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Sequential Approach Towards The Optimal Solution of Transportation Problem

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Abstract: In the course of time several methods and algorithms has been developed to solve transportation problems for more specific variations of its formulation. These approaches do not always find the true optimal solution. Instead, they will often consistently find good solutions to the problems. These good solutions are typically considered to be good enough simply because they are the best that can be found in a reasonable amount of time. Therefore, optimization often takes the role of finding the best solution possible in a reasonable amount of time. The proposed sequential approach is studied using modified Egerváry Theorem with numerical examples and comparative study on its algorithmic complexity. This methods gives a true optimal solution to the transportation problem with reasonable short time.

MSC: 90B06.

Keywords: Algorithmic complexity, Favorable matching, Perfect matching.

1. Introduction

The transportation theory is a branch of optimization in operation research to deal the study of optimal transportation and allocation of resources in a transportation network. In 1941, Hitchcock originally developed the basic transportation problem. In 1947, Koopmans independently study on the optimum utilization of transportation system. Subsequently the linear programming formulation and the associated systematic procedure for solution were given by Dantzig in 1951. Application of graph theory is one of the classical approach to obtain a perfect matching from a connected graph. Mohanta and Das [8], studied the optimal solution of assignment problem in the environment of graph theory by modifying the Egerváry Theorem. The same logical approach is being extended to the transportation network. This paper is being developed on the basis of the key idea studied by Mohanta [7] to obtained an optimal solution of transportation problem. In this paper we study the transportation theory on the basis of algorithmic graph theory though a logical sequence for the allocation of resources.

In section 2, we recall some basic information on favorable matching and perfect matching. Section 3 deals with sequential algorithmic approach in the environment connected bipartite graph. In section 4, we discuss proposed method with examples. In section 5, we study the algorithmic complexity of the proposed method. Section 6 deals with the result analysis and comparison.

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2. Preliminaries

The main objective of transportation problem is to transport a single homogeneous commodity that are initially stored in different sources to different destination as per their requirements in such a way that the transportation cost will be minimum. Application of graph theory can be modelled into a transportation network from a connected bipartite graph. From Jackson [2], we recited some of these results. Let G be a graph with a set of vertices $V(G)$ and a set of edges $E(G)$.

Definition 2.1 ([2]). *Let G be a graph and $M \subseteq E(G)$. Then M is matching in G if no two edges of M have a common end-vertex. we say that M is a maximum matching if it has maximum cardinality over all matching in G. A vertex $v \in V(G)$ is M-saturated if v is incident with an edge of M. we say that M is a perfect matching in G if every vertex of G is M-saturated. Thus, if M is a perfect matching, then $|M| = \frac{1}{2}|V(G)|$ and M is necessarily a maximum matching.*

Definition 2.2 ([2]). *Let G be a graph and $U \subseteq V(G)$. we say that U is a cover of G if every edges of G is incident with a vertex in U.*

Definition 2.3 ([2]). *The complete bipartite graph $K_{m;\ m}$ is the bipartite graph with bi-partition $\{X;\ Y\}$ where $|X| = m,\ \ |Y| = n$ and each vertex of X is adjacent to every vertex of Y.*

Definition 2.4 ([2]). *Let N be a network obtained from $K_{m;\ m}$ with partition $\{X;\ Y\}$ such that $|X| = m,\ \ |Y| = m$, $V(N) = X \cup Y$ and M be a perfect matching for N. Define $f:\ V(N) \to \mathbb{Z}$ such that $f(v)$ equal to minimum weight($w > 0$) of an edge incident on v and $w(xy) \geq f(x)$ for each $x \in X$ where xy is the edge in M. We define the depth of X by f_X such that $f_X = \sum\limits_{v\in X} f(v)$.*

Definition 2.5 ([2]). *If $f_X \geq f_Y$; then the set of vertices X is said to be favorable in N where f_X is the depth of X.*

Lemma 2.6 ([2]). *Let X be a favorable vertex matching and M be a perfect matching for N. Then $w(M) \geq f_X$.*

Definition 2.7 ([2]). *A favorable vertex matching X of N is said to be optimal; if the equality sub-graph G_X for f in N is the spanning sub-graph of N containing all edges for which $f_x = w(x)$ for each $x \in X$ where M be a perfect matching in N.*

Lemma 2.8 ([2]). *Let X be a optimal favorable vertex matching and M be a perfect matching for N in the equality sub-graph $G(f)$. Then $w(M) = f_X$ and X is a maximum size favorable vertex matching of N.*

Theorem 2.9 ([2]). *Let N be a weighted complete bipartite graph. Then the maximum weight of a perfect matching in N is equal to the minimum size of a feasible vertex labeling of N.*

Mohanta and Das [8], mainly focus to generate a set of edges (favorable matching) that will minimize the total weight of the network through a logical approach to obtain an optimal solution by modifying the classical Egerváry Theorem and extend it to the study of algorithm complexity of the proposed logical method. We revisited some of their results that will meet our requirement. Let N be a network obtained from $K_{m;\ m}$ with partition $\{X;\ Y\}$ such that $|X| = m,\ \ |Y| = m$, $V(N) = X \cup Y$ and M be a perfect matching for N. In the network N each vertex $x \in X$ is adjacent to all vertex $y \in Y$. For instance, let a vertex $x_1 \in X$ has m edges such as $x_1y_1,\ x_1y_2,\ \dots\ x_1y_m$ with each edge has an integer weight $w_{11},\ w_{12},\ \dots\ w_{1m}$ respectively. Let the weight of vertex x_1 is $w_1 = w(x_1) = \sum\limits_{1}^{m} w_{1j}$ is the total weight of all the edges incident on the vertex x_1 and the weight represent a single homogeneous components like distance or cost or time.

