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Hop Hub-Integrity of Graphs

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Abstract: The concept of hop hub-integrity is introduced as a new measure of the stability of a graph G and it is defined as $H_hI(G) = min\{|S| + m(G - S)\}$, where S is hop hub set and $m(G - S)$ is the order of a maximum component of $(G - S)$. In this paper, the hop hub-integrity of some graphs is obtained.The relations between hop hub-integrity and other parameters are determined.

MSC: 05C50, 05C99.

Keywords: Hub number, Hop hub number, Domination number, Connected hub number, Connected domination number.

1. Introduction

Let $G = (V, E)$ be a simple graph with n vertices and m edges. The symbols $\Delta(G), \delta(G), \alpha(G), \kappa(G), \lambda(G), \beta(G)$, and $\chi(G)$ denote the maximum degree, the minimum degree, the vertex cover number, the connectivity, the edge-connectivity, the independence number, and chromatic number of G, respectively. For graph theoretic terminology, we refer to [8].

In an analysis of the vulnerability of a communication network to disruption, two qualities that come to mind are the number of elements that are not functioning and the size of the largest remaining subnetwork within which mutual communication can still occur. In particular, in an adversarial relationship, it would be desirable for an opponent's network to be such that the two qualities can be made to be simultaneously small. The integrity of a graph $G = (V, E)$, which was introduced in [3] as a useful measure of the vulnerability of the graph, is defined as follows: $I(G) = min\{|S| + m(G - S) : S \subseteq V(G)\}$, where $m(G - S)$ denotes the order of the largest component. Barefoot, Entringer and Swart [4] defined the edge-integrity of a graph G with edge set $E(G)$ by $I'(G) = min\{|S| + m(G - S)\}$, $S \subseteq E(G)$. Unlike the connectivity measures, integrity shows not only the difficulty to break down the network but also the damage that has been caused. In [3] Barefoot et al. Gave some basic results on integrity. In [5] Moazzami et al. Compared the integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. To know more about integrity and edge-integrity one can see [1, 2, 4, 6]. A set $S \subseteq V(G)$ is called a dominating set of G if each vertex of $V - S$ is adjacent to at least one vertex of S. The domination number of a graph G denoted as $\gamma(G)$ is the minimum cardinality of a dominating set in G [9]. A dominating set D is a connected dominating set of G if the subgraph $< D >$ induced by D, is connected. The minimum cardinality of a connected dominating set of G is called the connected domination number of G which we denote by $\gamma_c(G)$.

Suppose that $H \subseteq V(G)$ and let $x, y \in V(G)$. An H-path between x and y is a path where all intermediate vertices are from H. (This includes the degenerate cases where the path consists of the single edge xy or a single vertex x if $x = y$, call

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such an H-path trivial). A set $H \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G) - H$, there is an H-path in G between x and y. The minimum cardinality of a hub set in G is called a hub number of G, and is denoted by $h(G)$ [24]. For more details on the hub studies we refer to [10–12].

In 2015, Sultan et al. [14] have introduced the concept of hub-integrity of a graph as a new measure of vulnerability which is defined as follows. The hub-integrity of a graph G denoted by $HI(G)$ is defined by, $HI(G) = \{min|S| + m(G - S)\}, S$ is a hub set of G, where $m(G - S)$ is the order of a maximum component of $G - S$. For more details on the hub-integrity see [15–19]. In 2021, A. S. Sand and S. S. Mahde [20], have introduced the concept of hop hub set of a graph which is defined as follows.

