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Minimal Feebly Semiseparated Čech-closure Spaces

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Abstract: In this paper we introduce the concept of Minimal feebly semiseparated, semiseparated Čech closure spaces. The relation between the underlying topological spaces of the minimal feebly semiseparated closure spaces are also investigated. Properties of sets (open, closed) in minimal feebly semiseparated closure spaces are investigated. A weak characterisation theorem for finite minimal feebly semisepaprted closure space is obtained.

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1. Introduction

Edward Čech in [4] introduced the concept of Čech closure operators on a set $\mathfrak{T}$, as a generalisation of Kuratowski closure Operators (topological closure operators). A Čech closure space $(\mathfrak{T}, \mu)$ is a set $\mathfrak{T}$ with a Čech closure operator μ, which need not be idempotent. He also showed that the set of all Čech closure operators on a set $\mathfrak{T}$ is a complete lattice. The notion of minimal topologies was implemented initially by A.S Parhomenko in 1939 [11], and has shown that compact Hausdorff spaces are minimal Hausdorff. It was proven by E. Hewitt that compact Hausdorff spaces are maximal compact and minimal Hausdorff [6]. R. E Larson [8] studied minimal T_0, minimal T_D,maximal separable and maximal second countable spaces. Many authors study minimal Hausdorffness, maximal compactness, maximal connectedness, and many other minimal and maximal topological properties [1–3, 10, 13]. According to W. J Thron, topological spaces do not constitute a natural boundary for validity of theorems, most of the results can be generalized to closure spaces [15]. Using this fact, in [16], we studied maximal and minimal Čech closure spaces. In this paper we extend the notion of minimal T_0 topological spaces to Čech closure spaces context.

2. Preliminaries

Let $\mathfrak{T}$ be a set and $P(\mathfrak{T})$ denotes the power set of $\mathfrak{T}$, a mapping $c : P(\mathfrak{T}) \longrightarrow P(\mathfrak{T})$ is called a Čech closure operator provided it meets the following conditions.

1. $\mu(\phi) = \phi$.

2. $A \subset \mu(A)$, $\forall A \subseteq \mathfrak{T}$.

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3. $\mu(A \cup B) = \mu(A) \cup \mu(B), \forall A, B \subseteq \mathfrak{T}$.

Then μ together with the underlying set $\mathfrak{T}$, is called a Čech closure space and is denoted by $(\mathfrak{T}, \mu)$. We name Čech closure space as closure space in this paper for convenience. If μ also satisfies $\mu(\mu(A)) = \mu(A), \forall A \subset \mathfrak{T}$, then $(\mathfrak{T}, \mu)$ is a topological space. Hence the we can say the concept of closure spaces is a generalization of topological spaces.

In a closure space $(\mathfrak{T}, \mu)$, a subset A of $\mathfrak{T}$ is closed provided $A = \mu(A)$ and a subset A of $\mathfrak{T}$ is open provided its complement $\mathfrak{T} - A$ is closed. In a closure space $(\mathfrak{T}, \mu)$, the set of all open sets is denoted by $\tau(\mu)$, and $\tau(\mu)$ is a topology on $\mathfrak{T}$, called the underlying topology of the closure space $(\mathfrak{T}, \mu)$. Let $(\mathfrak{T}, \mu_1), (\mathfrak{K}, \mu_2)$ are closure spaces, a function $\theta : \mathfrak{T} \longrightarrow \mathfrak{K}$ is said to be $\check{C}$-continuous (resp., $\check{C}$- homeomorphism) if $\theta(\mu_1(A)) \subseteq \mu_2(\theta(A))$ (resp., θ is a bijection with $\theta(\mu_1(A)) = \mu_2(\theta(A))$ for every $A \subseteq \mathfrak{T}$. If $(\mathfrak{T}, \mu_1)$ and $(\mathfrak{K}, \mu_2)$ are topological spaces under their closure operators then the definition of $\check{C}$-continuity is reduced to the corresponding definition of continuity. The following results are found in [9].

Theorem 2.1. *The composition of two $\check{C}$-continuous (resp., $\check{C}$-homeomorphism) is $\check{C}$-continuous (resp., $\check{C}$-homeomorphism).*

Theorem 2.2. *If $\theta : (\mathfrak{T}, \mu_1) \longrightarrow (\mathfrak{K}, \mu_2)$ is $\check{C}$-continuous (resp., $\check{C}$-homeomorphism), then $\theta : (\mathfrak{T}, \tau(\mu_1)) \longrightarrow (\mathfrak{K}, \tau(\mu_2))$ is continuous (resp., homeomorphism).*

In [14] it is proved that the converse of the above result not true in general.

