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# Some Graceful and $\alpha$-Graceful Labeling for Cycle Related Graphs

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**Abstract:** In this paper we present graceful and $\alpha$-graceful labeling for some cycle related graphs. We have proved that the graph obtained by adding two pendent vertices at distance two and one chord between them in a cycle $C_n$ and the graph obtained by adding arbitrary pendent vertices at two different places at distance two in a cycle $C_n$, when $n$ is odd are graceful graphs, while the graph obtained by adding alternate pendent vertices in a cycle $C_n$, when $n$ is even and the graph obtained by adding arbitrary pendent vertices at two different places at distance two in a cycle $C_n$, when $n$ is even are $\alpha$-graceful graphs.

**MSC:** 05C78.

**Keywords:** Labeling, Graceful Labeling, $\alpha$-Graceful Labeling, Cycle Related Graphs.


## 1. Introduction

We begin with a simple, undirected graph $G = (V, E)$ with $|V| = p$ vertices and $|E| = q$ edges. For all terminology and notations, we follow Harary [1]. First of all we define and recall some definitions, which are used in this paper.

**Definition 1.1.** *A function $f$ is called graceful labeling for a graph $G$, if $f : V \rightarrow \{0, 1, \ldots, q\}$ is injective and the induced function $f^* : E \rightarrow \{1, 2, \ldots, q\}$ defined as $f^*(e) = |f(u) - f(v)|$ is bijective for every edge $e = uv \in E$. A graph $G$ is called graceful graph, if it admits a graceful labeling.*

**Definition 1.2.** *A graceful labeling $f$ on a graph $G$ is said to be $\alpha$-graceful labeling, if there exists a non-negative integer $k$ less than $q$, the number of edges in $G$, with property that for every edge $uv \in E$ satisfies $\min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}$ in the graph $G$. A graph $G$ is called $\alpha$-graceful graph if it admits a $\alpha$-graceful labeling.*

**Definition 1.3.** *The graph $C_n^{2P+}$ is defined by adding two pendent vertices at the two vertices of a cycle $C_n$, which are at distance two and one chord between them in a cycle $C_n$.*

**Definition 1.4.** *The graph $C_n^{AltP}$ is defined by adding alternate pendent vertices in a cycle $C_n$, where $n \equiv 0 \pmod 2$.*

The graceful labeling was introduced by A. Rosa [2] during 1967. Golomb [3] named such labeling as graceful labeling, which was called earlier as $\beta$-valuation. Rosa [2] also defined $\alpha$-labeling. Any graph $G$, which admits a $\alpha$-labeling is necessarily a bipartite graph. Here we call such $\alpha$-labeling, as $\alpha$-graceful labeling. Kaneria and Makadia [5-6] proved that a star of

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cycle $C_n$ $(n \equiv 0\ (mod\ 4))$ is graceful graph and cycle of cycles $C_t\,(C_n)$, $t \equiv 0\ (mod\ 2)$, $n \equiv 0\ (mod\ 4)$ is graceful graph. Rosa [2] proved that cycle $C_n$ is graceful graph iff $n \equiv 0,\ 3\ (mod\ 4)$ and he also proved that cycle $C_n$ is $\alpha$-graceful graph iff $n \equiv 0\ (mod\ 4)$. Present work we investigate new graceful and $\alpha$-graceful graphs which is obtain from cycle $C_n$, for all n. For detail survey of graph labeling one can refer Gallian [4].

## 2. Main Results

**Theorem 2.1.** *$C_n^{2P+}$ is a graceful graph.*

*Proof.* Let $G$ be a graph obtained by adding two pendent vertices at the two vertices of $C_n$, which are at distance two and one chord between them i.e. $G = C_n^{2P+}$. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_i v_{i+1}/1 \le i < n\} \cup \{v_1 v_n\}$. Now we shall add two vertices at $v_1$ and $v_{n-1}$ and also add one chord between $v_2$ and $v_n$ to obtain the graph $G$. Let us call these two vertices by $v_0$ and $v_{n+1}$. i.e. $V(G) = \{v_0, v_1, \ldots, v_{n+1}\}$ and $E(G) = E(C_n) \cup \{v_0 v_1, v_2 v_n, v_{n-1} v_{n+1}\}$.
To define vertex labeling function $f : V(G) \to \{0, 1, 2, \ldots, n+2\}$, we take following cases.