Theorem 2.10 ([8]). *Let N be a weighted complete bipartite graph, the maximum weight of a perfect matching is $w(M)$. Then using proposed method to obtained an optimal favorable matching in N is S_X and $w(S_X) = w(M)$.*

Lemma 2.11 ([8]). *Let N be a transportation network obtained from $K_{m;\ m}$ with partition $\{X;\ Y\}$ such that $|X| = m,\ |Y| = m$, $V(N) = X \cup Y$ and each edge have an integer value c_{ij} and x_{ij} represent transportation cost per unit and amount of goods to be transport from x_i to y_j respectively. The capacity of vertices in X and Y represent by a_i and b_j respectively. Suppose we use the proposed sequential approach to construct a favorable matching in N. Then the number of times the method grows an alternating favorable matching is at most $\frac{1}{2}|V(N)|$, where $|V(N)|$ is cardinality of vertex set in N.*

3. Sequential Approach and Computation Procedure

Let N be a transportation network obtained from $K_{m;\ m}$ with partition $\{X;\ Y\}$ such that $|X| = m,\ |Y| = m$, $V(N) = X \cup Y$ and each edge have an integer value c_{ij} and x_{ij} represent transportation cost or time or distance per unit and amount of goods to be transport from x_i to y_j respectively. The capacity of vertices in X and Y represent by a_i and b_j respectively. Let M be a perfect matching of N obtained by MODI method. Now we define some of the following results to meet our requirements.

Definition 3.1. *Let N be a transportation network obtained from $K_{m;\ m}$ with partition $\{X;\ Y\}$ such that $|X| = m,\ |Y| = m$, $V(N) = X \cup Y$ and each edge have an integer value c_{ij} and x_{ij} represent transportation cost per unit and amount of goods to be transport from x_i to y_j respectively. The capacity of vertices in X and Y represent by a_i and b_j respectively. Let M be a perfect matching for N. Define $f:\ V(N) \to \mathbb{Z}$ such that*

$$f(x_i) = \begin{cases} min.(c_{ij}) \cdot x_{ij}\ , & a_i \leq b_j\ ; \\ \sum_{i=1}^{m} \eta_i \cdot \xi_i\ , & a_i > b_j\ . \end{cases} \tag{1}$$

where $x_{ij} = min.(a_i,\ b_j)$, $\eta_i = min./next\ min.(c_{ij})$ such that $\eta_1 \leq \eta_2\ \ldots\ \leq \eta_m$ and $\xi_1 = min.(a_i,\ b_j)$, $\xi_2 = min.(a_i - b_j,\ b_s)\ldots$ such that $1 \leq s \leq m$ and $j \neq s$.

Now $w(x_i) \geq f(x_i)$ for each $x_i \in X$ where $w(x_i) = c_{ij} \cdot x_{ij}$ is the weight of x_i and $x_{ij} \in M$. We define the depth of X by f_X such that $f_X = \sum_{x \in X} f(x)$. The tie between two minimum cost of a vertex can be settle as; if in i^{th} vertex there is a tie on minimum cost i.e., $c_{ij} = c_{i(j+1)}$ then preference should be given to that cost having more total opportunity cost of that cost i.e., $C(y_j) > C(y_{j+1})$; where $C(y_j) = \sum_{i=1}^{m} c_{ij}$ is the total opportunity cost for the cost c_{ij}.

Example 3.2.

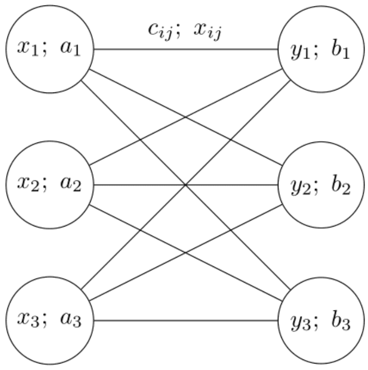

Figure 1. **Network representation of $K_{3;\ 3}$**

Let N is a network obtained from $K_{3;\ 3}$ "Figure: 1" with bi-partition $X = \{x_1,\ x_2,\ x_3\}$ and $Y = \{y_1,\ y_2,\ y_3\}$ and giving each edge have an integer value c_{ij} and x_{ij} represent transportation cost per unit and amount of goods to be transport from x_i to y_j respectively. The following matrix is a opportunity cost matrix,