A closure μ is said to be coarser (weaker) than a closure μ' on the same set $\mathfrak{T}$, if $\mu'(A) \subseteq \mu(A)$ for each subset A of $\mathfrak{T}$, we denote it by $\mu \leq \mu'$. If μ is coarser than μ' we also say μ' is finer than μ. This relation in the set of all closure operators on $\mathfrak{T}$ is a partial order. Thus the set of all closure operators on $\mathfrak{T}$ forms a lattice under the above partial order and is denoted by $LC(\mathfrak{T})$. The smallest element of this lattice is the indiscrete closure operator i, which is the closure associated with the indiscrete topology on $\mathfrak{T}$ and the largest is the discrete closure operator d, which is the closure operator associated with the discrete topology on $\mathfrak{T}$.

Theorem 2.3 ([4])**.** *The closure space $(\mathfrak{T}, \mu)$ is finer than the closure space $(\mathfrak{T}, \mu')$ if and only if $i : (\mathfrak{T}, \mu) \longrightarrow (\mathfrak{T}, \mu')$ is $\check{C}$-continuous, where i is the identity function on $\mathfrak{T}$.*

If $(\mathfrak{T}, \mu)$ be a closure space and $\mathfrak{Y} \subset \mathfrak{T}$, the closure μ' on $\mathfrak{Y}$ is defined as $\mu'(A) = \mathfrak{Y} \cap \mu(A)$ for all $A \subseteq \mathfrak{Y}$. The closure space $(\mathfrak{Y}, \mu')$ is called the subspace of $(\mathfrak{T}, \mu)$ or the relative closure space induced by μ on $\mathfrak{Y}$.

A closure space $(\mathfrak{T}, \mu)$ is said to be Hausdorff or separated if for any two distinct points x, y of $\mathfrak{T}$, there exist neighbourhoods $\mathfrak{U}$ of x and $\mathfrak{V}$ of y such that $\mathfrak{U} \cap \mathfrak{V} = \phi$. In a closure space $(\mathfrak{T}, \mu)$, a suset A of $\mathfrak{T}$ is called dense, if $\mu(A) = \mathfrak{T}$.

Definition 2.4 ([16])**.** *A property P of a closure space is called a closure property if it is preserved by $\check{C}$-homeomorphisms.*

Analogous to topological spaces, for a given closure property P, and a set $\mathfrak{T}$, we denote $LPC(\mathfrak{T})$, the collection of all closure operators on $\mathfrak{T}$, which have the property P, and $LPC(\mathfrak{T})$ is a partially ordered set by the relation 'coarser than'. A closure space $(\mathfrak{T}, \mu)$ is maximal with respect to P (P maximal) if μ is a maximal element in $LPC(\mathfrak{T})$. Similarly we define a closure space $(\mathfrak{T}, \mu)$ is minimal with respect to P (P minimal).

Theorem 2.5 ([16])**.** *A closure space $(\mathfrak{T}, \mu)$ is maximal P if and only if every $\check{C}$-continuous bijection from a space $(\mathfrak{K}, \mu')$ with property P to $(\mathfrak{T}, \mu)$ is a $\check{C}$-homeomorphism.*

In [16], we find some characterisation theorems for minimal and maximal closure spaces analogous to topological context. For more details, relevant to closure spaces we refer to [4, 9, 14, 15].

3. Minimal Feebly Semiseparated and Semiseparated Closure Spaces

E. Čech [4] has identified and discussed the separation properties in closure spaces. According to him, any two points can be separated by distinct neighbourhoods in a separated closure space. W. J Thron, David. N. Roth, Carlson, studied a number of separation properties like T_0, T_1, T_2, R_0, R_1, etc in closure spaces [12]. In this section we studied the notion of minimal feebly semiseparated and semiseparated closure spaces analogous to topological spaces. Properties of different subsets of a minimal feebly semiseparated space, when the underlying set is finite is studied in detail. Also we noticed that many of the results which holds in minimal T_0 topological spaces are not true in closure space context.