**Case 1:** $n \equiv 1\ (mod\ 4)$

$$f(v_i) = \begin{cases} \frac{i}{2} & ;\ if\ \ i = 0, 2, \ldots, \left(\frac{n-1}{2}\right) \\ \left(\frac{i+2}{2}\right) & ;\ if\ \ i = \left(\frac{n+3}{2}\right), \left(\frac{n+7}{2}\right), \ldots, (n-1) \\ n + \left(\frac{5-i}{2}\right) & ;\ if\ \ i = 3, 5, \ldots, (n-2) \\ n+3 & ;\ if\ \ i = 1 \\ n+2 & ;\ if\ \ i = n \\ \left(\frac{n+5}{2}\right) & ;\ if\ \ i = n+1 \end{cases}$$

**Case 2:** $n \equiv 2\ (mod\ 4)$

$$f(v_i) = \begin{cases} \frac{i}{2} & ;\ if\ \ i = 0, 2, 4, \ldots, \left(\frac{n+2}{2}\right) \\ \left(\frac{i+2}{2}\right) & ;\ if\ \ i = \left(\frac{n+6}{2}\right), \left(\frac{n+10}{2}\right), \ldots, n \\ n + \left(\frac{7-i}{2}\right) & ;\ if\ \ i = 1, 3, \ldots, \left(\frac{n}{2}\right) \\ n + \left(\frac{5-i}{2}\right) & ;\ if\ \ i = \left(\frac{n+4}{2}\right) \left(\frac{n+8}{2}\right), \ldots, (n+1) \end{cases}$$

**Case 3:** $n \equiv 3\ (mod\ 4)$

$$f(v_i) = \begin{cases} \frac{i+2}{2} & ;\ if\ \ i = 2, 4, \ldots, \left(\frac{n+1}{2}\right) \\ \left(\frac{i+4}{2}\right) & ;\ if\ \ i = \left(\frac{n+5}{2}\right), \left(\frac{n+9}{2}\right), \ldots, (n-1) \\ n + \left(\frac{7-i}{2}\right) & ;\ if\ \ i = 1, 3, \ldots, (n-2) \\ 1 & ;\ if\ \ i = n \\ \frac{n+7}{2} & ;\ if\ \ i = n+1 \\ 0 & ;\ if\ \ i = 0 \end{cases}$$

**Case 4:** $n \equiv 0\ (mod\ 4)$

**Subcase 4.1:** $n = 4$

$$f(v_i) = \begin{cases} \frac{i}{2} & ;\ if\ \ i = 0, 2, \ldots, \left(\frac{n}{2}\right) \\ \frac{i+2}{2} & ;\ if\ \ i = \left(\frac{n+4}{2}\right), \left(\frac{n+6}{2}\right), \ldots, n \\ n + \left(\frac{7-i}{2}\right) & ;\ if\ \ i = 1, 3, \ldots, n+1 \end{cases}$$

**Subcase 4.2:** $n \neq 4$

$$f(v_i) = \begin{cases} \frac{i}{2} & ;\ if\ \ i = 0, 2, \ldots, \left(\frac{n}{2}\right) \\ \left(\frac{i+2}{2}\right) & ;\ if\ \ i = \left(\frac{n+4}{2}\right), \left(\frac{n+6}{2}\right), \ldots, n \\ n + \left(\frac{7-i}{2}\right) & ;\ if\ \ i = 1, 3, \ldots, \left(\frac{n+2}{2}\right) \\ n + \left(\frac{5-i}{2}\right) & ;\ if\ \ i = \left(\frac{n+6}{2}\right), \left(\frac{n+10}{2}\right), \ldots, (n+1) \end{cases}$$

By defined pattern of function, it can be observe that $f$ is one-one, as there is no repeated vertex label. Now we shall prove $f^*$ is bijection. First of all we compute range of $f^*$ i.e. $f^*(E(G))$.