Definition 1.1. *A hub set S is a hop hub set of G if for every $v \in V - S$, there exists $u \in S$ such that $d(u, v) = 2$. The minimum cardinality of a hop hub set of G is called the hop hub number and is denoted by $h_h(G)$.*

This motivated us to introduce a new measure of stability of a graph G and it is called hop hub-integrity. The following results will be useful in the proof of our results

Theorem 1.2 ([24]). *For any connected graph G, $h(G) \leq \gamma_c(G) \leq h(G) + 1$.*

Theorem 1.3 ([24]). *For any graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

Theorem 1.4 ([20]). *For $n \geq 2$, $h_h(S'(P_n) = \begin{cases} 2, & \text{if } n = 2, 3, \\ n - 2, & \text{if } n \geq 4. \end{cases}$*

Theorem 1.5 ([20]). *For all $n \geq 2$, $h_h(S'(K_{1,n-1})) = 2$.*

Theorem 1.6 ([20]). *For all $n \geq 3$, $h_h(S'(C_n) = \begin{cases} 3, & \text{if } n = 3, \\ n - 2, & \text{if } n \geq 4. \end{cases}$*

Theorem 1.7 ([24]). *Let T be a tree with n vertices and l levels, Then $h(T) = n - l$.*

Theorem 1.8 ([20]). *For all $n, m \geq 2$, $h_h(S'(S_{n,m})) = 2$.*

Theorem 1.9 ([14]). *For any graph G, $\gamma(G) \leq HI(G)$.*

Theorem 1.10 ([13]). *If T is a binary tree order n with l terminal vertices, then T has $l - 1$ internal vertices.*

2. Main Results

Definition 2.1 ([20]). *A hub set S is a hop hub set of G if for every $v \in V - S$, there exists $u \in S$ such that $d(u, v) = 2$. The minimum cardinality of a hop hub set of G is called a hop hub number and is denoted by $h_h(G)$.*

Definition 2.2. *The hop hub-integrity of a graph G is denoted as $H_hI(G) = min\{|S| + m(G - S),\ S$ is a hop hub set$\}$, where $m(G - S)$ is the order of a maximum component of $G - S$.*

A H_hI-set of G is any subset S of $V(G)$ for which $H_hI(G) = min\{|S| + m(G - S)\}$. For any disconnected graph G having k components $G_1, G_2, ..., G_k$ of orders $p_1, p_2, ..., p_{k-1}, p_k$, respectively such that $p_1 \leq p_2 \leq ... \leq p_{k-1} \leq p_k$. We have $H_hI(G) = p_1 + p_2 + ... + p_{k-1} + H_hI(G_k)$. Also, by the definition of hop hub-integrity we obtain the obvious bound $H_hI(G) \geq HI(G) \geq I(G)$.

Proposition 2.3. *The hop hub-integrity of some specific classes of graphs are as below*

(1). For any complete graph K_n, $H_hI(K_n) = n$.

(2). For any path P_n with $n \geq 4$, $H_hI(P_n) = n - 1$.

(3). For the wheel graph $W_{1,n-1}$, $H_hI(W_{1,n-1}) = \lceil 2\sqrt{n-1} \rceil$.

(4). For the complete bipartite graph $K_{n,m}$, $H_hI(K_{n,m}) = 2 + min\{n-1, m-1\}$.

(5). For the double star $S_{n,m}$, $H_hI(S_{n,m}) = 3$.

(6). For any cycle C_n,

$$H_hI(C_n) = \begin{cases} n, & \text{if } n = 3, 4. \\ n-1, & \text{if } n \geq 5. \end{cases}$$

Remark 2.4. *In general, the inequality $H_hI(G') \leq H_hI(G)$ is not true for a subgraph G' of G, for the graph G and a subgraph G' shown in Figure 1, we have $H_hI(G) = 4$, while $H_hI(G') = 5$.*

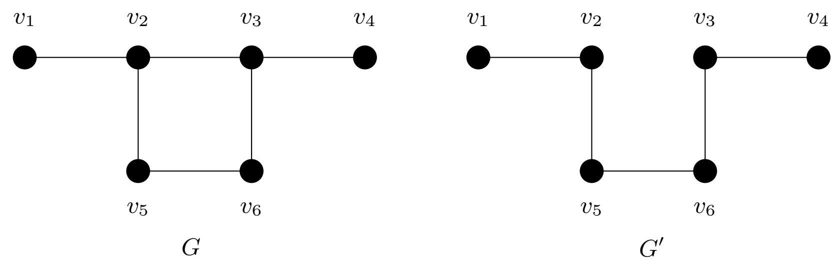


Figure 1

Proposition 2.5. *For any counted graph G, $2 \leq H_hI(G) \leq n$.*

The lower bound attains for K_2 and the upper bound attains for a complete graph K_n, $n \geq 2$.

