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Regularity of the Free Boundary in $div(a(x)\nabla u(x,y)) = -(h(x)\gamma(u))_x$ with $h'(x) < 0$

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Abstract: A free boundary problem of type $div(a(x)\nabla u) = -(h(x)\gamma(u))_x$ with $h_x < 0$ is considered. A regularity of the free boundary as a curve $y = \Phi(x)$ is established using a local monotony $bu_x - u_y < 0$ close to free boundary points.

MSC: 35A15, 35R35, 35JXX.

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1. Introduction

Our interest in this work came from the lubrication model studied by M. Chambat and G. Bayada in [2] where the pressure p is a periodic function solution of

$$div\big(h^3(x)\nabla p\big) = (h(x)\gamma)_x; \quad h(x) = 1 + \alpha \cos x; \quad \alpha \in (0,1)$$

and $0 \leqslant \gamma(p) \leqslant 1$ with $\gamma(p) = 1$ on $[p > 0]$. The authors established the existence of a solution and the uniqueness under regularity assumptions on the free boundary.

Recent works [4–6], showed a continuity of the free boundary in

$$div(a(x,y)\nabla u) = -(h(x,y)\gamma(u))_x \text{ when } h_x(x,y) \geqslant 0.$$

This monotony on h led to a monotony of γ which allowed the characterization of the free boundary as a function $x = \phi(y)$. The continuity of ϕ is established under assumptions relating $a(x,y)$ with $h(x,y)$ in [4], and under $C^{0,\alpha}_{loc}$ regularity on a in [5]. These work brought answers to the Lubrication free boundary problem in half of the domain since

$$h'(x) = -\alpha \sin(x) \text{ is negative on } (0,\pi) \text{ and positive on } (\pi, 2\pi).$$

In an attempt to explore the situation where $h_x < 0$, we assume, in this paper, that a and h are independent of y. We look for a monotone solution u in the y-direction. The free boundary is then defined as a function $y = \Phi(x)$. We establish its

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continuity by using techniques developed in [1, 9] for obstacle problems where a solution is more regular at the free boundary points than it is in our case. The idea is to construct a cone with a vertex at a free boundary point while controlling a part of $[u > 0]$ in that cone.

For simplicity, we set the domain

$$\Omega = (0,1) \times (0,1), \quad \text{denote:} \quad \Gamma_0 = (0,1) \times \{0\}, \quad \Gamma_1 = (0,1) \times \{1\},$$

and formulate the problem as:

$$(P) \begin{cases} \text{Find } (u,\gamma) \in H^1(\Omega) \times L^\infty(\Omega) \text{ such that:} \\ (i) \quad u \geqslant 0, \quad 0 \leqslant \gamma \leqslant 1, \quad u(\gamma - 1) = 0 \text{ a.e. in } \Omega \\ (ii) \quad u = \varphi \text{ on } \partial\Omega \\ (iii) \quad \int_\Omega \big(a(x)\nabla u + \gamma h(x) e_x\big).\nabla\xi dxdy = 0 \quad \forall \ \xi \in H_0^1(\Omega) \end{cases}$$

where $e_x = (1,0)$, $\varphi \in C^{0,1}(\overline{\Omega})$ with

$$\varphi(x,y) = \left\| \begin{array}{llllll} 0 & \text{on} & \Gamma_0, & \theta_0(y) & \text{on} & \{0\} \times [0,1] \\ & & & \text{and} & & \\ u_a & \text{on} & \Gamma_1, & \theta_1(y) & \text{on} & \{1\} \times [0,1] \end{array} \right.$$

with θ_i being regular and nondecreasing functions satisfying $0 \leqslant \theta_i(y) \leqslant u_a$, $i = 1,2$, and u_a is a positive constant. The function h is $C^2([0,1])$ and satisfies for some positive constants $\overline{h}$ and λ:

$$|h(x)| \leqslant \overline{h}, \qquad -\overline{h} \leqslant h'(x) < -\lambda < 0, \qquad |h''(x)| \leqslant \overline{h} \quad \text{for} \quad x \in [0,1]. \tag{1}$$

The matrix a depends only on the x-variable and satisfies:

$$a \in W^{2,\infty}(0,1) \cap C^{1,1}[0,1] \tag{2}$$

$$m|\xi|^2 \leqslant a_{ij}\xi_i\xi_j \leqslant M|\xi|^2 \quad \forall \ \xi \in \mathbb{R}^2, \quad m > 0, \quad M > 0. \tag{3}$$

The existence of a solution to (P) follows the proof in [3].