**Case 1:** $n \equiv 1\ (mod\ 4)$. Observe that,

$$f^*(v_0v_1) = n+3,$$
$$f^*(v_1v_2) = n+2,$$
$$f^*(v_2v_3) = n,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f^*\left(v_{\frac{n-1}{2}}v_{\frac{n+3}{2}}\right) = \frac{n+5}{2},$$
$$f^*\left(v_{\frac{n+3}{2}}v_{\frac{n+7}{2}}\right) = \frac{n+1}{2},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f^*(v_{n-2}v_{n-1}) = 3.$$

Therefore,

$$\{f^*(v_iv_{i+1})/i = 0,1,\ldots,(n-2)\} = \left\{3,4,\ldots,\left(\frac{n+1}{2}\right),\left(\frac{n+5}{2}\right),\ldots,n+3\right\} \quad and$$
$$f^*(v_{n-1}v_{n+1}) = 2, \quad f^*(v_1v_n) = 1, \quad f^*(v_2v_n) = n+1, \quad f^*(v_{n-1}v_n) = \frac{n+3}{2}$$
$$i.e. \quad f^*(E(G)) = \{1,2,3,\ldots,n+3\}.$$

Hence, $f^*$ is onto in this case.

**Case 2:** $n \equiv 2\ (mod\ 4)$. Observe that,

$$f^*(v_0v_1) = n+3, \quad f^*(v_1v_2) = n+2,\ldots,f^*\left(v_{\frac{n}{2}}v_{\frac{n+2}{2}}\right) = \frac{n+6}{2},$$
$$f^*\left(v_{\frac{n+2}{2}}v_{\frac{n+4}{2}}\right) = \frac{n+2}{2}, \quad f^*\left(v_{\frac{n+4}{2}}v_{\frac{n+6}{2}}\right) = \frac{n-2}{2},\ldots,f^*(v_{n-1}v_n) = 2.$$

Therefore,

$$\{f^*(v_iv_{i+1})/i = 0,1,\ldots,(n-1)\} = \{2,3,\ldots,\left(\frac{n-2}{2}\right),\left(\frac{n+2}{2}\right),\left(\frac{n+6}{2}\right),\ldots,n+3\} \quad and$$
$$f^*(v_{n-1}v_{n+1}) = 1, \quad f^*(v_1v_n) = \frac{n+4}{2}, \quad f^*(v_2v_n) = \frac{n}{2}$$
$$i.e. \quad f^*(E(G)) = \{1,2,3,\ldots,n+3\}$$

Hence, $f^*$ is onto in this case.

**Case 3:** $n \equiv 3\ (mod\ 4)$. Observe that,

$$f^*(v_0v_1) = n+3, \quad f^*(v_1v_2) = n+1,\ldots,f^*\left(v_{\frac{n+1}{2}}v_{\frac{n+3}{2}}\right) = \frac{n+3}{2},$$
$$f^*\left(v_{\frac{n+3}{2}}v_{\frac{n+5}{2}}\right) = \frac{n-1}{2},\ldots,f^*(v_{n-2}v_{n-1}) = 3.$$

Therefore,

$$f^*(v_iv_{i+1})/i = 0,1,\ldots,(n-2)\} = \{3,4,\ldots,\left(\frac{n-1}{2}\right),\left(\frac{n+3}{2}\right),\ldots,n+3\} \quad and$$

$$f^*(v_{n-1}v_{n+1}) = 2,\ f^*(v_1v_n) = n+2,\ f^*(v_2v_n) = 1,\ f^*(v_{n-1}v_n) = \frac{n+1}{2}$$
$$i.e.\ \ f^*(E(G)) = \{1, 2, 3, \ldots, n+3\}$$

Hence, $f^*$ is onto in this case.

**Case 4:** $n \equiv 0\ (mod\ 4)$

**Subcase 4.1:** $n = 4$. Observe that,

$$f^*(v_0v_1) = n+3,\ f^*(v_1v_2) = n+2,\ f^*(v_2v_3) = n+1,\ f^*(v_1v_n) = n,$$
$$f^*(v_{n-1}v_n) = n-1,\ f^*(v_{n-1}v_{n+1}) = 1\ \ and\ \ f^*(v_2v_n) = 2.$$
$$i.e.\ \ f^*(E(G)) = \{1, 2, 3, \ldots, n+3\}$$

Hence, $f^*$ is onto in this case.