Definition 3.1 ([4]). *A closure space $(\mathfrak{X}, \mu)$ is feebly semiseparated if $x \neq y \in \mathfrak{X}$, then $x \notin \mu(\{y\})$ or $y \notin \mu(\{x\})$, semiseparated if $x \notin \mu(\{y\})$ and $y \notin \mu(\{x\})$.*

The terms T_0, T_1 are synonyms for feebly semiseparated and semiseparated closure spaces.

Definition 3.2. *A closure space $(\mathfrak{X}, \mu)$ is minimal feebly semiseparated if $(\mathfrak{X}, \mu)$ is feebly semiseparated and there is no coarser closure operator μ' on $\mathfrak{X}$, which is feebly semiseparated. Similarly, we define a minimal semiseparated closure space.*

Example 3.3.

(a). If $\mathfrak{X}$ is a two point set $\mathfrak{X} = \{u, v\}$, then define, $\mu(\{u\}) = \{u\}$, $\mu(\{v\}) = \mu(\mathfrak{X}) = \mathfrak{X}$, $\mu(\phi) = \phi$ then μ is a closure operator on $\mathfrak{X}$, which is feebly semiseparated as well as minimal feebly semiseparated.

(b). Let $\mathfrak{X} = \{u, v, w\}$, define $\mu(\{u\}) = \{u, v\}$, $\mu(\{v\}) = \{v, w\}$, $\mu(\{w\}) = \{u, w\}$, $\mu\{u, v\} = \mu\{v, w\} = \mu\{u, w\} = \mu(\mathfrak{X}) = \mathfrak{X}$, and $\mu(\phi) = \phi$. Then $(\mathfrak{X}, \mu)$ is minimal feebly semiseparated.

(c). Let $\mathfrak{X} = \{u, v, w\}$, define a closure μ on $\mathfrak{X}$ as follows, $\mu(\{u\}) = \{u\}$, $\mu(\{v\}) = \{v, w\}$, $\mu(\{w\}) = \{u, w\}$, $\mu\{u, w\} = \{u, w\}$, $\mu\{u, v\} = \mu\{v, w\} = \mu(\mathfrak{X}) = \mathfrak{X}$, and $\mu(\phi) = \phi$. Then $(\mathfrak{X}, \mu)$ is feebly semiseparated, but not minimal.

Remark 3.4. *A closure space $(\mathfrak{X}, \mu)$ is minimal feebly semiseparated doesn't imply $(\mathfrak{X}, \tau_\mu)$ is minimal T_0. In Example 3.3 (b), $(\mathfrak{X}, \mu)$ is minimal feebly semiseparated. But the underlying topology τ_μ is the indiscrete topology on $\mathfrak{X}$, which is not T_0.*

Remark 3.5. *In [5], Doyle proved that a T_0 topology on a finite set, in which open or closed sets are nested is minimal T_0. Using Example 3.3 (c), we can say this is not true in closure space context. But the following proposition prove, in a minimal feebly semiseparated finite closure space, open sets as well as closed sets are nested.*

Proposition 3.6. *For a finite minimal feebly semiseparated closure space $(\mathfrak{X}, \mu)$, the following are true*

(1). The closed sets in $(\mathfrak{X}, \mu)$ are nested.

(2). The open sets in $(\mathfrak{X}, \mu)$ are nested.

Proof. Assume $(\mathfrak{X}, \mu)$ is minimal feebly semiseparated. To prove (1), on the contrary assume, the closed sets are not nested. Then there exist $A, B \subset \mathfrak{X}$, closed and $A \nsubseteq B$ and $B \nsubseteq A$.Then there exist at least an element $a \in \mathfrak{X}$ such that $a \in A, a \notin B$ and there exist an element $b \in \mathfrak{X}$ such that $b \in B$, $b \notin A$.Define a closure μ' on X as $\mu'(a) = \mu(a) \cup \{b\}$, and $\mu'(x) = \mu(x)$, and for all $S \subseteq \mathfrak{X}$,

$$\mu'(S) = \begin{cases} \phi \ ; if \ S = \phi \\ \cup_{x \in S}\{\mu'(x)\} \ ; otherwise \end{cases}$$

Then clearly $\mu' \leq \mu$.We claim $(\mathfrak{X}, \mu')$ is feebly semiseparated. Let us take $x_1, x_2 \in \mathfrak{X}$, $x_1 \neq x_2$.

Case 1: $x_1 = a$ $x_2 = b$. Then $x_2 = b \in \mu(a) \cup \{b\} = \mu'(x_1 = a)$. Also $a \notin B$, so $a \notin \mu(b) = \mu'(b)$. Thus $x_1 \notin \mu'(x_2)$ or

$x_2 \notin \mu'(x_1)$.