Sources	Destinations			Supply	f_x
	y_1	y_2	y_3		
x_1	6	4	1	50	95
x_2	3	8	7	40	200
x_3	4	4	2	60	170
Demand	20	95	35		
f_y	60	380	35		

Table 1. **Transportation table**

Now the following

$$\begin{aligned}
f(x_1) &= \sum_{i=1}^{3} \eta_i \cdot \xi_i = c_{13} \cdot x_{13} + c_{12} \cdot x_{12} = (1 \times 35) + (4 \times 15) = 95;\\
f(x_2) &= \sum_{i=1}^{3} \eta_i \cdot \xi_i = c_{21} \cdot x_{21} + c_{23} \cdot x_{23} = (3 \times 20) + (7 \times 20) = 200;\\
f(x_3) &= \sum_{i=1}^{3} \eta_i \cdot \xi_i = c_{33} \cdot x_{33} + c_{32} \cdot x_{32} = (2 \times 35) + (4 \times 25) = 170;\\
\therefore \quad f_X &= 95 + 200 + 170 = 465.\\
f(y_1) &= min.(c_{i1}) \cdot x_{i1} = c_{21} \cdot x_{21} = 3 \times 20 = 60;\\
f(y_2) &= \sum_{i=1}^{3} \eta_i \xi_i = c_{12} \cdot x_{12} + c_{32} \cdot x_{32} = (4 \times 50) + (4 \times 45) = 380;\\
f(y_3) &= min.(c_{i3}) \cdot x_{i3} = c_{13} \cdot x_{13} = 1 \times 35 = 35;\\
\therefore \quad f_Y &= 60 + 380 + 35 = 475.
\end{aligned}$$

Let $M = \{x_{12},\ x_{13},\ x_{21},\ x_{22},\ x_{32}\}$ *be a perfect matching by MODI method with maximum weight in* N *is*

$$\begin{aligned}
w(\mathrm{M}) &= c_{12} \cdot x_{12} + c_{13} \cdot x_{13} + c_{21} \cdot x_{21} + c_{22} \cdot x_{22} + c_{32} \cdot x_{32},\\
&= (4 \times 15) + (1 \times 35) + (3 \times 20) + (8 \times 20) + (4 \times 60) = 555.
\end{aligned}$$

clearly $f_Y > f_X$, *so the set of vertex* Y *is favorable for matching in* N.

Definition 3.3. *Let* N *be a transportation network obtained from a complete bipartite graph* $K_{m;\ m}$ *with bi-partition* $\{X;\ Y\}$ *such that* $|X| = m,\ |Y| = m$, $V(N) = X \cup Y$ *and* M *be a perfect matching for* N. *Define* E_X *be the set of edges generates by* f *over* X *is*

$$\mathrm{E_X} = \{\xi_i :\ \xi_i \text{ is an edge or edges associated with } f(x_i), 1 \leq i \leq m\} \tag{2}$$

such that

$$f_X = \sum_{v \in X} f(v) = \sum_{\xi \in \mathrm{E_X}} w(\xi) = w(\mathrm{E_X}). \tag{3}$$

and similarly E_Y *be the set of edges generates by* f *over* Y *is*

$$\mathrm{E}_Y = \{\xi_j :\ \xi_j \text{ is an edge or edges associated with } f(y_j), 1 \leq j \leq m\} \tag{4}$$

such that

$$f_Y = \sum_{v \in Y} f(v) = \sum_{\xi \in \mathrm{E_Y}} w(\xi) = w(\mathrm{E_Y}). \tag{5}$$

Where

$$\xi_i = \begin{cases} \xi_1 = min.(a_i,\ b_j), & a_i \leq b_j;\\ \xi_1 = min.(a_i,\ b_j),\ \xi_2 = min.(a_i - b_j,\ b_s)\ \dots\ s.\ t.\ 1 \leq s \leq m \text{ and } j \neq s, & a_i > b_j. \end{cases} \tag{6}$$

by setting for a vertex x_i.

Definition 3.4. *Let N be a network obtained from a complete bipartite graph $K_{m;\ m}$ with bi-partition $\{X;\ Y\}$ such that $|X| = m,\ \ |Y| = m$, $V(N) = X \cup Y$ and $f : V(N) \to \mathbb{Z}$ such that*

$$f(x_i) = \begin{cases} min.(c_{ij}) \cdot x_{ij}\ , & a_i \le b_j\ ; \\ \sum_{i=1}^{m} \eta_i \xi_i\ , & a_i > b_j\ . \end{cases} \tag{7}$$

where $x_{ij} = \min.(a_i,\ b_j)$, $\eta_i = min./next\ min.(c_{ij})$ such that $\eta_1 \le \eta_2\ \dots\ \le \eta_m$ and $\xi_1 = min.(a_i,\ b_j)$, $\xi_2 = min.(a_i - b_j,\ b_s)\ \dots$ such that $1 \le s \le m$ and $j \ne s$. The depth of X by f_X such that $f_X = \sum_{v \in X} f(v)$. The function f generates a favorable matching E_X be a set of edges obtained from a set of favorable vertices X and M be a perfect matching for N. Then $w(M) \ge w(E_X)$; where $w(E_X) = f_X$.