**Subcase 4.2:** $n \neq 4$. Observe that,

$$f^*(v_0v_1) = n+3,\ f^*(v_1v_2) = n+2, \ldots, f^*\left(v_{\frac{n}{2}}v_{\frac{n+2}{2}}\right) = \frac{n+6}{2},$$
$$f^*\left(v_{\frac{n+2}{2}}v_{\frac{n+4}{2}}\right) = \frac{n+2}{2},\ f^*\left(v_{\frac{n+4}{2}}v_{\frac{n+6}{2}}\right) = \frac{n-2}{2}, \ldots, f^*(v_{n-1}v_n) = 2.$$

Therefore,

$$\{f^*(v_iv_{i+1})/i = 0, 1, \ldots, (n-1)\} = \{2, 3, \ldots, \left(\frac{n-2}{2}\right), \left(\frac{n+2}{2}\right), \left(\frac{n+6}{2}\right), \ldots, n+3\}$$

and $f^*(v_{n-1}v_{n+1}) = 1$, $f^*(v_1v_n) = \frac{n+4}{2}$, $f^*(v_2v_n) = \frac{n}{2}$ i.e. $f^*(E(G)) = \{1, 2, 3, \ldots, n+3\}$. Hence, $f^*$ is onto in this case. Thus, we proved that $f^*$ is an onto map in each cases. Further domain of $f^*$ and range of $f^*$ have same cardinality, gives $f^*$ is one-one. Therefore, $f^*$ is bijection. Thus, $f$ is graceful labeling for $G$. Therefore, $G = C_n^{2P+}$ is graceful graph. □

**Illustration 2.2.** *Graph $C_{10}^{2P+}$ and its graceful labeling shown in Figure 1.*

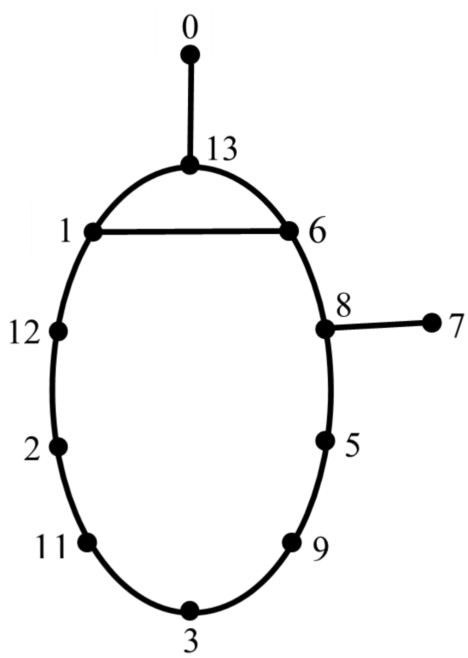


**Figure 1.** **Graceful labeling for $G = C_{10}^{2P+}$**

**Theorem 2.3.** $C_n^{AltP}$ *is $\alpha$-graceful graph when $n \equiv 0\ (mod\ 2)$.*

*Proof.* Let $G$ be a graph obtained by adding alternate pendent vertices in cycle $C_n$ when $n \equiv 0\ (mod\ 2)$ i.e. $G = C_n^{AltP}$. Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and $E(C_n) = \{v_i v_{i+1}/1 \leq i < n\} \cup \{v_1 v_n\}$. Now we shall add $\frac{n}{2}$ vertices at $v_1, v_3, \dots, v_{n-1}$ to obtain graph $G$. Let us call these vertices by $v_{n+1}, v_{n+2}, \dots, v_{\frac{3n}{2}}$ i.e. $V(G) = \left\{v_1, v_2, \dots, v_n, v_{n+1}, \dots, v_{\frac{3n}{2}}\right\}$ and $E(G) = E(C_n) \cup \left\{v_1 v_{n+1}, v_3 v_{n+2}, \dots, v_{n-1} v_{\frac{3n}{2}}\right\}$.

To define vertex labeling function $f : V(G) \to \{0, 1, 2, \dots, \frac{3n}{2}\}$, we take following cases.

