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On Ideals and Multiplicative (Generalized) - (Φ, Φ) - Derivations

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Abstract: Let P be a prime ring. I is a nonzero ideal of P. Φ is an automorphism on P. A mapping $M : P \to P$ is called Multiplicative (generalized) (Φ, Φ)-derivation if there exist a map $d : P \to P$ such that $M(a, b) = M(a)\Phi(b) + \Phi(a)d(b)$ holds for all $a, b \in P$. The objective of the present paper is to study the following identities (i). If $M(ab) + M(a)M(b) = 0$ for all $a, b \in I$ then $\Phi(I)[M(a), M(b)] = 0$ for all $a \in I$ (ii). Let M_1 and M_2 be two multiplicative (generalized)-(Φ, Φ) derivations on P associated with the maps d_1 and d_2 on P respectively. If $M_1(ab) = \Phi(b) \circ M_2(a)$ for all $a, b \in I$ then R is abelian or commutative or $\Phi(I)\,[\Phi(I), M_2(I)] = 0$ (iii). If $M_1(ab) = [\Phi(b), M_2(a)]$ for all $a, b \in I$ then either $\Phi(I)\,[\Phi(I), M_2(I)] = (0)$ or R is commutative.

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1. Introduction

A ring P is prime if for any $a, b \in P$, $aPb = (0)$ implies that either $a = 0$ or $b = 0$. $M(a, b) = M(a)\Phi(b) + \Phi(a)d(b)$ for all $a, b \in P$ is called Multiplicative (generalized) -(Φ, Φ)-derivation, where $M : P \to P$ is a mapping and P is a prime ring. In 2014 [1] Dhara proved few identities connected to Multiplicative (generalized) $-(\sigma, \sigma)$ derivations where σ is an epimorphism. Furthermore accurately they demonstrated succeeding outcomes. Let R be a semi prime ring, I a nonzero left ideal of R and σ any epimorphism of R Suppose that F is a Multiplicative (generalized) -(σ, σ) derivation associate with the map d. If $F(xy) - F(x)F(y) = 0$ holds for all $x, y \in I$ then $\sigma(I)d(I) = (0)$ and $\sigma(I)[F(x), \sigma(x)] = (0)$ for all $x \in I$. In 2020 [2] chirag Garg showed few results associated to Left ideals and Multiplicative (generalized) -(α, β)-derivations. Particularly, they proved the subsequent result. Let R be a prime ring and L be a non zero left ideal of R. Suppose that F is Multiplicative (generalized) -(α, β)-derivation on R associated with the map d on R. If $F(xy) + F(x)F(y) = 0$ for all $x, y \in L$ then either $\sigma(L)[F(x), \alpha(x)] = (0)$ or $\beta(L)[F(x), \beta(x)] = (0)$ for all $x \in L$. Considering exceeding results we initiate our theorems We will frequently use the basic commutator and Skew-commutator identities

(i). $[xy, z] = x[y, z] + [x, z]y$.

(ii). $[x, yz] = y[x, z] + [x, y]z$.

(iii). $x \circ yz = (x \circ y) - y[x, z] = y(xoz) + [x, y]z$.

(iv). $xy \circ z = x(yoz) - [x, z]y = (x \circ z)y + x[y, z]$ for all $x, y \in P$.

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2. Main Results

Theorem 2.1. *In a prime ring P and I is a nonzero ideal of P if $M(ab) + M(a)M(b) = 0$ for all $a, b \in I$, where M is a multiplicative (generalized) -(Φ, Φ) derivation on P related with the map d on P then $\Phi(I)[M(a), M(b)] = 0$ for all $a \in I$.*

Proof. From the hypothesis we have

(1). $M(ab) + M(a)M(b) = 0$ for all $a, b \in I$.

Restore bc in the place of b in (1).

(2). $M(abc) + M(a)M(bc) = 0$ for all $a, b, c \in I$.

(3). $M(ab)\Phi(c) + \Phi(ab)d(c) + M(a)\{M(b)\Phi(c) + \Phi(b)d(c)\} = 0$ for all $a, b, c \in I$.

(4). $M(ab)\Phi(c) + \Phi(ab)d(c) + M(a)M(b)\Phi(c) + M(a)\Phi(b)d(c) = 0$ for all $a, b, c \in I$.

(5). $\{M(ab) + M(a)M(b)\}\Phi(c) + \Phi(ab)d(c) + M(a)\Phi(b)d(c) = 0$ for all $a, b, c \in I$. Using (1) we have

(6). $\Phi(ab)d(c) + M(a)\Phi(b)d(c) = 0$ for all $a, b, c \in I$. Substitute au in the place of a in (6).

(7). $\Phi(aub)d(c) + M(au)\Phi(b)d(c) = 0$ for all $a, b, c, u \in I$.