Theorem 2.6. *Let T be a tree with n vertices and l terminals vertices, such that internal vertices $p \geq 2$. Then $H_hI(G) = n - l + 1$.*

Proof. Let $H_hI(T) = |S| + m(S - T)$. The set $n - l$ of all internals vertices in T forms a hop hub set, since the unique path between any two terminals never passes through another terminal. Note that any proper subset of $n - l$ cannot be a hop hub set. So $|S| = n - l$, since every internal vertex is a cut-vertex. If we delete of all $n - l$ vertices, we get one competent or more than two competent of order 1. So, $H_hI(T) = |S| + m(T - S) = n - l + 1$. □

Theorem 2.7. *For any tree T, $H_hI(T) \geq \alpha(T) + 1$.*

Proof. Let S' be a minimum covering set of T. Then

$$\begin{aligned} H_hI(T) &= |S| + m(T - S) \\ &\geq |S'| + m(T - S')(Because \quad S \geq S') \\ &\geq |S'| + 1 \\ &\doteq \alpha(T) + 1. \end{aligned}$$

□

Corollary 2.8. *For any graph G, $H_hI(G) \geq \gamma(G) + 1$.*

Proof. By using Theorem 1.9 and from $H_hI(G) \geq HI(G)$ we get the result. □

Corollary 2.9. *For any graph G, $H_hI(G) \geq h_h(G)$, and if G is complete then is equality .*

Definition 2.10 ([13]). *A tree is called a binary tree if it has one vertex of degree 2 and each of the remaining vertices of degree 1 or 3. Clearly, P_3 is the smallest binary tree .*

Theorem 2.11. *If a tree T is a binary tree of order n. Then $H_hI(G) = \lceil n/2 \rceil$.*

Proof. Let $H_hI(T) = |S| + m(T - S)$. Since the hop hub set in any binary tree is p internal vertices, by Theorem 1.10, $|S| = p = l - 1$, where l is the set of its number of terminal vertices of T . If we remove $l - 1$ internal vertices from binary tree T we get a totally disconnected graph. So, $m(T - S) = 1$. Therefore, $H_hI(T) = l - 1 + 1 = l$. Since the number of terminal vertices in any binary tree equal $\lceil n/2 \rceil$, it follows that $l = \lceil n/2 \rceil$, Therefore $H_hI(T) = l = \lceil n/2 \rceil$. □

Theorem 2.12. *Let $G \cong K_n - e$, $e \in E(G)$. Then $H_hI(\overline{G}) = n$.*

Proof. If $G \cong K_n - e$, then $\overline{G} \cong K_2 \cup (n-2)K_1$. By definition of hop hub-integrity of disconnected graph, we have

$$\begin{aligned} H_hI(\overline{G}) &= n - 2 + H_hI(K_2) \\ &= n - 2 + 2 \\ &= n. \end{aligned}$$

□

Corollary 2.13. *Let $G \cong K_n - e$, $e \in E(G)$. Then $H_hI(\overline{G}) = H_hI(G) + 1$. Therefore, $H_hI(\overline{G}) - H_hI(G) = 1$.*

2.1. Some properties of hop hub-integrity of line graphs

Definition 2.14 ([8]). *The line graph $L(G)$ of G has the edges of G as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G.*

Proposition 2.15.