**Case 1:** $n \equiv 0\ (mod\ 3)$

$$f(v_i) = \begin{cases} \frac{3n}{2} - \left(\frac{i-1}{2}\right) & ;\ if\ \ i = 1, 3, \dots, (n-1) \\ i - 1 & ;\ if\ \ i = 2, 4, \dots, \frac{2n}{3} \\ i & ;\ if\ \ i = \left(\frac{2n+6}{3}\right), \left(\frac{2n+12}{3}\right), \dots, n \\ 2(i-n) - 2 & ;\ if\ \ i = n+1, n+2, \dots, \frac{4n}{3} \\ 2(i-n) - 1 & ;\ if\ \ i = \left(\frac{4n+3}{3}\right), \left(\frac{4n+6}{3}\right), \dots, \frac{3n}{2} \end{cases}$$

**Case 2:** $n \equiv 1\ (mod\ 3)$

$$f(v_i) = \begin{cases} \frac{3n}{2} - \left(\frac{i-1}{2}\right) & ;\ if\ \ i = 1, 3, \dots, (n-1) \\ i & ;\ if\ \ i = 2, 4, \dots, \left(\frac{n-4}{3}\right) \\ i + 1 & ;\ if\ \ i = \left(\frac{n+2}{3}\right), \left(\frac{n+8}{3}\right), \dots, (n-2) \\ 2(i-n) - 1 & ;\ if\ \ i = n+2, n+3, \dots, \left(\frac{7n+2}{6}\right) \\ 2(i-n) & ;\ if\ \ i = \left(\frac{7n+8}{6}\right), \left(\frac{7n+14}{6}\right), \dots, \frac{3n}{2} \\ 0 & ;\ if\ \ i = n+1 \\ 1 & ;\ if\ \ i = n \end{cases}$$

**Case 3:** $n \equiv 2\ (mod\ 3)$

$$f(v_i) = \begin{cases} \frac{3n}{2} - \left(\frac{i-1}{2}\right) & ;\ if\ \ i = 1, 3, \dots, (n-1) \\ i & ;\ if\ \ i = 2, 4, \dots, \left(\frac{n-2}{3}\right) \\ i + 1 & ;\ if\ \ i = \left(\frac{n+4}{3}\right), \left(\frac{n+10}{3}\right), \dots, (n-2) \\ 2(i-n) - 1 & ;\ if\ \ i = n+2, n+3, \dots, \left(\frac{7n-2}{6}\right) \\ 2(i-n) & ;\ if\ \ i = \left(\frac{7n+4}{6}\right), \left(\frac{7n+10}{6}\right), \dots, \frac{3n}{2} \\ 0 & ;\ if\ \ i = n+1 \\ 1 & ;\ if\ \ i = n \end{cases}$$

By defined pattern of function $f$, it can be observe that $f$ is one-one, as there is no repeated vertex label. Now we shall prove $f^*$ is bijection. First of all we compute range of $f^*$ i.e. $f^*(E(G))$.

**Case 1:** $n \equiv 0\ (mod\ 3)$. Observe that,

$$f^*(v_1 v_{n+1}) = \frac{3n}{2},\ f^*(v_1 v_2) = \frac{3n}{2} - 1, \dots, f^*\left(v_{\frac{2n}{3}} v_{\frac{2n+3}{3}}\right) = \frac{n+2}{2},$$
$$f^*\left(v_{\frac{2n+3}{3}} v_{\frac{4n+3}{3}}\right) = \frac{n-2}{2}, \dots, f^*\left(v_{n-2} v_{\frac{3n}{2}}\right) = 2, f^*(v_{n-1} v_n) = 1\ \ and$$
$$f^*(v_1 v_n) = \frac{n}{2}\ \ i.e.\ \ f^*(E(G)) = \left\{1, 2, 3, \dots, \frac{3n}{2}\right\}$$

Hence, $f^*$ is onto in this case.

**Case 2:** $n \equiv 1\ (mod\ 3)$. Observe that,

$$f^*(v_1 v_{n+1}) = \frac{3n}{2},\ f^*(v_1 v_n) = \frac{3n}{2} - 1,\ f^*(v_1 v_2) = \frac{3n}{2} - 2,\ f^*(v_2 v_3) = \frac{3n}{2} - 3, \dots,\ f^*\left(v_{\frac{n-1}{3}} v_{\frac{7n+2}{6}}\right) = n+1,$$

$$f^*\left(v_{\frac{n-1}{3}}v_{\frac{n+2}{3}}\right) = n-1, \ldots, f^*\left(v_{n-2}v_{n-1}\right) = 2,\ f^*\left(v_{n-1}v_{\frac{3n}{2}}\right) = 1 \ \ and$$
$$f^*(v_{n-1}v_n) = n \ \ i.e.\ \ f^*(E(G)) = \left\{1, 2, 3, \ldots, \frac{3n}{2}\right\}$$

Hence, $f^*$ is onto in this case.