(8). $\Phi(aub)d(c) + \{M(a)\Phi(u) + \Phi(a)d(u)\}\Phi(b)d(c) = 0$ for all $a, b, c, u \in I$.

(9). $\Phi(aub)d(c) + M(a)\Phi(u)\Phi(b)d(c) + \Phi(a)d(u)\Phi(b)d(c) = 0$ for all $a, b, c, u \in I$. Another time substitute ub in the place of b in (6)

(10). $\Phi(aub)d(c) + M(a)\Phi(ub)d(c) = 0$ for all $a, b, c, u \in I$.

(11). $\Phi(aub)d(c) + M(a)\Phi(u)\Phi(b)d(c) = 0$ for all $a, b, c, u \in I$. Subtract (9) from (11),

(12). $\Phi(aub)d(c) + M(a)\Phi(u)\Phi(b)d(c) + \Phi(a)d(u)\Phi(b)d(c) - \Phi(aub)d(c) - M(a)\Phi(u)\Phi(b)d(c) = 0$ for all $a, b, c, u \in I$.

(13). $M(a)\Phi(u)\Phi(b)d(c) + \Phi(a)d(u)\Phi(b)d(c) - M(a)\Phi(u)\Phi(b)d(c) = 0$ for all $a, b, c, u \in I$.

(14). $\{M(a)\Phi(u) + \Phi(a)d(u) - M(a)\Phi(u)\}\Phi(b)d(c) = 0$ for all $a, b, c, u \in I$.

(15). $\{M(au) - M(a)\Phi(u)\}\Phi(b)d(c) = 0$ for all $a, b, c, u \in I$. By the primness of P, we have,

(16). $M(au) - M(a)\Phi(u) = 0$ or

(17). $\Phi(b)d(c) = 0$.

Case 1: If $M(au) - M(a)\Phi(u) = 0$ for all $a, u \in I$. In particular, for all $a, b \in I$, we have $M(ab) - M(a)\Phi(b) = 0$.

(18). $M(ab) = M(a)\Phi(b)$ for all $a, b \in I$. From (1), we have $M(ab) = -M(a)M(b)$. Replace bc in the place of b,

(19). $M(abc) = -M(ab)M(c) = -M(a)M(bc)$ for all $a, b, c \in I$. Above equation can be written as $M(ab)M(c) = M(a)M(bc)$. Using (18),

(20). $M(a)\Phi(b)M(c) = M(a)M(b)\Phi(c)$.

(21). $M(a)\{\Phi(b)M(c) - M(b)\Phi(c)\} = 0$. Replace a by apu, $p \in P$, $u \in I$

$$M(apu)\{\Phi(b)M(c) - M(b)\Phi(c)\} = 0 \text{ Using (19)}$$
$$M(a)M(pu)\{\Phi(b)M(c) - M(b)\Phi(c)\} = 0$$
$$M(a)M(p)\Phi(u)\{\Phi(b)M(c) - M(b)\Phi(c)\} = 0$$
$$M(a)M(p)\Phi(u)\Phi(b)M(c) + M(a)M(p)\Phi(u)M(b)\Phi(c) = 0$$

(22). $M(a)M(p)\Phi(u)M(b)\Phi(c) - M(a)M(p)\Phi(u)\Phi(b)M(c) = 0$. Replace b by c

(23). $M(a)M(p)\Phi(u)M(c)\Phi(c) - M(a)M(p)\Phi(u)\Phi(c)M(c) = 0$.

(24). $M(a)M(p)\Phi(u)[M(c), \Phi(c)] = 0$.

(25). $\Phi(u)[M(c), \Phi(c)] = 0$. In particular $\Phi(I)[M(a), \Phi(a)] = 0$.

Case 2: Now when $\Phi(b)d(c) = 0$. We get $M(ab) = M(a)\Phi(b)$ for all $a, b \in I$. And proceeding in the similar way as before time we obtain $\Phi(I)[M(a), \Phi(a)] = 0$ for all $a \in I$. Therefore the proof of the theorem is completed.

□

Theorem 2.2. *Φ is an automorphism on P. In a prime ring P, let I be a nonzero ideal of P. M_1 and M_2 be two multiplicative (generalized)-(Φ, Φ) derivations on P associated with the maps d_1 and d_2 on P respectively. If $M_1(ab) = \Phi(b) \circ M_2(a)$ for all $a, b \in I$ then R is abelian or commutative or $\Phi(I)\,[\Phi(I), M_2(I)] = 0$.*

Proof. From the hypothesis

(26). $M_1(ab) = \Phi(b) \circ M_2(a)$. For all $a, b \in I$, substitute bc in the place of b in (26)

(27). $M_1(abc) = \Phi(bc) \circ M_2(a)$.