- *In the star $K_{1,n-1}$, $H_hI(L(K_{1,n-1}) = n - 1$.*
- *In the cycle C_n, $H_hI(L(C_n)) = H_hI(C_n)$.*
- *In the path P_n, $n \geq 4, H_hI(L(P_n) = H_hI(P_{n-1})$.*
- *In the double star $S_{n,m}$, $n, m \geq 2$, $H_hI(L(S_{n,m}) = 3$.*

Remark 2.16. *The hop hub-integrity of a graph G and hop hub-integrity of line graph are not comparable. For this situation consider the graphs in the following cases:*

- *In the star $K_{1,n-1}$, $H_hI(L(K_{1,n-1})) > H_hI(K_{1,n-1})$.*
- *In the cycle C_n, $H_hI(L(C_n)) = H_hI(C_n)$.*
- *In the path P_n, $n \geq 4, H_hI(L(P_n)) < H_hI(P_n)$.*

Proposition 2.17. *For any path P_n, $n \geq 5$, $H_hI(L(P_n)) + H_hI(\overline{L(P_n)}) = 2n - 4$.*

Theorem 2.18. *Let $G \cong K_n - e$, $e \in E(G)$. Then $H_hI(L(\overline{G})) = 1$.*

Proof. Since $G \cong K_n - e$, then $\overline{G} \cong K_2 \cup (n-2)K_1$, and $L(\overline{G}) \cong K_1$. Thus $H_hI(L(\overline{G})) = 1$. □

Proposition 2.19. *If G is regular graph of degree 2, then $H_hI(G) = H_hI(L(G))$.*

Proof. G is regular of degree 2, hence $G \cong C_n$, and $H_hI(C_n) = H_hI(L(C_n))$, so the result. □

Corollary 2.20. *Let G be a connected graph and let $\alpha(G) = 1$, Then $H_hI(L(G)) = n$.*

Proof. Suppose $\alpha(G) = 1$, then $G \cong K_{1,n-1}$. Then $L(G) = K_n$, so proof follows from Proposition 2.3. □

Proposition 2.21. *If $H_hI(L(G)) = |E(G)|$, then $G \cong K_{1,n-1}$ or $G \cong C_3$.*

Theorem 2.22. *For any subset D of vertices in a graph $L(G)$, $H_hI(L(G) - D) \geq H_hI(L(G)) - |D|$.*

Proof. Let S be a H_hI- set of $L(G)$, let $D \leq V(L(G))$ and S^* be a H_hI-set of $L(G) - D$ such that $S^{**} = S^* \cup D$. Then $|S^{**}| = |S^*| + |D|$ and $L(G) - S^{**} = L(G) - (S^* \cup D) = (L(G) - D) - S^*$. Therefore

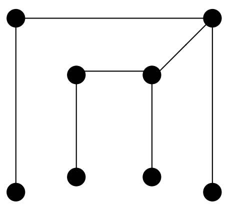
$$\begin{aligned} H_hI(L(G)) &= |S| + m(L(G) - S) \\ &\leq |S^{**}| + m(L(G) - S^{**}) \\ &= |S^*| + |D| + m[(L(G) - D) - S^*] \\ &= HI(L(G) - D) + |D|. \end{aligned}$$

□

Corollary 2.23. $H_hI(L(K_{1,n-1})) + H_hI(\overline{L(K_{1,n-1})}) = 2n - 2.$

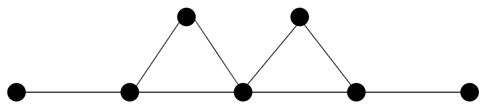
Proof. Since $L(K_{1,n-1}) \cong K_{n-1}$, it follows from Proposition 2.3 that $H_hI(K_{n-1}) = n - 1$, and $\overline{L(K_{1,n-1})} \cong \overline{K_{n-1}}$, so $H_h\overline{I(K_{n-1})} = n - 1$, hence the result. □

Remark 2.24. *If G is a connected graph, and $|E(L(G))| < |E(G)|$, then $H_hI(L(G)) < H_hI(G)$. We note that $|E(L(G))| < |E(G)|$ obtained only in a path graph, hence the result. But the converse is not true, for example, the graphs shown in Figure 2, and Figure 3.*



G

Figure 2



$L(G)$

Figure 3

$H_hI(G) = 5$ and $H_hI(L(G)) = 4$, while $|E(L(G))| > |E(G)|$.