Case 2: Since $(\mathfrak{T}, \mu)$ is feebly semiseparated, and $\mu'(x) = \mu(x)$, $\forall x \neq a$, thus in all other caes $x_1 \notin \mu'(x_2)$ or $x_2 \notin \mu'(x_1)$. Hence $(\mathfrak{T}, \mu')$ is feebly semiseparated, which is not possible. So what we assumed is wrong, thus closed sets in $(\mathfrak{T}, \mu)$ are nested.

Similarly, we can prove the open sets in $(\mathfrak{T}, \mu)$ are nested. □

Proposition 3.7. *Let $(\mathfrak{T}, \mu)$ be a finite minimal feebly semiseparated closure space, then every non empty proper closed set in $(\mathfrak{T}, \mu)$, if it exist is a point closure.*

Proof. Given $(\mathfrak{T}, \mu)$ is minimal feebly semiseparated closure space. Assume there exist a closed set $\mathcal{F}$, $|\mathcal{F}| > 1$, such that $\mu(x) \neq \mathcal{F}$ for any $x \in \mathfrak{T}$.Then $\mathcal{F} = \bigcup_{f_i \in \mathcal{F}} \mu(f_i)$. Define a closure μ^* on $\mathfrak{T}$ as follows, $\mu^*(f) = \mathcal{F}$ for some fixed $f = f_i \in \mathcal{F}$, and $\mu^*(x) = \mu(x)$ $\forall x \neq f \in \mathfrak{T}$. Then $\mu^* \leq \mu$. Using similar arguments in the proof of Proposition 3.6, we can prove μ^* is feebly semiseparated. Which is not possible. Hence every non empty proper closed sets in $(\mathfrak{T}, \mu)$, if it exist is a point closure. □

Remark 3.8. *The converse of the above result need not be true in closure spaces (Example 3.3(c)).*

Theorem 3.9. *Let $(\mathfrak{T}, \mu)$ be a finite minimal feebly semiseparated closure space, then every non empty open sets in $(\mathfrak{T}, \mu)$ are dense subsets of $(\mathfrak{T}, \mu)$.*

Proof. On the contrary assume there exist an open set $\mathcal{U}$ (say) in $(\mathfrak{T}, \mu)$, which is not dense in $(\mathfrak{T}, \mu)$. Define a closure μ' on $\mathfrak{T}$ as follows for every $x \in \mathfrak{T}$, $\mu'(x) = \mu(x)$ if $x \in \mu(U)$, $\mu'(x) = \mu(x) \cup \mathcal{U}$ if $x \notin \mu(\mathcal{U})$ and for all $A \subseteq \mathfrak{T}$,

$$\mu(A) = \begin{cases} \bigcup_{a \in A} \mu(a)\,;\ if\ A \neq \phi \\ \phi\,;\ if\ A = \phi \end{cases}$$

Then $\mu' \leq \mu$. We claim μ' is feebly semiseparated. For., let $x \neq y \in X$ then we have the following cases.

Case 1: If $x, y \in \mu(U)$, then $\mu'(x) = \mu(x)$ and $\mu'(y) = \mu(y)$. Since (X, μ) is feebly semi separated, $x \notin \mu(y) = \mu'(y)$ or $y \notin \mu(x) = \mu'(x)$.

Case 2: If $x, y \notin \mu(U)$, then clearly $x, y \notin U$, hence $x \notin \mu(y) \cup U = \mu'(y)$ or $y \notin \mu(x) \cup U = \mu'(x)$.

Case 3: If $x \in \mu(U)$ and $y \notin \mu(U)$. Then $\mu'(x) = \mu(x)$ and $\mu'(y) = \mu(y) \cup U$. Since (X, μ) is feebly semi separated, $x \notin \mu(y)$ or $y \notin \mu(x) = \mu'(x)$. Thus $x \notin \mu(y) \cup U = \mu'(y)$ or $y \notin \mu'(x)$.