Definition 3.5. *A favorable matching S of N is said to be optimal; if the equality sub-graph G_S for f in N is the spanning sub-graph of N containing all edges ϵ for which $w(\epsilon) = w(e)$ for each $\epsilon \in S$ where $e \in M$ be a perfect matching in N.*

Lemma 3.6. *Let S be an optimal favorable matching and M be a perfect matching for a transportation network N. Then $w(M) = w(S)$ and S is a maximum size favorable matching of N with $|S| = 2m - 1$.*

Proof. Let N be a transportation network obtained from a complete bipartite graph $K_{m;\ m}$ with bi-partition $\{X;\ Y\}$ such that $|X| = m,\ \ |Y| = m$, $V(N) = X \cup Y$ and M be a perfect matching for N. Let S be the set of edges generates by f over X is

$$S = \{\xi_i : \xi_i \text{ is an edge or edges associated with } f(x_i), 1 \le i \le m\} \tag{8}$$

such that

$$f_X = \sum_{v \in X} f(v) = \sum_{\xi \in S} w(\xi) = w(S). \tag{9}$$

Where

$$\xi_i = \begin{cases} \xi_1 = min.(a_i,\ b_j), & a_i \le b_j; \\ \xi_1 = min.(a_i,\ b_j),\ \xi_2 = min.(a_i - b_j,\ b_s)\ \dots \text{ s. t. } 1 \le s \le m \text{ and } j \ne s, & a_i > b_j. \end{cases} \tag{10}$$

by setting for a vertex x_i. Since S be an optimal favorable matching, so for each $\xi_i \in S$ and $e \in M$ we have $w(\xi_i) = w(e)$. Now the following

$$\sum_{\xi_i \in S} w(\xi_i) = \sum_{e \in M} w(e), \quad \Rightarrow \quad w(S) = w(M).$$

Again each vertex of favorable vertex set X of the transportation network N will give(s) at least one edge to the optimal favorable matching S satisfying the demand and supply of N. Since the problem is balance, so the favorable matching S will have exactly $m + m - 1 = 2m - 1$ number of edges satisfying each demand and supply of the network. □

Example 3.7. *In Example 3.2; $M = \{x_{12},\ x_{13},\ x_{21},\ x_{22},\ x_{32}\}$ be the perfect matching for the transportation network N with total weight $w(M) = 555$. Let us suppose that $E_Y = \{x_{12},\ x_{13},\ x_{21},\ x_{32}\}$ be the initial favorable matching for the N obtained by set of favorable vertices Y. Now we can verify the definitions as follows:*

$$\begin{aligned} f(y_1) &= c_{21} \cdot x_{21} = 3 \times 20 = w(x_{21}) = w(\xi_1);\ \xi_1 = x_{21}, \\ f(y_2) &= c_{12} \cdot x_{12} + c_{32} \cdot x_{32} = (4 \times 50) + (4 \times 45) = w(x_{12}) + w(x_{32}) = w(\xi_2) + w(\xi_3); \quad where\ \xi_2 = x_{12},\ \xi_3 = x_{32}, \\ f(y_3) &= c_{13} \cdot x_{13} = 1 \times 35 = w(x_{13}) = w(\xi_4);\ \xi_4 = x_{13}, \\ \therefore\quad f_Y &= f(y_1) + f(y_2) + f(y_3) + f_(y_4) = w(\xi_1) + w(\xi_2) + w(\xi_3) + w(\xi_4) = w(\mathrm{E_Y}) = 475. \end{aligned}$$

We may also write, weight of a vertex $w(y_j) = f(y_j);\ j = 1,\ 2,\ 3,\ 4$ *is the sum of weight of edges in* E_Y *incident on the vertex such as follows*

$$\begin{aligned} w(y_1) &= w(x_{21}) = c_{21} \cdot x_{21} = 3 \times 20, \\ w(y_2) &= w(x_{12}) + w(x_{32}) = c_{12} \cdot x_{12} + c_{32} \cdot x_{32} = (4 \times 50) + (4 \times 45), \\ w(y_3) &= w(x_{13}) = c_{13} \cdot x_{13} = 1 \times 35. \end{aligned}$$

Where weight of an edge is $w(x_{ij}) = c_{ij} \cdot x_{ij}$ *for all edges in* $x_{ij} \in E_Y$*. Now we have* $w(M) = 555$ *and* $w(E_Y) = 475$ *satisfying* $w(M) > w(E_Y)$*. The inequality in Definition 3.4 holds good for each edges in favorable matching* E_Y*. The equality sub-graph* $G(E_Y)$ *of initial favorable matching* E_Y *of the transportation network* N *is as follows:*