2.2. Hop hub-integrity of splitting graph

Vaidya and Kothari [22] have discussed domination integrity of a graph obtained by duplication of an edge by vertex and duplication of vertex by an edge in path and cycle. Also Vaidya and Kothari [23] have discussed domination integrity of splitting graph of path and cycle. Sultan and Veena [16] have discussed hub-integrity of splitting graph of some standard graphs. In the present work, we investigate hop hub-integrity of splitting graphs in some standard graphs.

Definition 2.25 ([23]). *For a graph G, the splitting graph $S'(G)$ of graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$ where $N(v)$ and $N(v')$ are the neighborhood sets of v and v', respectively.*

Theorem 2.26. *For $n \geq 2$, $H_hI(S'(P_n)) = \begin{cases} 3, & \text{if } n = 2, \\ 4, & \text{if } n = 3, \\ n, & \text{if } n \geq 4. \end{cases}$*

Proof. Let $\{u_1, u_2, ..., u_n\}$ be the vertices of path P_n and $\{v_1, v_2, ..., v_n\}$ be the new vertices corresponding to $\{u_1, u_2, ..., u_n\}$ which are added to obtain $S'(P_n)$ as shown in Figure 4. We have the following cases:

Case 1: $n = 2$. From Theorem 1.4, we have $h_h(S'(P_2)) = 2$ and $H = \{u_1, u_2\}$ is a hop hub set of $S'(P_2)$. Then $m(S'(P_2) - H) = 1$. This implies that $H_hI(S'(P_2)) = h_h(S'(P_2)) + m(S'(P_2) - H) = 2 + 1 = 3$. Clearly there does not exist any hop hub set S_1 of $S'(P_2)$ such that $|S_1| + m(S'(P_2) - S_1) \leq h_h(S'(P_2)) + m(S'(P_2) - H)$. Hence, $H_hI(S'(P_2)) = 3$.

Case 2: $n = 3$. From Theorem 1.4, we have $h_h(S'(P_3)) = 2$ and $H = \{u_1, u_2\}$ is a hop hub-set of $S'(P_3)$. Then $m(S'(P_3) - H) = 2$. This implies that $H_hI(S'(P_3)) = h_h(S'(P_3)) + m(S'(P_3) - H) = 2 + 2 = 4$. Moreover, for any hop hub set S of $S'(P_3)$ we have, $|S| + m(S'(P_3) - S) \geq |H| + m(S'(P_3) - H)$. Hence $H_hI(S'(P_3)) = 4$.

Case 3: $n \geq 4$. From Theorem 1.4, we have $h_h(S'(P_n)) = n - 2$. Let $H = \{u_2, u_3, ..., u_{n-1}\}$ be a hop hub-set of graph $S'(P_n)$. Then $m(S'(P_n) - H) = 2$. Therefore,

$$H_hI(S'(P_n)) \leq h_h(S'(P_n)) + m(S'(P_n) - H) = n - 2 + 2 = n. \tag{1}$$

For showing that the number $|H| + m(S'(P_n) - H)$ is minimum. The minimality of both $|H|$ and $m(S'(P_n) - H)$ is taken into consideration. The minimality of $|H|$ is guaranteed as H is hop hub-set. It remains to show that if S is any hop hub set other then H, $|S| + m(S'(P_n) - S) \geq n$. If $m(S'(P_n) - S) = 1$, then $|S| \geq n > n - 1$, consequently $|S| + m(S'(P_n) - S) \geq n + 1$. If $m(S'(P_n) - S) \geq 2$, then trivially $|S| + m(S'(P_n) - S) \geq n$. Hence for any hop hub set S,

$$|S| + m(S'(P_n) - S) \geq n. \tag{2}$$

From (1) and (2), $H_hI(S'(P_n)) = n$.