Thus in all cases $x \notin \mu'(y)$ or $y \notin \mu'(x)$. Hence $(\mathfrak{T}, \mu')$ is feebly semiseparated. Which is not possible since $(\mathfrak{T}, \mu)$ is minimal feebly semiseparated.So our assumption is wrong. Hence every non empty open sets in $(\mathfrak{T}, \mu)$ are dense subsets. □

Example 3.10. *Let $\mathfrak{T} = \{a, b, c, d\}$, define a closure μ on $\mathfrak{T}$ as follows $\mu(\{a\}) = \{a, b\}$, $\mu(\{b\}) = \{b, c\}$, $\mu(\{c\}) = \{c, d\}$, $\mu(\{d\}) = \{a, d\}$, and for all $A \subseteq \mathfrak{T}$,*

$$\mu(A) = \begin{cases} \bigcup_{a \in A} c(a)\,;\ if\ A \neq \phi \\ \phi\,;\ if\ A = \phi \end{cases}$$

Then $(\mathfrak{T}, \mu)$ is feebly semiseparated, in which all open sets are dense. But $(\mathfrak{T}, \mu)$ is not minimal feebly semiseparated. Using this example we can say the converse of the above result is not always true.

From Theorem 3.9, we have the following weak characterisation theorem for a finite minimal feebly semiseparated closure space.

Theorem 3.11. *Let $(\mathfrak{X}, \mu)$ be a finite feebly semiseparated closure space, then $(\mathfrak{X}, \mu)$ is minimal feebly semiseparated if and only if every non empty proper open sets in $(\mathfrak{X}, \mu)$ are dense.*

Proof. The first part of the theorem follows from Theorem 3.9. For the converse assume there exist an open set $\mathcal{U} \neq \mathfrak{X}$ in $(\mathfrak{X}, \mu)$, which is not dense. Then by a similar step used in Theorem 3.9, we can construct a closure operator μ' on $\mathfrak{X}$ coarser than μ, which is feebly semiseparated. Hence $(\mathfrak{X}, \mu)$ is not minimal feebly semiseparted. □

There may exist closure spaces,which do not have proper non empty open sets, but the space is minimal feebly semiseparated. Hence the above theorem, we call a weak characterisation theorem for finite minimal feebly semisepaprted closure spaces.

Corollary 3.12. *If μ is a topological closure operator such that $(\mathfrak{X}, \mu)$ is minimal T_0 and $\mathfrak{X}$ is finite, then $(\mathfrak{X}, \mu)$ is minimal feebly semiseparated.*

Proposition 3.13. *If $(\mathfrak{X}, \mu)$ be a finite, minimal feebly semiseparated closure space then there exist exactly one singleton set, which is closed or no singleton sets is closed.*

Proof. Let $(\mathfrak{X}, \mu)$ be a finite minimal feebly semiseparated closure space. Assume there exist $a \neq b \in \mathfrak{X}$ such that $\mu(\{a\}) = \{a\}$ and $\mu(\{b\}) = \{b\}$. Then $\{a\} \nsubseteq \{b\}$, and $\{b\} \nsubseteq \{a\}$. Thus there exist closed sets in $(\mathfrak{X}, \mu)$, which are not nested. Which is not possible due to Proposition 3.6, thus there exist exactly one singleton set, which is closed or no singleton sets is closed in $(\mathfrak{X}, \mu)$. □

Remark 3.14. *The converse of the above result not true, for consider the closure operator on $\mathfrak{X} = \{u, v, w\}$ defined as $\mu(\{u\}) = \{u\}$, $\mu(\{v\}) = \{v, w\}$, $\mu(\{w\}) = \{u, w\}$, $\mu\{u, w\} = \{u, w\}$, $\mu\{u, v\} = \mu\{v, w\} = \mu(\mathfrak{X}) = \mathfrak{X}$ and $\mu(\phi) = \phi$. Here the only singleton set which is closed is $\{u\}$, but $(\mathfrak{X}, \mu)$ is not minimal feebly semiseparated.*

Corollary 3.15. *If μ is a topological closure operator on a finite set, which is minimal T_0 then there exist exactly one singleton set, which is closed in $(\mathfrak{X}, \mu)$.*

Proof. Since μ is a minimal topological closure operator on $\mathfrak{X}$, so μ is not the indiscrete closure operator. Now the result follows from Proposition 3.13. □

Remark 3.16. *In [8] it is proved that a finite topological space is minimal T_0 if and only if finite union of point closures are point closures. But for a closure space context this result need not be true in general (see Example 3.3 (b)).*

Proposition 3.17. *Let $(\mathfrak{X}, \mu)$ be a finite, minimal feebly semiseparated closure space, $A \subseteq \mathfrak{X}$ then $(A, \mu_{|A})$ is minimal feebly semiseparated.*

Proof. Given $(\mathfrak{X}, \mu)$ be a finite, minimal feebly semiseparated closure space. We first prove $(A, \mu_{|A})$ is feebly semiseparated, where $A \subseteq \mathfrak{X}$. For $a \neq b$ in A , we have $a \notin \mu(b)$ or $b \notin \mu(a)$. Then $a \notin A \cap \mu\{b\}$ or $b \notin A \cap \mu\{a\}$, thus $a \notin \mu_{|A}\{b\}$ or $b \notin \mu_{|A}\{a\}$. Hence $(A, \mu_{|A})$ is feebly semiseparated.