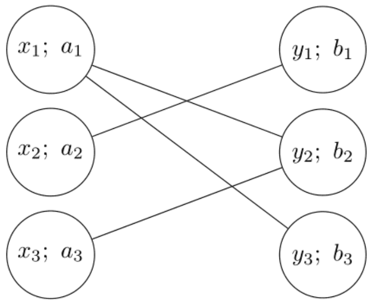

Figure 2. **Equality sub-graph** $G(E_Y)$

Theorem 3.8 (Extension of Egerváry Theorem on Transportation Network)**.** *Let* N *be a transportation network obtained from a weighted complete bipartite graph* $K_{m;\ m}$ *and* M *is a perfect matching obtained by MODI method with maximum weight is* $w(M)$*. Let* E_X *be an optimal favorable matching in* N *is obtained by using the proposed method. Then* $w(E_X) = w(M)$*.*

Proof. The Definitions 3.1; 3.3; 3.4; 3.5 and Lemma 3.6 together will established the statement. □

3.1. Proposed method

Suppose N is a transportation network obtained from $K_{m;\ m}$ with partition $\{X;\ Y\}$ such that $|X| = m,\ |Y| = m$, $V(N) = X \cup Y$ and each edge have an integer value c_{ij} and x_{ij} represent transportation cost per unit and amount of goods to be transport from x_i to y_j respectively. The capacity of vertices in X and Y represent by a_i and b_j respectively. Let M be a perfect matching for N. The algorithm iterative constructs a sequence of favorable matching $E_{X_1}; E_{X_2},\ dots\ E_{X_m}$ for N such that $w(E_{X_{i+1}}) > w(E_{X_i})$ and a sequence of matching E_{X_i} such that E_{X_i} is a favorable matching in the equality sub-graph $G_{E_{X_i}}(f)$ for all $1 \leq i \leq m$. It stops when it finds a optimal favorable matching E_{X_i}.

Basis step:

- Compute δ_x for each $x \in X$, if X is a favorable vertex matching; else compute δ_y for each $y \in Y$, where

$$\begin{aligned} \delta_i = \delta(x_i) &= \{\max.\, c_{ij} - (min.\ c_{ij} + next\ \min.\, c_{ij}) :\ \text{for all } j\},\ 1 \leq i \leq m, \\ \delta_j = \delta(y_i) &= \{\max.\, c_{ij} - (min.\ w_{ij} + next\ \min.\, c_{ij}) :\ \text{for all } i\},\ 1 \leq j \leq m. \end{aligned}$$

- Construct a new favorable matching, $E_{X^*} = \{\xi_i :\ \xi_i$ is or are associated with $f(x_i);\ 1 \leq i \leq m\}$ starting from the vertex having min. δ_x followed by next $x \in X$. Where ξ_i are edges as in Definition 3.3.

- let us suppose that a tie on min. δ_x in the i^{th} vertex; preference should be given to that one having min. c_{ij}, for all j; but in case of tie on both min. δ_x and min. c_{ij} preference should be given to that min. c_{ij} having greater total cost of vertex, i.e. $C(y_j) = \sum_{i=1}^{m} c_{ij}$ as compare to other.

Recursive step: Suppose that we have constructed a favorable matching E_{X_i} of N with total weight $w(E_{X_i})$ and maximum matching M in G for some $i \geq 1$.

- if $w(M) \neq w(E_{X_i})$, then construct a new favorable matching $E_{X_{i+1}}$ for N as follows: in i^{th} vertex; set $\delta = min.\ (c_{ij})$, for all j. Now modify our δ_x for each $x \in X$ such that

$$\delta_x = \begin{cases} \delta_x + \delta, & \delta_x < 0; \\ \delta_x - \delta, & \delta_x > 0; \\ \delta_x, & \delta_x = 0. \end{cases} \tag{11}$$

- Construct favorable matching, $E_{X_{i+1}} = \{\xi_i :\ \xi_i$ is or are associated with $f(x_i);\ 1 \leq i \leq m\}$ starting from the larger δ_x to small for each $x \in X$.

- if $w(M) = w(E_{X_{i+1}})$ or $i = m$; then stop and out put $E_{X_{i+1}}$ and $w(E_{X_{i+1}})$; else iterate.

Remark 3.9. *Some key points to be noted about the method:*

a. *The method must terminate since each iteration decreases the number of vertices, and depth of favorable matching is bounded above by the weight of perfect matching of N.*

b. *When the algorithm terminates it outputs an optimal favorable matching E_{X_i} and a perfect matching M_i in the equality sub-graph $G_{E_X}(f)$ such that $w(M_i) = w(E_{X_i})$.*

c. *When a transportation network is unbalanced. We can modify the network by adding dummy vertices with each edge having cost $c(xy) = 0$ and having capacity $\sum b_j - \sum a_i$ or $\sum a_i - \sum b_j$ whichever is deemed fit the problem. Now we can apply the logical approach to find the optimal favorable matching.*

d. *Maximization problem can be solve by converting it in to minimization.*

4. Examples

Numerical examples on transportation problem are studied to illustrate the process of calculation for the proposed sequential approach.