We introduce, for $\epsilon \in (0, \min(1, u_a))$, the penalization problem:

$$(P_\epsilon) \begin{cases} \text{Find } u_\epsilon^\eta \in H^1(\Omega) \text{ such that :} \\ (i) \qquad u_\epsilon = \varphi \text{ on } \partial\Omega \\ (ii) \quad \int_\Omega \big(a(x)\nabla u_\epsilon + h(x) H_\epsilon(u_\epsilon) e_x\big)\nabla\xi dxdy = 0 \quad \forall \ \xi \in H_0^1(\Omega) \end{cases}$$

with

$$H_\epsilon(t) = \left\| \begin{array}{ll} 0 & \text{if } t < 0 \\ t/\epsilon & \text{if } 0 \leqslant t \leqslant \epsilon \\ 1 & \text{if } t > \epsilon \end{array} \right.$$

We show, as in [3], that there exists a unique solution for (P_ϵ) satisfying:

$$u_\epsilon \rightharpoonup u \quad \text{in } H^1(\Omega), \quad H_\epsilon(u_\epsilon) \rightharpoonup \gamma \quad \text{in} \quad L^2(\Omega)$$

and that (u, γ) is a solution of (P).

Taking u_ϵ^- (resp.$(u_\epsilon - u_a)^+$) as a test functions in (P_ϵ), shows that $u_\epsilon \geqslant 0$ (resp.$u_\epsilon \leqslant u_a$). Then, comparing $u_\epsilon^\eta = u_\epsilon(x, y+\eta)$ with u_ϵ as in [7], we obtain $(u_\epsilon)_y \geqslant 0$ and finally get

$$0 \leqslant u \leqslant u_a, \quad \frac{\partial u}{\partial y} \geqslant 0 \quad \text{a.e in } \Omega. \tag{4}$$

In all what follows, we consider only monotone solutions of (P). As a consequence, we deduce that:

- $\forall\ (x_0, y_0) \in [u > 0] = [u(x, y) > 0] \cap \Omega, \exists\ \delta > 0$ such that $u(x, y) > 0$ for $(x, y) \in B_\delta(x_0, y_0) \cup (x_0 - \delta, x_0 + \delta) \times [y_0, 1]$
- $\Phi : (0, 1) \longrightarrow [0, 1)$ is well defined by $\Phi(x) = \inf\{y \in (0, 1) \,/\, u(x, y) > 0\}$ and is upper semi-continuous (u.s.c) on (0,1).
- $[u > 0] = [y > \Phi(x)]$.

Now, we list some properties of the solutions of (P). We have

- $div(a(x)\nabla u) = -(h\gamma)_x$ in $\mathcal{D}'(\Omega)$.
- $u \in C^{0,\alpha}_{loc}(\Omega \cup \Gamma_0 \cup \Gamma_1)$ ([8, Theorem 8.24, p 202]).
- $[u > 0]$ is an open set.
- If $a \in C^{1,1}[0, 1]$ and $h \in C^2(0, 1)$, then $u \in C^2_{loc}([u > 0])$ ([8, Theorem 8.10, p 186]).
- $div(a(x)\nabla u) \geqslant -(h)_x\chi([u > 0])$ in $\mathcal{D}'(\Omega)$.
- $div(a(x)\nabla u) \geqslant 0$ and $(h\chi)_x \leqslant 0$ in $\mathcal{D}'(\Omega)$.

Remark 1.1. *The above last inequalities are obtained by taking* $\pm(H_\epsilon(u)\xi)$, $\xi \in \mathcal{D}(\Omega)$, $\xi \geqslant 0$ *as a test function in* (P). *The* $C^{0,\alpha}_{loc}$ *regularity holds because* $h\gamma \in L^q(\Omega)$ *for* $q > 2$.

The main result of this paper is the following:

Theorem 1.2. *Assume the interior of the set* $(0, 1) \cap [\Phi(x) > 0]$ *non empty. Then* Φ *is continuous at each interior point of* $(0, 1) \cap [\Phi(x) > 0]$.

To prove the theorem, we work close to a free boundary point P_0. We construct a half cone with vertex at P_0. This is possible by establishing a local monotony $b\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y} \leqslant 0$.

2. Local Monotony

For the purpose of clarity, we establish the monotonicity result through the following steps.

Step 1: First, we have

Lemma 2.1.

$$\frac{\partial u}{\partial y} > 0 \ \ in \ \ [u > 0]. \tag{5}$$

Proof. We have

$$div(a(x)\nabla\Big(\frac{\partial u}{\partial y}\Big)) = \frac{\partial}{\partial y}\Big(div(a(x)\nabla u)\Big) = \frac{\partial}{\partial y}(h'(x)) = 0 \quad \text{in } [u > 0] \quad \text{and} \quad \frac{\partial u}{\partial y} \geqslant 0.$$

By the strong maximum principle ([8, Theorem 9.6 p.225]), one has

$$\frac{\partial u}{\partial y} > 0 \quad \text{in} \quad [u > 0] \qquad \text{or} \qquad \frac{\partial u}{\partial y} = 0 \quad \text{in} \quad [u > 0].$$

