-closed set in X. For $A < A$, by definition $bCl(A) < A$. But $A < bCl(A)$, which implies $A = bCl(A)$. Hence A is fb-closed in X. □

Theorem 2.17. *A fuzzy set A is fbg-closed if and only if AqB implies $bCl(A)qB$ for every fb-closed set B of X.*

Proof. Suppose A be a fbg-closed set of X. Let B be a fb-closed set in X such that AqB. Then by definition that implies $A < 1 - B$ and $1 - B$ is fb-open set of X. Therefore $bCl(A) < bCl(1 - B) < 1 - B$ as A is fbg-closed. Hence $bCl(A)qB$. Conversely, let D be fb-open set in X such that $A < D$. Then by definition $Aq(1 - D)$ and $1 - D$ is fb-closed set in X. By hypothesis, $bCl(A)q(1 - D)$, which implies $bCl(A) < D$. Hence A is fbg-closed. □

Theorem 2.18. *Let A be fbg-closed set in (X, τ) and xp be a fuzzy point of (X, τ) such that $x_p q bCl(A)$ then $bCl(x_p)qA$.*

Proof. Let A be fbg-closed and x_p be a fuzzy point of X. Suppose $bCl(x_p)qA$, then by definition $bCl(x_p) < 1 - A$ that implies $A < 1 - bCl(x_p)$. So $bCl(A) < 1 - bCl(x_p) < 1 - x_p$ because $1 - bCl(xp)$ is fb-open in X and A is fbg-closed in X. Hence $x_p q bCl(A)$, which is a contradiction. □

Theorem 2.19. *Let (Y, T_y) be a subspace of (X, τ) and A be a fuzzy set of Y. If A is fbg-closed in X, then A is fbg-closed in Y.*

Definition 2.20. *Let A be a fuzzy set in Fuzzy topological Space X and x_p be a fuzzy point of X, then A is called fuzzy b-generalized neighbourhood (briefly fbg-neighbourhood) of x_p if and only if there exists a fbg-open set B of X such that $x_p \in B < A$.*

Definition 2.21. *Let A be a fuzzy set in Fuzzy topological Space X and x_p be a fuzzy point of X, then A is called fuzzy b-generalized q-neighbourhood (briefly fbgq-neighbourhood) of x_p if and only if there exist a fbg-open set B such that $x_p qB < A$.*

Theorem 2.22. *A is fbg-open set in X if and only if for each fuzzy point $x_p \in A$, A is a fbg-neighbourhood of x_p.*

Theorem 2.23. *If A and B are fbg-neighborhood of x_p then $A \vee B$ is also a fbg-neighbourhood of x_p.*

Theorem 2.24. *Let A be a fuzzy set of a Fuzzy topological Space X. Then a fuzzy point $x_p \in bCl(A)$ if and only if every fbgq-neighbourhood of x_p is quasi-coincident with A.*

Definition 2.25. *A Fuzzy topological Space (X, τ) is called a fuzzy $bT^*_{1/2}$ space (in short $fbT^*_{1/2}$ space) if every fbg-closed set in X is fuzzy closed.*

Theorem 2.26. *A Fuzzy topological Space (X, τ) is $fbT_{1/2}$ space if and only if every fuzzy set in (X, τ) is both fb-open and fbg-open.*

Proof. Let X be $fbT_{1/2}$ space and Let A be fbg-open set in X. Then $1 - A$ is fbg-closed X. By definition every fbg-closed set in X is fb-closed, so $1 - A$ is fb-closed and hence A is fb-open in X. Conversely, Let A be fbg-closed. Then $1 - A$ is fbg-open which implies $1 - A$ is fb-open. Hence A is fb-closed. Every fbg-closed set in X is fb-closed. Therefore X is $fbT_{1/2}$ space. □

Theorem 2.27. *A Fuzzy topological Space (X, τ) is $fbT_{1/2}$ space if and only if every fuzzy set in (X, τ) is both fuzzy open and fbg-open.*

Remark 2.28. *A Fuzzy topological Space (X, τ) is*

(i). $fbT_{1/2}$ space if every fbg-open set in X is f b-open.

(ii). $fbT^*_{1/2}$ space if every fgb-open set in X is f- open.

Remark 2.29. *In a Fuzzy topological Space (X, τ)*

(i). Every $fT_{1/2}$ space is $fbT_{1/2}$ space.

(ii). Every $fbT_{1/2}$ space is $fgbT_{1/2}$ space.

(iii). Every $fbT^*_{1/2}$ space is $fgbT_{1/2}$ space.

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