Example 4.1. *Let N be a transportation network obtained from the complete bipartite graph $K_{3;\ 4}$ with partition source and destination $\{S;\ D\}$ such that $|S| = 3,\ |D| = 4$, $V(N) = S \cup D$ and each edge have an integer value c_{ij} and x_{ij} represent transportation cost per unit and amount of goods to be transport from x_i to y_j respectively. The capacity of vertices in source S and destination D represent by a_i and b_j respectively. Let $M = \{x_{12},\ x_{13},\ x_{14},\ x_{23},\ x_{31},\ x_{34}\}$ be a perfect matching for N obtained by MODI method with maximum weight $w(M) = 149$. The following transportation table represents cost per unit of goods to be transport from source S to destination D.*

Sources	Destinations				Supply
	D_1	D_2	D_3	D_4	
S_1	6	3	5	4	22
S_2	5	9	2	7	15
S_3	5	7	8	6	8
Demand	7	12	17	9	

Table 2. **Transportation table**

This is a balance transportation problem with total supply equal to total demand i.e., $\sum a_i = \sum b_j = 45$. *Now let us calculate depth of each source and destination to find the favorable set of vertex for favorable matching as follows*

$$
\begin{aligned}
f(S_1) &= \sum_{i=1}^{4} \eta_i \cdot \xi_i = c_{12} \cdot x_{12} + c_{14} \cdot x_{14} = (3 \times 12) + (4 \times 9) + (5 \times 1) = 77, \\
f(S_2) &= \sum_{i=1}^{4} \eta_i \cdot \xi_i = c_{23} \cdot x_{23} = (2 \times 15) = 30; \\
f(S_3) &= \sum_{i=1}^{4} \eta_i \cdot \xi_i = c_{31} \cdot x_{31} + c_{34} \cdot x_{34} = (5 \times 7) + (6 \times 1) = 41; \\
\therefore \quad f_S &= \sum_{S_i \in S} f(S_i) = 77 + 30 + 41 = 148. \\
f(D_1) &= \min.\,(c_{i1}) \cdot x_{i1} = c_{31} \cdot x_{31} = (5 \times 7) = 35; \\
f(D_2) &= \min.\,(c_{i2}) \cdot x_{i2} = c_{12} \cdot x_{12} = (3 \times 12) = 36; \\
f(D_3) &= \sum_{i=1}^{3} \eta_i \cdot \xi_i = c_{23} \cdot x_{23} + c_{13} \cdot x_{13} = (2 \times 15) + (5 \times 2) = 40; \\
f(D_4) &= \min.\,(c_{i4}) \cdot x_{i4} = c_{14} \cdot x_{14} = (4 \times 9) = 36; \\
\therefore \quad f_D &= \sum_{D_i \in D} f(D_i) = 35 + 36 + 40 + 36 = 147.
\end{aligned}
$$

Since $f_S > f_D$; *so* $E_S = \{x_{12},\ x_{13},\ x_{14},\ x_{23},\ x_{31},\ x_{34}\}$ *be the initial favorable matching for* N. *Now let us calculate* δ_s *for each* $s \in S$ *and represent in the transportation table as follows:*

Sources	Destinations				Supply	δ_s
	D_1	D_2	D_3	D_4		
S_1	6	3	5	4	22	-1
S_2	5	9	2	7	15	2
S_3	5	7	8	6	8	-3
Demand	7	12	17	9		

Table 3. **Transportation table**

Since $\delta_s = -3$, *is the smallest value associated with* S_3; *so our favorable matching will start from this source as follows*

$$S_3 : \sum min.\,(c_{3j}) \cdot x_{3j} = c_{31} \cdot x_{31} + c_{34} \cdot x_{34} = (5 \times 7) + (6 \times 1) = 41,$$

where $x_{31} = min.\,(a_3,\ b_1) = 7$; $x_{34} = min.\,(a_3 - b_1,\ b_4) = 1$; *here capacity of* S_3 *and demand of* D_1 *reduces to zero. So they delete from table for next calculation. But demand of* D_4 *reduces to* $b_4 = 8$,

$$S_1 : \sum min.\,(c_{1j}) \cdot x_{1j} = c_{12} \cdot x_{12} + c_{14} \cdot x_{14} + c_{13} \cdot x_{13} = (3 \times 12) + (4 \times 8) + (5 \times 2) = 78,$$

where $x_{12} = min.\,(a_1,\ b_2) = 12$; $x_{14} = min.\,(a_1 - b_2,\ b_4) = 8$; $x_{13} = min.\,(a_1 - b_2 - b_4,\ b_3) = 2$, *here capacity of* S_1 *and demand of* D_2, D_4 *reduces to zero. So they delete from table for next calculation. But demand of* D_3 *reduces to* $b_3 = 15$,

$$S_2 : \sum min.\,(c_{2j}) \cdot x_{2j} = c_{23} \cdot x_{23} = (2 \times 15) = 30;$$

where $x_{23} = min.\ (a_2,\ b_3) = 15$, *here capacity of* S_2 *and demand of* D_3 *reduces to zero. After first iteration our favorable matching is* $E_S = \{x_{31},\ x_{34},\ x_{12},\ x_{14},\ x_{13},\ x_{23}\}$ *with depth* $f_S = 149 = w(E_S) = w(M)$ *and* $|E_S| = m+n-1 = 6$. *Where* $M = \{x_{12},\ x_{13},\ x_{14},\ x_{23},\ x_{31},\ x_{34}\}$ *is a perfect matching of* N *obtained by MODI method. Hence* E_S *is optimal favorable matching for the transportation network* N. *Hence the algorithm will terminate.*