But if $\dfrac{\partial u}{\partial y} = 0$ in $[u > 0]$, then $u = u(x) = u(x,1) = u_a$ since $u \in C^0(\Omega \cup \Gamma_1)$. This leads to $0 = div(a(x)\nabla u) = -h'(x) > 0$ in $[u > 0]$ which is not possible. □

Step 2: Next, let $x_0 \in (0,1)$ with $y_0 = \Phi(x_0) > 0$. Set $\epsilon_0 = (1 - y_0)/6$, $\delta_0 = \min(x_0, 1 - x_0)/6$. Since Φ is u.s.c, then for $\epsilon \in (0, \epsilon_0)$, $\exists \ \delta \in (0, \delta_0)$ such that $\Phi(x) < \Phi(x_0) + \epsilon$ for any $x \in (x_0 - \delta, x_0 + \delta)$. Using the continuity of u up to the boundary $y = 1$, we can find $0 < \rho < 1 - (y_0 + 3\epsilon_0)$ such that $u > 0$ on $[x_0 - \delta, x_0 + \delta] \times [1 - \rho, 1]$. Set

$$F = [x_0 - \delta/2, x_0 + \delta/2] \times [y_0 + 2\epsilon, 1 - \rho]$$
$$G = (x_0 - \delta, x_0 + \delta) \times \left(y_0 + \epsilon, 1 - \frac{\rho}{2}\right)$$

Note that: $G \subset [u > 0] = [y > \Phi(x)]$.

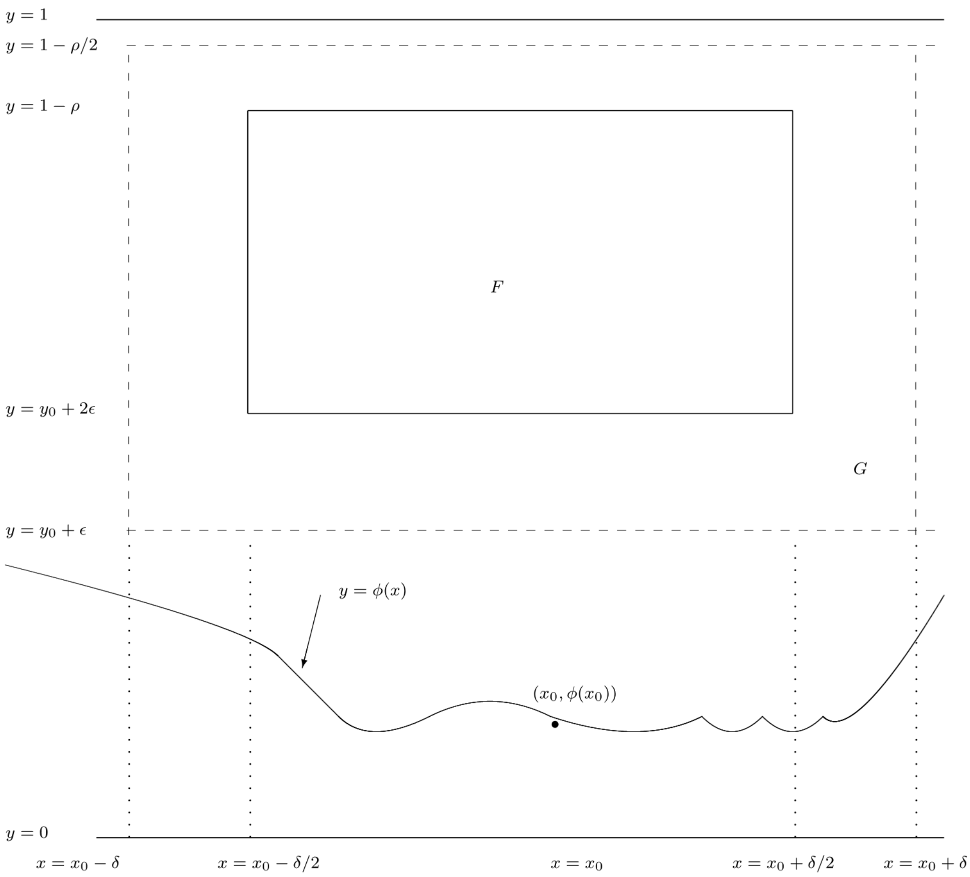

Figure 1: Set F and set G

Step 3: Since $u \in C^2([u > 0]$, then we have in G:

$$div(a(x)\nabla u) = -h'(x)$$
$$div\left(a(x)\nabla\left(\frac{\partial u}{\partial y}\right)\right) = 0$$

$$div\left(a(x)\nabla\left(\frac{\partial u}{\partial x}\right)\right)=\frac{\partial}{\partial x}\Big(div(a(x)\nabla u)\Big)-\Big(\frac{\partial a_{11}}{\partial x}\frac{\partial^2 u}{\partial x^2}+\frac{\partial a_{22}}{\partial x}\frac{\partial^2 u}{\partial y^2}+\frac{\partial(a_{12}+a_{21})}{\partial x}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 a_{11}}{\partial x^2}\frac{\partial u}{\partial x}+\frac{\partial^2 a_{12}}{\partial x^2}\frac{\partial u}{\partial y}\Big).$$