To prove $(A, \mu_{|A})$ is minimal feebly semiseparated, assume $(A, \mu_{|A})$ is not minimal. Then there exist a closure $\mu'_{|A}$ (say) such that $\mu'_{|A} \leq \mu_{|A}$ and $\mu'_{|A}$ is feebly semiseparated. Define a closure μ' on X as follows

$$\mu'(a) = \begin{cases} \mu'_{|A}(a) \cup \mu(a); & \text{if } a \in A \\ \mu(a)\ ; & \text{otherwise} \end{cases}$$

and for all $A \neq \phi \subseteq \mathfrak{X}$, $\mu'(A) = \cup_{a\in A}\mu'(a)$, $\mu'(\phi) = \phi$. Then μ' is feebly semiseparated. Which is not possible. Hence $(A, \mu_{|A})$ is minimal feebly semiseparated. □

The following examples will give a non minimal feebly semiseparated and a minimal feebly semiseparated closure space on an infinite set.

Example 3.18. *On $\mathbb{N}$, the set of natural numbers, define a closure μ as follows , $\forall n \in \mathbb{N}, \mu(n) = \{n, n+1\}$ and for all $A \subseteq \mathbb{N}$,*

$$\mu(A) = \begin{cases} \bigcup_{a \in A} \mu(a)\,;\ if\ A \neq \phi \\ \phi\,;\ if\ A = \phi \end{cases}$$

Then $(\mathbb{N}, \mu)$ is a feebly semiseparated closure space, which is not minimal feebly semiseparated.

Example 3.19. *On $\mathbb{N}$, the set of natural numbers, define a closure μ as follows, $\mu(1) = \mathbb{N}$, for all $n \neq 1 \in \mathbb{N}$, $\mu(n) = \mathbb{N} - \{1, 2, ..., n-1\}$ and for all $A \subseteq \mathbb{N}$,*

$$\mu(A) = \begin{cases} \bigcup_{a \in A} \mu(a)\,;\ if\ A \neq \phi \\ \phi\,;\ if\ A = \phi \end{cases}$$

Then $(\mathbb{N}, \mu)$ is a minimal feebly semiseparated closure space (not unique), since the only closure coarser than μ is the indiscrete closure on $\mathbb{N}$.

The following results are found in [4, 14].

Proposition 3.20. *For a closure space $(\mathfrak{T}, \mu)$, the following are equivalent.*

(1). $(\mathfrak{T}, \mu)$ is semiseparated.

(2). For any $x \in \mathfrak{T}$, $\{x\}$ is closed.

(3). Every finite subsets of $\mathfrak{T}$ is closed.

Proposition 3.21. *A closure space $(\mathfrak{T}, \mu)$ is semiseparated if and only if $(\mathfrak{T}, \tau_\mu)$ is T_1.*

Now using Proposition 3.20, we can conclude, the closure μ, is finer than the cofinite closure on $\mathfrak{T}$. Also from Proposition 3.21, the only minimal semiseparated closure operator on a set $\mathfrak{T}$ is the cofinite closure on $\mathfrak{T}$. If $\mathfrak{T}$ is finite then the semiseparated as well as minimal semiseparated closures will coincide with the discrete closure operator d.

4. Concluding Remarks

The concepts, minimal feebly semiseparated closure spaces are introduced. We proved, not every minimal feebly semiseparated closure space is minimal T_0. Properties of sets (open, closed) in minimal feebly semiseparated closure spaces are investigated. A weak characterisation theorem for finite minimal feebly semisepaprted closure space is obtained. These are some questions for further investigation.

1. Check the validity of results in finite minimal feebly semiseparated closure spaces in general context.
2. Find some characterisation theorems, which is valid for a general minimal feebly semiseparated closure space.
3. Extend the study in to other closure separation properties.
4. Extend this study in fuzzy context is also recommended.

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