Example 4.2. *Let* N *be a transportation network obtained from the complete bipartite graph* $K_{4;\ 4}$ *with partition source and destination* $\{S;\ D\}$ *such that* $|S| = 4,\ \ |D| = 4,\ \ V(N) = S \cup D$ *and each edge have an integer value* c_{ij} *and* x_{ij} *represent transportation cost per unit and amount of goods to be transport from* x_i *to* y_j *respectively. The capacity of vertices in source* S *and destination* D *represent by* a_i *and* b_j *respectively. Let* $M = \{x_{12},\ x_{14},\ x_{23},\ x_{24},\ x_{31},\ x_{34},\ x_{41}\}$ *be a perfect matching for* N *obtained by MODI method with maximum weight* $w(M) = 965$. *The following transportation table represents cost per unit of goods to be transport from source* S *to destination* D.

Sources	Destinations				Supply
	D_1	D_2	D_3	D_4	
S_1	6	1	9	3	70
S_2	11	5	2	8	55
S_3	10	12	4	7	70
S_4	0	0	0	0	20
Demand	85	35	50	45	

Table 4. **Transportation table**

Here S_4 *is a dummy source used in the transportation network with capacity* $a_4 = 20$. *Now this is a balance transportation problem with total supply equal to total demand i.e.,* $\sum a_i = \sum b_j = 215$. *Now let us calculate depth of each source and destination to find the favorable set of vertex for favorable matching as follows*

$$
\begin{aligned}
f(S_1) &= \sum_{i=1}^{4} \eta_i \cdot \xi_i = c_{12} \cdot x_{12} + c_{14} \cdot x_{14} = (1 \times 35) + (3 \times 35) = 140;\\
f(S_2) &= \sum_{i=1}^{4} \eta_i \cdot \xi_i = c_{23} \cdot x_{23} + c_{24} \cdot x_{24} = (2 \times 50) + (8 \times 5) = 140;\\
f(S_3) &= \sum_{i=1}^{4} \eta_i \cdot \xi_i = c_{33} \cdot x_{33} + c_{34} \cdot x_{34} = (4 \times 50) + (7 \times 20) = 340;\\
f(S_4) &= \sum_{i=1}^{4} \eta_i \cdot \xi_i = c_{41} \cdot x_{41} = (0 \times 20) = 0;\\
\therefore\ \ f_S &= \sum_{S_i \in S} f(S_i) = 140 + 140 + 340 = 620.\\
f(D_1) &= \sum_{i=1}^{4} \eta_i \cdot \xi_i = c_{41} \cdot x_{41} + c_{11} \cdot x_{11} = (0 \times 20) + (6 \times 65) = 390;\\
f(D_2) &= \sum_{i=1}^{4} \eta_i \cdot \xi_i = c_{42} \cdot x_{22} + c_{22} \cdot x_{22} = (0 \times 20) + (1 \times 15) = 15;\\
f(D_3) &= \sum_{i=1}^{4} \eta_i \cdot \xi_i = c_{43} \cdot x_{43} + c_{23} \cdot x_{23} = (0 \times 20) + (2 \times 30) = 60;\\
f(D_4) &= \sum_{i=1}^{4} \eta_i \cdot \xi_i = c_{44} \cdot x_{44} + c_{14} \cdot x_{14} = (0 \times 20) + (3 \times 25) = 75;\\
\therefore\ \ f_D &= \sum_{D_i \in D} f(D_i) = 390 + 15 + 60 + 75 = 540.
\end{aligned}
$$

Since $f_S > f_D$; *so* $E_S = \{x_{12},\ x_{14},\ x_{23},\ x_{24},\ x_{33},\ x_{34},\ x_{41}\}$ *be the initial favorable matching for* N. *Now let us calculate* δ_s *for each* $s \in S$ *and represent in the transportation table as follows:*

Sources	Destinations				Supply	δ_s
	D_1	D_2	D_3	D_4		
S_1	6	1	9	3	70	5
S_2	11	5	2	8	55	4
S_3	10	12	4	7	70	1
S_4	0	0	0	0	20	0
Demand	85	35	50	45		