Using the assumptions on a and h, we deduce that the function

$$w=u+\tau\frac{\partial u}{\partial x}-E\frac{\partial u}{\partial y} \tag{6}$$

satisfies

$$\begin{aligned}div(a(x)\nabla w)&=div(a(x)\nabla u)+\tau div\left(a(x)\nabla\Big(\frac{\partial u}{\partial x}\Big)\right)-E\,div\left(a(x)\nabla\Big(\frac{\partial u}{\partial y}\Big)\right)\\&=-h'(x)+\tau\,div\left(a(x)\nabla\Big(\frac{\partial u}{\partial x}\Big)\right)-0\geqslant\lambda-|\tau|C_1\quad\text{in}\quad G\end{aligned}$$

where $C_1=C(\overline{h},|a|_{1,1},|u|_{C^2(\overline{G})},|a|_{W^{2,\infty}})$ is a constant depending on ϵ. Thus, for $|\tau|<\dfrac{\lambda}{2C_1}$, we have

$$div(a(x)\nabla w)>\frac{\lambda}{2}\quad\text{in}\quad G. \tag{7}$$

Step 4:. Now, let $\zeta\in C^\infty(\mathbb{R}^2)$ satisfying $\zeta=0$ on F, $0\leqslant\zeta\leqslant 1$ and $\zeta\geqslant 1$ on ∂G. We have

$$div(a(x)\nabla\zeta)=a_{11}\frac{\partial^2\zeta}{\partial x^2}+a_{22}\frac{\partial^2\zeta}{\partial y^2}+(a_{12}+a_{21})\frac{\partial^2\zeta}{\partial x\partial y}+\frac{\partial a_{11}}{\partial x}\frac{\partial\zeta}{\partial x}+\frac{\partial a_{12}}{\partial x}\frac{\partial\zeta}{\partial y}$$

Then,

$$|div(a(x)\nabla\zeta)|\leqslant C_2=\max_{\overline{G}}|div(a(x)\nabla\zeta)|$$

We choose $\mu\in\left(0,\dfrac{\lambda}{2C_2}\right)$ so that

$$\mu\;div(a(x)\nabla\zeta)\leqslant\frac{\lambda}{2}\quad\text{in}\quad G$$

From (7), we deduce that

$$div(a(x)\nabla(w-\mu\zeta))\geqslant 0\quad\text{in}\quad G. \tag{8}$$

Step 5: Set

$$\begin{aligned}k&=\max_{\overline{G}}|u|+\max_{\overline{G}}|\nabla u|, &\qquad \tau_0&=\frac{\mu}{1+k},\\ E_0&=\frac{k}{\beta(\tau_0)}, &\qquad \beta(\tau_0)&=\min_{[u\geqslant\tau_0]\cap\overline{G}}\Big(\frac{\partial u}{\partial y}\Big)\end{aligned}$$

By the extreme value theorem, the minimum value $\beta(\tau_0)$ is attained if the closed bounded set $[u\geqslant\tau_0]\cap\overline{G}\neq\emptyset$ and is strictly positive by Lemma 2.1; that is

$$\beta(\tau_0)=\frac{\partial u}{\partial y}(m_0)>0;\qquad\qquad m_0\in[u\geqslant\tau_0]\cap\overline{G}.$$

To show that $w-\mu\zeta\leqslant 0$ on ∂G, we discuss the following:

- If $\partial G\cap[u\geqslant\tau_0]=\emptyset$, then $u<\tau_0$ on ∂G. As a consequence, we have

$$\begin{aligned}w=u+\tau\frac{\partial u}{\partial x}-E\frac{\partial u}{\partial y}&\leqslant\tau_0+|\tau|k\;\;\text{since}\;\;\frac{\partial u}{\partial y}\geqslant 0\\&\leqslant\tau_0+\tau_0k=\mu\;\;\text{on}\;\;\partial G\;\;\text{if}\;\;|\tau|<\tau_0.\end{aligned}$$

- If $\partial G \cap [u \geqslant \tau_0] \neq \emptyset$, then
  - on $\partial G \cap [u < \tau_0]$, we have, as in the previous case $w \leqslant \mu$ on $\partial G \cap [u < \tau_0]$ if $|\tau| < \tau_0$.
  - on $\partial G \cap [u \geqslant \tau_0]$,

$$\begin{aligned} w = u + \tau\frac{\partial u}{\partial x} - E\frac{\partial u}{\partial y} &\leqslant k + |\tau|k - E\beta(\tau_0) \text{ since } \frac{\partial u}{\partial y} \geqslant \beta(\tau_0) > 0 \\ &\leqslant k + \tau_0 k - E_0\beta(\tau_0) \text{ for } E \geqslant E_0 \\ &= \tau_0 k \leqslant \tau_0(1+k) = \mu \text{ on } \partial G \cap [u \geqslant \tau_0] \text{ if } |\tau| < \tau_0 \text{ and } E \geqslant E_0. \end{aligned}$$