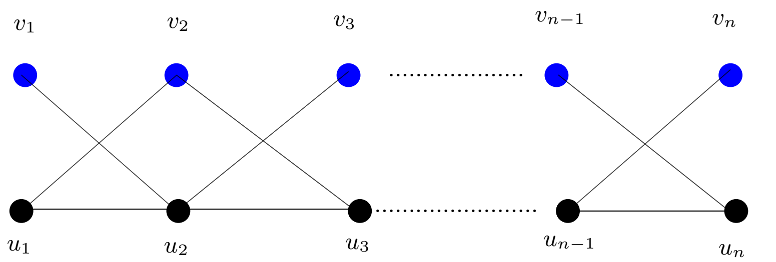


Figure 4 :Splitting graph of P_n

□

Theorem 2.27. *For all $n \geq 3$, $H_hI(S'(C_n)) = \begin{cases} 4, & \text{if } n = 3, \\ n+1, & \text{if } n \geq 4. \end{cases}$*

Proof. Let $\{u_1, u_2, ..., u_n\}$ be the vertices of cycle C_n and $\{v_1, v_2, ..., v_n\}$ be the new vertices corresponding to$\{u_1, u_2, ..., u_n\}$ which are added to obtain $S'(C_n)$ as shown in Figure 5. We have the three following cases:

Case 1: For $n = 3$. From Theorem 1.6, we have $h_h(S'(C_3)) = 3$, and $H = \{u_1, u_2, u_3\}$ is a hop hub-set of $S'(C_3)$. Then $m(S'(C_3) - H) = 1$.

This implies that $H_hI(S'(C_3)) = h_h(S'(C_3)) + m(S'(C_3) - H) = 3 + 1 = 4$. Clearly there does not exist any hop hub set S_1 of $S'(C_3)$ such that $|S_1| + m(S'(C_3) - S_1) \leq h_h(S'(C_3)) + m(S'(C_3) - H)$. Hence, $H_hI(S'(C_3)) = 4$.

Case 2: $n \geq 4$. From Theorem 1.6, we have $h_h(S'(C_n)) = n-2$ and $H = \{u_1, u_2, ..., u_{n-2}\}$ is a hop hub-set of $S'(C_n)$. Then $m(S'(C_n) - H) = 6$. Therefore

$$H_hI(S'(C_n)) \leq h_h(S'(C_n)) + m(S'(C_n) - H) = n - 2 + 6 = n + 4. \tag{3}$$

If S_1 is any hop hub set of $S'(C_n)$ other than H with $m(S'(C_n) - S_1) = 4$ or 5, then $|S_1| \geq h_h(S'(C_n)) = n-2$. This implies that

$$|S_1| + m(S'(C_n) - S_1) \geq h_h(S'(C_n)) + 4 = n - 2 + 4 = n + 2. \tag{4}$$

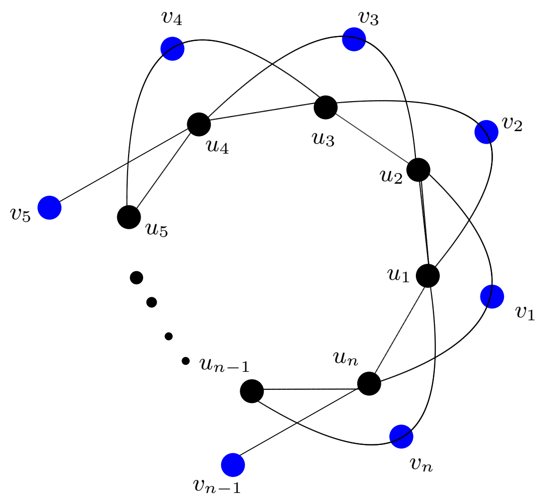
If S_2 is any hop hub set of $S'(C_n)$ other than H with $m(S'(C_n) - S_2) = 2$ or 3, then $|S_2| \geq n - 1$. This implies that

$$|S_2| + m(S'(C_n) - S_2) \geq n - 1 + 3 = n + 2. \tag{5}$$

Let $S_3 = \{u_1, u_2, ..., u_n\}$, a hop hub set of $S'(C_n)$, then $m(S'(C_n) - S_3) = 1$. This implies that

$$|S_3| + m(S'(C_n) - S_3) = n + 1. \tag{6}$$

Hence from (3), (4), (5) and (6), $H_hI(S'(C_n)) = n + 1$.