(28). $M_1(ab)\Phi(c) + \Phi(ab)d(c) = \Phi(b)\Phi(c) \circ M_2(a)$.

(29). $M_1(ab)\Phi(c) + \Phi(ab)d(c) = (\Phi(b) \circ M_2(a))\,\Phi(c) + \Phi(b)\,[\Phi(c), M_2(a)]$ using(26)

(30). $M_1(ab)\Phi(c) + \Phi(ab)d(c) = M_1(ab)\Phi(c) + \Phi(b)[\Phi(c), M_2(a)]$.

(31). $\Phi(ab)d(c) = \Phi(b)[\Phi(c), M_2(a)]$. For all $a, b, c \in$ I, substitute ub in the place of b in (31). We obtain

(32). $\Phi(aub)d(c) = \Phi(ub)\,[\Phi(c), M_2(a)]$.

(33). $\Phi(a)\Phi(u)\Phi(b)d(c) = \Phi(u)\Phi(b)\,[\Phi(c), M_2(a)]$. For all $a, b, c, u \in I$. Left multiply (31) with $\Phi(u)$

(34). $\Phi(u)\Phi(a)\Phi(b)d(c) = \Phi(u)\Phi(b)\,[\Phi(c), M_2(a)]$. Subtract (33) from (34) we have

(35). $\Phi(a)\Phi(u)\Phi(b)d(c) - \Phi(u)\Phi(a)\Phi(b)d(c) = 0$.

(36). $[\Phi(a), \Phi(u)]\Phi(b)d(c) = 0$. For all $a, b, c, u \in I$, substitute pb in the place of b in (36)

(37). $[\Phi(a), \Phi(u)]\Phi(pb)d(c) = 0$, where $p \in P$, $b \in I$, $pb \in I$. Using primeness of R, we have either

$$\Phi(b)d(c) = 0 \tag{38}$$

or

$$[\Phi(a), \Phi(u)] = 0. \tag{39}$$

Case 1: If $\Phi(b)d(c) = 0$, from (31), $\Phi(a)\Phi(b)d(c) = \Phi(b)\,[\Phi(c), M_2(a)]$.

(40). $\Phi(b)\,[\Phi(c), M_2(a)] = 0$. Therefore $\Phi(I)\,[\Phi(I), M_2(I)] = 0$ for all $a, b, c \in I$.

Case 2: If $[\Phi(a), \Phi(u)] = 0$ for all $a, u \in I$. In particular $[\Phi(a), \Phi(b)] = 0$ for all a,b $\in I$.

(41). $\Phi(a)\Phi(b) - \Phi(b)\Phi(a) = 0$. Substitute pb in the place of b in (41), where $p \in P$

(42). $\Phi(a)\Phi(pb) - \Phi(pb)\Phi(a) = 0$.

(43). $\Phi(a)\Phi(p)\Phi(b) - \Phi(p)\Phi(b)\Phi(a) = 0$. Left multiply (41) with $\Phi(p)$.

(44). $\Phi(p)\Phi(a)\Phi(b) - \Phi(p)\Phi(b)\Phi(a) = 0$. Subtract (43) from (44), we have $\Phi(a)\Phi(p)\Phi(b) - \Phi(p)\Phi(a)\Phi(b) = 0$.

(45). $[\Phi(a)\Phi(p)]\Phi(b) = 0$. Since I is nonzero and primness of R forces that

$$\Phi(a)\Phi(p) - \Phi(p)\Phi(a) = 0$$

$$[\Phi(a), p]_{\Phi,\Phi} = 0 \quad \text{for all } a \in I,\ p \in P$$

Applying Lemma 2 [12], let R be a prime ring, I be a left ideal of R. If $[x, r] = 0$ for all $x \in I$, $r \in R$, then R is commutative. Hence R is commutative. Therefore the proof of the theorem is completed. □

Theorem 2.3. *In a prime ring P, I a nonzero ideal of R. Let M_1, M_2 be two Multiplicative (generalized) -(Φ, Φ) derivations on R associated with the maps d_1 and d_2 on P (respectively). If $M_1(ab) = [\Phi(b), M_2(a)]$ for all $a, b \in I$, then either $\Phi(I)\,[\Phi(I), M_2(I)] = (0)$ or R is commutative.*

Proof. From the hypothesis

(46). $M_1(ab) = [\Phi(b), M_2(a)]$ for all $a, b \in I$. Substitute be in the place of b

(47). $M_1(abc) = [\Phi(bc), M_2(a)]$.

(48). $M_1(ab)\Phi(c) + \Phi(ab)d(c) = [\Phi(b)\Phi(c), M_2(a)]$.