Step 6: Now, since $div(a(x)\nabla(w - \mu\zeta)) \geqslant 0$ in G and $w - \mu\zeta \leqslant 0$ on ∂G, we deduce, by the maximum principle, that $w - \mu\zeta \leqslant 0$ in G. In particular, we have $w \leqslant 0$ in F. Hence

$$\tau\frac{\partial u}{\partial x} - E\frac{\partial u}{\partial y} \leqslant -u < 0 \text{ in } F \text{ if } |\tau| < \tau_0 \text{ and } E \geqslant E_0.$$

Setting $b = \dfrac{\tau}{E}$ and $b_0 = \dfrac{\tau_0}{E_0}$, we finally established, through the 6 steps, the following result:

Theorem 2.2. *There exists a constant $b_0 > 0$ depending on x_0, y_0, ϵ such that*

$$b\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} < 0 \textit{ in } F \textit{ for } |b| < b_0.$$

3. Set of Directions

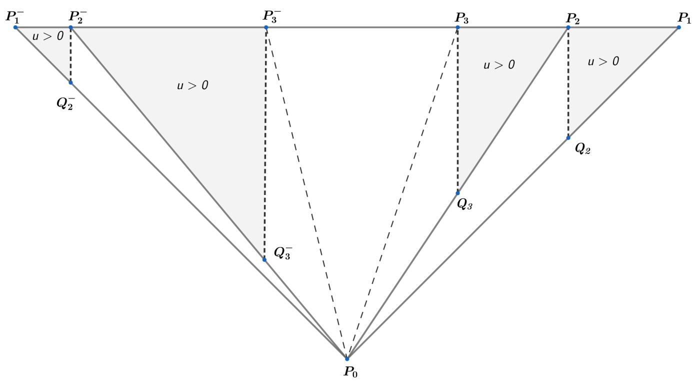

Figure 2: Lines P_0P_n and $P_n^-P_0$

In this section, we describe how we form particular lines passing through the vertex $P_0 = (x_0, y_0)$ and crossing the horizontal line $y = y_0 + 2\epsilon = y_\epsilon$.

Line segment P_0P_1**:** It is possible to find $x_1 \in (x_0, x_0 + \delta/4)$ such that

$$\frac{x_1 - x_0}{y_\epsilon - y_0} = \frac{x_1 - x_0}{2\epsilon} = \kappa b_0 < b_0$$

It suffices, to choose

$$\kappa = \frac{x_1 - x_0}{2\epsilon}.\frac{1}{b_0} \text{ such that } |x_1 - x_0| = 2\epsilon b_0\kappa < \frac{\delta}{4}; \quad \kappa \in \Big(0, \min\Big(\frac{1}{2}, \frac{\delta}{8\epsilon b_0}\Big)\Big).$$

Set $P_1 = (x_1, y_\epsilon)$ and $b_1 = \kappa b_0$. The line joining P_0 and P_1 has the slope

$$\frac{y_\epsilon - y_0}{x_1 - x_0} = b_1^{-1}$$

and can be described by the following two parameterizations:

$$P_0P_1 : \quad \begin{cases} x = x_1 - tb_1 \\ & t \in [0, 2\epsilon] \\ y = y_\epsilon - t \end{cases} \qquad \text{or} \qquad \begin{cases} x = x_0 + sb_1 \\ & s \in [0, 2\epsilon] \\ y = y_0 + s \end{cases}$$

Define the function

$$f_{b_1}(t) = u(x_1 - tb_1, y_\epsilon - t).$$

We have

$$f_{b_1}(0) = u(x_1, y_\epsilon) > 0, \quad f'_{b_1}(0) = \Big(- b_1 \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \Big)(x_1, y_\epsilon) < 0$$

By continuity, we have $f'_{b_1}(t) < 0$ for t small. Let

$$t_2 = \sup\, U; \quad U = \{t \in [0, 2\epsilon] \ \text{ such that } \ f_{b_1}(s) > 0 \ \ \forall \ \ s \in [0, t)\}.$$

Since $U \neq \emptyset$ and bounded, then t_2 is well defined and by continuity of f_{b_1}, we have $f_{b_1}(t_2) = 0$. Note that $t_2 > \epsilon$ since $u > 0$ in $(x_0 - \delta, x_0 + \delta) \times [y_0 + \epsilon, y_\epsilon]$. If $t_2 = 2\epsilon$, then $u > 0$ along the line segment $P_0P_1 \setminus \{P_0\}$. If $t_2 \neq 2\epsilon$, set

$$x_2 = x_1 - t_2 b_1; \qquad y_2 = y_\epsilon - t_2; \qquad Q_2 = (x_2, y_2),$$

then form the second line segment.