□

Figure 5 :$S'(C_n)$

Theorem 2.28. *For all $n \geq 4$, $H_hI(S'(K_{1,n-1})) = 4$.*

Proof. Let $\{u, u_1, ..., u_{n-1}\}$ be the vertices of star $K_{1,n-1}$ and $\{v, v_1, ..., v_{n-1}\}$ be the new vertices corresponding to $\{u, u_1, ..., u_{n-1}\}$ which are added to obtain $S'(K_{1,n-1})$ as shown in Figure 7. From Theorem 1.5, we have $h_h(S'(K_{1,n-1})) = 2$ and $m(S'(K_{1,n-1}) - H) = n$, then

$$H_hI(S'(K_{1,n-1})) \leq h_h(S'K_{1,n-1}) + m(S'(K_{1,n-1}) - H) = n + 2. \tag{7}$$

If $H = \{u, v, u_1\}$ is a hop hub-set of $S'(K_{1,n-1})$. Then $m(S'(K_{1,n-1}) - H) = 1$. Therefore,

$$H_hI(S'(K_{1,n-1})) = |H| + m(S'(K_{1,n-1}) - H) = 3 + 1 = 4. \tag{8}$$

To show that the number $|H| + m(S'(K_{1,n-1}) - H)$ is minimum, it is assumed that S is any hop hub set other than H and $m(S'(K_{1,n-1}) - S) > 1$, and $|S| \geq 3$, then $|S| + m(S'(K_{1,n-1}) - S) > 1 + 3 = 4$. Hence for any hop hub set S,

$$|S| + m(S'(K_{1,n-1}) - S) > h_h(S'(K_{1,n-1})) + 1. \tag{9}$$

From (8) and (9), we have $H_hI(S'(K_{1,n-1})) = 4$.

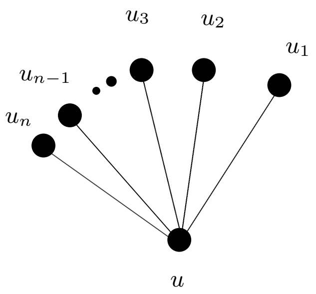


Figure 6 :$K_{1,n-1}$

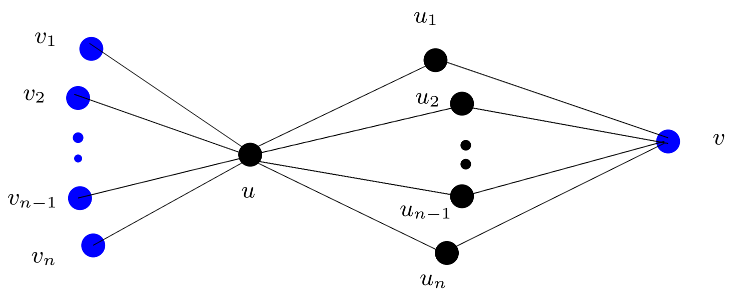


Figure 7 :$S'(K_{1,n-1})$

□

Theorem 2.29. *For all $n, m \geq 2$, $H_hI(S'(S_{n,m})) = 5$.*

Proof. Let $\{u, u_1, u_2, ..., u_{n-1}, v, v_1, v_2, ..., v_{m-1}\}$ be the vertex set of double star $S_{n,m}$ and $\{u', u'_1, u'_2, ..., u'_{n-1}, v', v'_1, v'_2, ..., v'_{m-1}\}$ be the new vertices corresponding to $\{u, u_1, u_2, ..., u_{n-1}, v, v_1, v_2, ..., v_{m-1}\}$ which are added to obtain $S'(S_{n,m})$ as shown in Figure 9. Consider $S = \{u, v\}$, a hop hub set of $S'(S_{n,m})$.