Table 5. **Transportation table**

Since $\delta_s = 0$ *is the smallest value associated with* S_4; *so our favorable matching will start from this source as follows*

$$f(S_4) = \sum_{i=1}^{4} \eta_i \cdot \xi_i = c_{41} \cdot x_{41} = (0 \times 20) = 0;$$

here capacity of S_4 *reduces to zero, so it may be deleted from the table. But demand of* D_1 *reduces to* $b_1 = 65$,

$$f(S_1) = \sum_{i=1}^{4} \eta_i \cdot \xi_i = c_{12} \cdot x_{12} + c_{14} \cdot x_{14} = (1 \times 35) + (3 \times 35) = 140;$$

here capacity of S_1 *and demand of* D_2 *reduces to zero, so it may be deleted from the table. But demand of* D_4 *reduces to* $b_4 = 10$,

$$f(S_2) = \sum_{i=1}^{4} \eta_i \cdot \xi_i = c_{23} \cdot x_{23} + c_{24} \cdot x_{24} = (2 \times 50) + (8 \times 5) = 140;$$

here capacity of S_2 *and demand of* D_3 *reduces to zero, so it may be deleted from the table. But demand of* D_4 *reduces to* $b_4 = 5$,

$$f(S_3) = \sum_{i=1}^{4} \eta_i \cdot \xi_i = c_{34} \cdot x_{34} + c_{31} \cdot x_{31} = (7 \times 5) + (10 \times 65) = 685;$$

here capacity of S_3 *and demand of* D_1, D_4 *reduces to zero, so it may be deleted from the table.But demand of* D_4 *reduces to* $b_4 = 5$,

$$\therefore f_S = \sum_{S_i \in S} f(S_i) = 140 + 140 + 685 = 965.$$

After first iteration our favorable matching is $E_S = \{x_{41},\ x_{12},\ x_{14},\ x_{23},\ x_{24},\ x_{31},\ x_{34}\}$ *with depth* $f_S = 965 = w(E_S) = w(M)$ *and* $|E_S| = m + n - 1 = 7$. *Where* M *is a perfect matching of* N *obtained by MODI method. Hence* E_S *is optimal favorable matching for the transportation network* N. *Hence the algorithm will terminate.*

5. Algorithm Complexity

In this section our keen interest is study the efficiency of the algorithm (time complexity). The complexity of an algorithm is simply the number of computational steps that it takes to transform the input data to the result of a computation. Now we mainly focus on our proposed algorithm and for this purpose we have the following results.

Theorem 5.1. *Let* N *be a transportation network obtained from* $K_{m;\ m}$ *with partition* $\{X;\ Y\}$ *such that* $|X| = m,\ \ |Y| = m$, $V(N) = X \cup Y$ *and each edge have an integer value* c_{ij} *and* x_{ij} *represent transportation cost per unit and amount of goods to be transport from* x_i *to* y_j *respectively. The capacity of vertices in* X *and* Y *represent by* a_i *and* b_j *respectively. Then the proposed method finds a favorable matching in* N *in time* $O(|V(N)|^2)$ *where* $|V(N)|$ *is cardinality of vertex set in* N; *under the assumption that all elementary arithmetic operations take constant time.*

Proof. Let N be a transportation network obtained from $K_{m;\ m}$ with partition $\{X;\ Y\}$ such that $|X| = m$, $|Y| = m$, $V(N) = X \cup Y$ and each edge have an integer value c_{ij} and x_{ij} represent transportation cost per unit and amount of goods to be transport from x_i to y_j respectively with cardinality of vertex set $|V(N))| = 2m$. The algorithm proceeds by growing alternating favorable matching E_{X_i} for $1 \leq i \leq m$. Growing an alternating favorable matching in an equality sub-graph $G(E_{X_i})$ by breadth first search takes $2m$ time. Now growing an m-alternating favorable matching in an equality sub-graph $G(E_{X_i})$ takes at most $2m \times m = 2m^2$ time. Thus the total time spent on alternating favorable vertex matching is $O(2m^2) = O(m^2) = O(|V(N)|^2)$. This established the result. □

6. Result Analysis

In this section the results obtained by proposed method are compared with results obtained by MODI methods with their optimal solutions. The following table "Table 6" summarize all the results.

Methods	Optimal($w(E_X)$ or $w(M)$)	
	Ex.1	Ex.2
Proposed method	149	965
MODI method	149	965

Table 6. **Comparison table**

The optimal favorable matching of a transportation network N by proposed logical method and MODI method are coincide with the same numerical value. The time complexity of proposed logical method is fairly less than as compare to complexity of MODI method. Here total number of algebraic calculations needed to convert the input data to the optimal solution is multiple of n^2, i.e., $O(n^2)$ under the assumption that all algebraic calculations can take equal time.

7. Conclusions

A large number of real world problems can be modeled as an transportation problem because of its combinatirial nature. Till date several methods and algorithms has been develop to solve the transportation problem. But, it is very important to choose the perfect method or approach to deal the problem, to an obtained optimal solution or closer to optimal solution depending on the nature of complexity of the problem. In recent trends some approaches are top choice for the solution of an transportation problem because they produce good but not certainly optimal solution. In this context our proposed method produce good as well as optimal solution in reasonable short amount of time.

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