Line segment P_0P_2**:** Set $P_2 = (x_2, y_\epsilon)$. Then, we have

$$\begin{aligned} \frac{x_2 - x_0}{y_\epsilon - y_0} &= \frac{1}{2\epsilon}((x_1 - t_2 b_1) - x_0) \\ &= \frac{1}{2\epsilon}((x_0 + 2\epsilon b_1 - t_2 b_1) - x_0) \ \text{ since } \ x_1 = x_0 + 2\epsilon b_1 \\ &= \Big(\frac{2\epsilon - t_2}{2\epsilon}\Big) b_1 = b_2 < b_1 < b_0. \end{aligned}$$

The line joining P_0 and P_2 has the slope

$$\frac{y_\epsilon - y_0}{x_2 - x_0} = b_2^{-1}$$

and can be described by the following two parameterizations:

$$P_0P_2 : \quad \begin{cases} x = x_2 - tb_2 \\ & t \in [0, 2\epsilon] \\ y = y_\epsilon - t \end{cases} \qquad \text{or} \qquad \begin{cases} x = x_0 + sb_2 \\ & s \in [0, 2\epsilon] \\ y = y_0 + s \end{cases}$$

Define the function

$$f_{b_2}(t) = u(x_1 - tb_2, y_\epsilon - t).$$

We have

$$f_{b_2}(0) = u(x_1, y_\epsilon) > 0, \ \ f'_{b_2}(0) = \Big(- b_2 \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \Big)(x_1, y_\epsilon) < 0$$

By continuity $f'_{b_2}(t) < 0$ for small t. Let $t_3 \in (0, 2\epsilon]$ such that

$$f_{b_2}(t_3) = 0 \text{ and } f_{b_2}(t) > 0 \text{ for } t \in [0, t_3).$$

Note that $t_3 > \epsilon$ since $u > 0$ in $(x_0 - \delta, x_0 + \delta) \times [y_0 + \epsilon, y_\epsilon]$. If $t_3 = 2\epsilon$, then $u > 0$ along the line segment $P_0P_2 \setminus \{P_0\}$. If $t_3 \neq 2\epsilon$, set

$$x_3 = x_2 - t_3 b_2; \qquad y_3 = y_\epsilon - t_3; \qquad Q_3 = (x_3, y_3),$$

then we form the line segment P_0P_3.

Line segment P_0P_n: By repeating the previous process, we will obtain a sequence of points $P_n = (x_n, y_\epsilon)$ and $Q_{n+1} = (x_{n+1}, y_\epsilon - t_{n+1})$ satisfying:

$$\begin{aligned}
x_1 - x_0 &= 2\epsilon b_1 = 2\epsilon b_0 \kappa & x_1 - x_2 &= t_2 b_1 \geqslant \epsilon b_1 \\
x_2 - x_0 &= 2\epsilon b_2 & x_2 - x_3 &= t_3 b_2 \geqslant \epsilon b_2 \\
&\vdots & &\vdots \\
x_n - x_0 &= 2\epsilon b_n & x_n - x_{n+1} &= t_{n+1} b_n \geqslant \epsilon b_n
\end{aligned}$$

The sequence (x_n) is convergent since it is decreasing and bounded. Indeed, we have

$$0 \leqslant 2\epsilon b_{n+1} \leqslant 2\epsilon b_n \leqslant 2\epsilon b_0 \kappa \leqslant \epsilon b_0 \text{ and } t_{n+1} \geqslant \epsilon.$$

We also have

$$\begin{aligned}
2\epsilon b_0 \kappa = x_1 - x_0 &\geqslant x_1 - x_{n+1} \\
&= (x_1 - x_2) + (x_2 - x_3) + \ldots + (x_n - x_{n+1}) \\
&\geqslant \epsilon b_1 + \epsilon b_2 + \ldots + \epsilon b_n = s_n \geqslant 0
\end{aligned}$$

(s_n) is a bounded increasing sequence. Thus convergent. As a consequence

$$\epsilon b_n = s_n - s_{n-1} \longrightarrow 0 \text{ and } x_n = x_0 + 2\epsilon b_n \longrightarrow x_0 \text{ as } n \longrightarrow +\infty.$$

Similarly, we obtain a sequence of points $P_1^-, P_2^-, \ldots, P_n^-, \ldots$ to the left of P_0.