Case 1: $n = m = 2$. From Theorem 1.8, we have $h_h(S'(S_{2,2})) = 2$ and $S = \{u, v\}$ is a hop hub-set of $S'(S_{2,2})$. Then $m(S'(S_{2,2}) - S) = 3$. Therefore

$$H_hI(S'(S_{2,2})) \leq h_h(S'(S_{2,2})) + m(S'(S_{2,2}) - S) = 5. \tag{10}$$

Consider S_1 is any hop hub set of $S'(S_{2,2})$ other than S with $m(S'(S_{2,2}) - S_1) = 2$, then $|S_1| \geq 4$. This implies that

$$|S_1| + m(S'(S_{2,2}) - S_1) \geq 2 + 4 = 6. \tag{11}$$

Let $S_2 = \{u, v, u', v'\}$ be a hop hub set of $S'(S_{2,2})$, then $m(S'(S_{2,2}) - S_2) = 1$. This implies that

$$|S_2| + m(S'(S_{2,2}) - S_2) = 4 + 1 = 5. \tag{12}$$

Hence from (10), (11) and (12), $H_hI(S'(S_{2,2})) = 5$.

Case 2: $n \geq 2, m > 2$ or $n > 2, m \geq 2$.

From Theorem 1.8, $h_h(S'(S_{n,m})) = 2$, and $S = \{u, v\}$ is a hop-set of $S'(S_{n,m})$. Then $m(S'(S_{n,m}) - S) = max\{n+1, m+1\}$. Therefore

$$H_hI(S'(S_{n,m})) \leq h_h(S'(S_{n,m})) + m(S'(S_{n,m}) - S) = 2 + max\{n + 1, r + 1\}. \tag{13}$$

Consider $S_1 = \{u, v, u', v'\}$ a hop hub set of $S'(S_{n,m})$, then $m(S'(S_{n,m}) - S_1) = 1$. This implies that

$$|S_1| + m(S'(S_{n,m}) - S_1) = 4 + 1 = 5. \tag{14}$$

We claim that S_1 is a minimum hop hub set. Since u is adjacent to $\{v, v', u_1, ..., u_n, u'_1, ..., u'_n\}$, and removal of u from S_1 leads to nonexistence of S_1-path between u_i and u'_i, it follows that S_1 is a minimum hop hub set. Hence from (13) and (14), $H_hI(S'(S_{n,m})) = 5$.

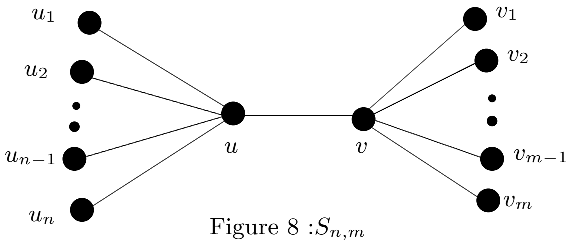


Figure 8 :$S_{n,m}$

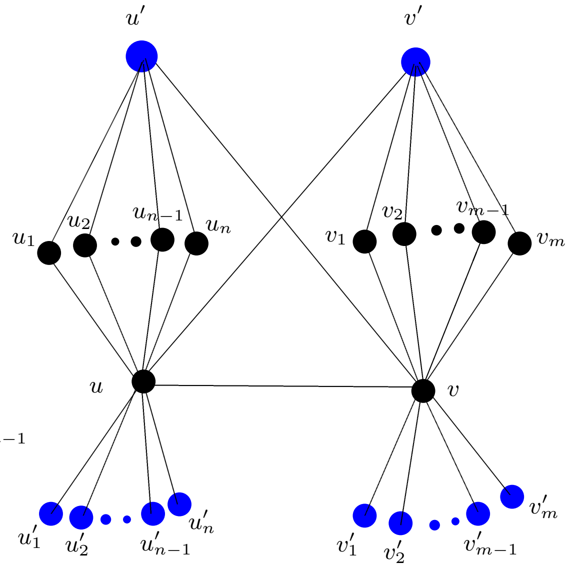


Figure 9 :$S'(S_{n,m})$

□

Theorem 2.30. *For any wheel* $W_{1,n-1}$, $H_hI(S'(W_{1,n-1})) = n+1$.

Proof. Since $S'(W_{1,n-1})$ contains a wheel graph $W_{1,n-1}$ as its subgraph. If we choose the set S as all vertices of $W_{1,n-1}$ of $S'(W_{1,n-1})$, then there exist n components each contains only one vertex. So $H_hI(S'(W_{1,n-1})) = n+1$. □

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