(49). $M_1(ab)\Phi(c) + \Phi(ab)d(c) = \Phi(b)\,[\Phi(c), M_2(a)] + [\Phi(b), M_2(a)]\,\Phi(c)$. Using (46)

(50). $M_1(ab)\Phi(c) + \Phi(ab)d(c) = \Phi(b)\,[\Phi(c), M_2(a)] + M_1(ab)\Phi(c)$. We have

(51). $\Phi(ab)d(c) = \Phi(b)\,[\Phi(c), M_2(a)]$ for all $a, b, c \in I$

Equation (51) is same as (31) in Theorem 3.2 we go-ahead in the same way as in Theorem 3.2. And we obtain the required result. □

At present we windup this segment with an example, which exhibit that the primness of the ring in our results is essential.

Example 2.4. *Consider the ring* $P = \left\{ \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} / u, v, w \in Z \right\}$, *where Z is the set of integers Let* $I = \left\{ \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} / u, v \in Z \right\}$ *be a left ideal. Let $\Phi : P \to P$ defined as* $\Phi \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} = \begin{pmatrix} u & -v \\ 0 & w \end{pmatrix}$ *be an autoorphism Define* *a mapping* $d_1 : P \to P$ *on* M_1 *as* $d_1 \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} = \begin{pmatrix} 0 & v/2 \\ 0 & 0 \end{pmatrix}$ *and* $M_1 \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} = \begin{pmatrix} u & -v/2 \\ 0 & 0 \end{pmatrix}$ *respectively. Again define mapping* d_2 *on* M_2 *as* $d_2 \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} = \begin{pmatrix} 0 & v/2 \\ 0 & 0 \end{pmatrix}$ *and* $M_1 \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} = \begin{pmatrix} u/2 & 0 \\ 0 & 0 \end{pmatrix}$. *We observe that* M_1, M_2 *are multiplicative (generalized) -*(Φ, Φ) *-derivations on* P *related to the maps* d_1 *and* d_2 *on* P. *We can substantiate that* $M_1(ab) + M_1(a)M_1(b) = 0, M_1(ab) = \Phi(b) \circ M_2(a)$ *for all* $a, b \in I$. *We realize that* $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} P \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (0)$ *but* $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *are nonzero elements of* P. *It implies that* P *is not prime ring. In this illustration we also observe that* R *is not commutative.*

$$\Phi(I)\,[\Phi(I), M_2(I)] \neq (0)$$

$$\Phi(I)\,[M_1(a), \Phi(a)] \neq 0 \ \ \textit{for} \ \ a \in I$$

$$\Phi(I)\,[M_1(a), \Phi(a)] \neq 0 \ \ \textit{for some} \ \ a \in I$$

References

[1] Basudeb Dhara, Sukhendu Kar and Deepankar Das, *A multiplicative (generalized)-*(σ, σ)*-derivation acting as (anti) homomorphism in semiprime rings*, Palestine Journal of Mathematics, 3(2)(2014), 240-246.

[2] Hidetoshi Marubayashi, Mohammad Ashraf, Nadeem-ur Rehman and Shakir Ali, *On generalized* (α, β) *derivations in prime rings*, Algebra Colloquium, 17(1)(2010), 865-874.

[3] S. Khan, *On semi prime rings with multiplicative (generalized)-derivations*, Contributions to Algebra and Geometry, 57(1)(2016), 119-128.

[4] S. Ali, B. Dhara, N. A. Dar and A. N. Khan, *On Lie ideals with multiplicative (Generalized)-derivations in prime and semi prime rings*, Contributions to Algebra and Geometry, 56(2015), 325-337.

[5] B. Dhara and S. Ali, *On multiplicative (generalized)-derivations in prime and semi prime rings*, Aequationes Math., 86(1-2)(2013), 65-79.

[6] O. Golbasi and E. Koc, *Generalized derivations of Lie ideals in prime rings*, Turk. J. Math., 35(2011), 23-28.

[7] N. Rehman, *On commutativity of rings with generalized derivations*, Math. J. Okayama Univ., 44(2002), 43-49.

[8] M. Ashraf and N. Rehman, *On commutativity of rings with derivations*, Results Math., 42(2002), 3-8.

[9] E. Albas, *Generalized derivations on ideals of prime rings*, Miskolc Mathematical Notes, 14(1)(2013), 3-9.

[10] H. Goldmann and P. Semrl, *Multiplicative derivations on* $C(X)$, Monatsh. Math., 121(1996), 189-197.

[11] H. E. Bell and M. N. Daif, *On derivations and commutativity in prime rings*, Acta Math. Hungar., 66(1995), 337-343.

[12] Mohammad Ashraf, Asma Ali and Shakir Ali, *Some Commutativity Theorems for Rings with Generalized Derivations*, Southeast Asian Bulletin of Mathematics, 31(3)(2007), 415-421.