Line segment $P_1^- P_0$: With $x_1^- - x_0 = -2\epsilon\kappa b_0$, we have

$$\frac{x_1^- - x_0}{y_\epsilon - y_0} = \frac{x_1^- - x_0}{2\epsilon} = -\kappa b_0 > -b_0, \quad x_1^- \in (x_0 - \delta/4, x_0)$$

Since

$$\kappa = \frac{x_1^- - x_0}{2\epsilon} \cdot \frac{1}{-b_0} \text{ is such that } |x_1^- - x_0| = 2\epsilon b_0 \kappa < \frac{\delta}{4} \text{ for } \kappa \in \Big(0, \min\Big(\frac{1}{2}, \frac{\delta}{8\epsilon b_0}\Big)\Big).$$

Set $P_1^- = (x_1^-, y_\epsilon)$ and $b_1 = \kappa b_0$. The line joining P_1^- and P_0 has the slope

$$\frac{y_\epsilon - y_0}{x_1^- - x_0} = -b_1^{-1}$$

and can be described by the following two parameterizations:

$$P_1^- P_0 : \quad \begin{cases} x = x_1^- + tb_1 \\ \qquad\qquad\qquad t \in [0, 2\epsilon] \\ y = y_\epsilon - t \end{cases} \qquad \text{or} \qquad \begin{cases} x = x_0 - sb_1 \\ \qquad\qquad\qquad s \in [0, 2\epsilon] \\ y = y_0 + s \end{cases}$$

Define the function

$$g_{b_1}(t) = u(x_1^- + tb_1, y_\epsilon - t).$$

We have

$$g_{b_1}(0) = u(x_1^-, y_\epsilon) > 0, \;\; g'_{b_1}(0) = \Big(b_1 \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\Big)(x_1^-, y_\epsilon) < 0$$

By continuity $g'_{b_1}(t) < 0$ for small t. Let $t_2^- \in (0, 2\epsilon]$ such that $g_{b_1}(t_2^-) = 0$ and $g_{b_1}(t) > 0$ for $t \in [0, t_2^-)$. Note that $t_2^- > \epsilon$ since $u > 0$ in $(x_0 - \delta, x_0 + \delta) \times [y_0 + \epsilon, y_\epsilon]$. If $t_2^- = 2\epsilon$, then $u > 0$ along the line segment $P_1^- P_0 \setminus \{P_0\}$. If $t_2^- \neq 2\epsilon$, set $x_2^- = x_1^- + t_2^- b_1$; $y_2^- = y_\epsilon - t_2^-$; $Q_2^- = (x_2^-, y_2^-)$, then we form the following line segment $P_2^- P_0$ with $P_2^- = (x_2^-, y_\epsilon)$.

Line segment $P_n^- P_0$: We obtain a sequence of points $P_n^- = (x_n^-, y_\epsilon)$ and $Q_{n+1}^- = (x_{n+1}^-, y_\epsilon - t_{n+1}^-)$ satisfying:

$$0 \leqslant x_0 - x_n^- = 2\epsilon b_n \leqslant 2\epsilon b_1 = 2\epsilon b_0 \kappa \leqslant \epsilon b_0; \;\; x_{n+1}^- - x_n^- = t_{n+1}^- b_n \geqslant \epsilon b_n$$

and $x_n^- = x_0 - 2\epsilon b_n \longrightarrow x_0$ as $n \longrightarrow +\infty$.

4. Proof of Continuity

Proof of Theorem 1.2. Let x_0 be in the interior of $(0,1) \cap [\Phi(x) > 0]$. Define $\epsilon_0, \delta_0, \rho$, and the sets F and G as in Step 2 of section 2. Then, construct points $P_1^-, P_2^-, \ldots, P_n^-, \ldots$ to the left of P_0 and points $P_1, P_2, \ldots, P_n, \ldots$ to the right of P_0 (as in section 3) such that $u > 0$ above the line segments $[P_i^- Q_{i+1}^-)$ and $[P_i Q_{i+1})$; $i = 1, 2, \ldots, n, \ldots$.

For $x \in (x_0 - \dfrac{\delta}{8}, x_0 + \dfrac{\delta}{8}) \setminus \{x_0\}$, there exists $n \geqslant 1$ such that $x_{n+1} \leqslant x < x_n$ or $x_n^- < x \leqslant x_{n+1}^-$ since $x_n = x_0 + 2\epsilon b_n$ and $x_n^- = x_0 - 2\epsilon b_n$. Now, because $u(x, \Phi(x)) = 0$, the point $(x, \Phi(x))$ will be under the line segment

$$[P_n Q_{n+1}) : \quad x - x_0 = b_n(y - y_0) \qquad \text{or} \qquad [P_n^- Q_{n+1}^-) : \;\; x - x_0 = -b_n(y - y_0).$$

We discuss two situations:

i). $\Phi(x) \geqslant y_0 = \Phi(x_0)$. We have

$$\Phi(x) - \Phi(x_0) < b_n^{-1}(x - x_0) \;\text{ if }\; x_{n+1} \leqslant x < x_n$$
$$\Phi(x) - \Phi(x_0) < b_n^{-1}(x_0 - x) \;\text{ if }\; x_n^- < x \leqslant x_{n+1}^-$$

* For $x_{n+1} \leqslant x < x_n$, we have

$$\begin{aligned} 0 < x - x_0 &= (x - x_n) + (x_n - x_0) \\ &\leqslant (x_n - x_{n+1}) + (x_n - x_0) \\ &= t_n b_n + 2\epsilon b_n \\ b_n^{-1}(x - x_0) &\leqslant t_n + 2\epsilon \leqslant 2\epsilon + 2\epsilon = 4\epsilon \end{aligned}$$

* For $x_n^- < x \leqslant x_{n+1}^-$, we have

$$\begin{aligned} 0 < x_0 - x &= (x_0 - x_{n+1}^-) + (x_{n+1}^- - x) \\ &\leqslant (x_0 - x_{n+1}^-) + (x_{n+1}^- - x_n^-) \\ &= 2\epsilon b_{n+1} + t_{n+1}^- b_n \\ b_n^{-1}(x_0 - x) &\leqslant 2\epsilon b_n^{-1}.b_{n+1} + t_{n+1}^- \leqslant 2\epsilon + 2\epsilon = 4\epsilon \ \text{ since } \ b_{n+1} \leqslant b_n. \end{aligned}$$

Thus $\Phi(x) - \Phi(x_0) \leqslant 4\epsilon$.

ii). $\Phi(x) < y_0 = \Phi(x_0)$.

Set $x_0' = x$ and $y_0' = \Phi(x) = y$. The point $P_0' = (x_0', y_0')$ will play the role of (x_0, y_0) in section 3 and we will form with the same process the sequence of points $P_1^{-\prime}, P_2^{-\prime}, \dots, P_n^{-\prime}, \dots$ to the left of P_0' and points $P_1', P_2', \dots, P_n', \dots$ to the right of P_0'. It is sufficient that we start with P_1' (resp. $P_1^{-\prime}$) such that the corresponding b_1' (resp. $b_1^{-\prime}$) is less than b_0.

Line segments $P_0'P_1'$ and $P_1^{-\prime}P_0'$: We choose $P_1' = (x_1', y_\epsilon) = P_1 = (x_1, y_\epsilon)$ and $P_1^{-\prime} = (x_1^{-\prime}, y_\epsilon) = P_1^- = (x_1^-, y_\epsilon)$. This ensures that we remain working on the interval $[x_1^-, x_1]$ and x_0 is in this interval. We have

$$\begin{aligned} |x_0 - x_0'| &\leqslant 2\epsilon b_1 \ \text{ since } \ |x_0 - x_1| = |x_1^- - x_0| = 2\epsilon b_1 \\ |x_1' - x_0'| &= |x_1 - x_0 + x_0 - x_0'| \leqslant |x_1 - x_0| + |x_0 - x_0'| \leqslant 4\epsilon b_1 \ \text{ since } \ x_1 = x_0 + 2\epsilon b_1 \\ y_\epsilon - y_0' &\geqslant y_\epsilon - y_0 = 2\epsilon \ \text{ since } \ y_0' < y_0 \\ b_1' &= \frac{x_1' - x_0'}{y_\epsilon - y_0'}. \end{aligned}$$

So

$$|b_1'| = \frac{|x_1' - x_0'|}{y_\epsilon - y_0'} \leqslant \frac{4\epsilon b_1}{2\epsilon} = 2b_1 = 2\kappa b_0 < b_0 \quad \Longleftrightarrow \quad \kappa < \frac{1}{2}$$

which is satisfied by the first choice of κ. Similarly, we have

$$-b_1^{-\prime} = \frac{x_1^- - x_0'}{y_\epsilon - y_0'}; \qquad |b_1^{-\prime}| = \frac{|x_1^- - x_0'|}{y_\epsilon - y_0'} = \frac{|x_0 - 2\epsilon b_1 - x_0'|}{y_\epsilon - y_0'} \leqslant \frac{4\epsilon b_1}{2\epsilon} = 2b_1.$$

Arguing as in i), we obtain

$$\Phi(x_0) - \Phi(x) \leqslant \max\left(\frac{1}{b_q^{-\prime}}, \frac{1}{b_m'}\right)|x - x_0| \leqslant 4\epsilon,$$

depending if x_0 is to the right of x; $x_{m+1}' \leqslant x_0 < x_m'$, or x_0 is to the left of x; $x_q^{-\prime} < x_0 \leqslant x_{q+1}^{-\prime}$. Finally, we have

$$|\Phi(x) - \Phi(x_0)| \leqslant 4\epsilon \ \text{ for } \ |x - x_0| < \frac{\delta}{8}.$$

This completes the proof of the continuity of Φ.

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