**Case 3:** $n \equiv 2\ (mod\ 3)$. Observe that,

$$f^*\left(v_1v_{n+1}\right) = \frac{3n}{2},\ f^*\left(v_1v_n\right) = \frac{3n}{2}-1,\ f^*\left(v_1v_2\right) = \frac{3n}{2}-2,\ f^*\left(v_2v_3\right) = \frac{3n}{2}-3, \ldots, f^*\left(v_{\frac{n-2}{3}}v_{\frac{n+1}{3}}\right) = n+1,$$
$$f^*\left(v_{\frac{n+1}{3}}v_{\frac{7n+4}{6}}\right) = n-1, \ldots, f^*\left(v_{n-2}v_{n-1}\right) = 2,\ f^*\left(v_{n-1}v_{\frac{3n}{2}}\right) = 1 \ \ and$$
$$f^*(v_{n-1}v_n) = n \ \ i.e.\ \ f^*(E(G)) = \left\{1, 2, 3, \ldots, \frac{3n}{2}\right\}$$

Hence, $f^*$ is onto in this case.

Thus, we proved that $f^*$ is an onto map in each cases. Further domain of $f^*$ and range of $f^*$ have same cardinality, gives $f^*$ is one-one. Therefore, $f^*$ is bijection. Thus, $f$ is graceful labeling for $G = C_n^{AltP}$, $n \equiv 0\ (mod\ 2)$. By taking $k = n$, it can be observe that $C_n^{AltP}$, $n \equiv 0\ (mod\ 2)$ is bipartite graph and for any $uv \in E(G)$, $\min\{f(u), f(v)\} \le k < \max\{f(u), f(v)\}$ and hence, $f$ is an $\alpha$-labeling in these cases. Therefore, $C_n^{AltP}$, $n \equiv 0\ (mod\ 2)$ is $\alpha$-graceful graph. □

**Illustration 2.4.** *Graph $C_{10}^{AltP}$ and its $\alpha$-graceful labeling shown in figure 2.*

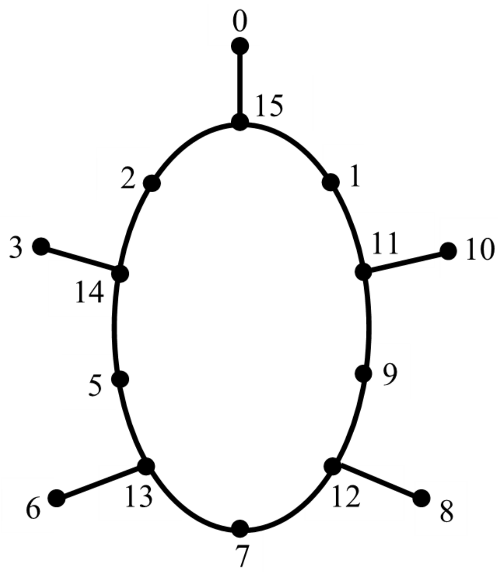


**Figure 2.** **$\alpha$-graceful labeling for $G = C_{10}^{AltP}$ [here $k = n = 10$]**

**Theorem 2.5.** *The graph obtained by adding arbitrary pendent vertices at two different places among at first place $t$ and second place $m$ in cycle $C_n$ at distance two is graceful graph when $n \equiv 1\ (mod\ 2)$ and it is $\alpha$-graceful graph when $n \equiv 0\ (mod\ 2)$, where $\max\{t, m\} \ge \left[\frac{n}{2}\right]$.*

*Proof.* Let $G$ be a graph obtained by adding arbitrary pendent vertices at two different places among at first place $t$ and second place at $m$ in cycle $C_n$ at distance two. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_iv_{i+1}/1 \le i < n\} \cup \{v_1v_n\}$. Now we shall add two different arbitrary pendent vertices at $v_1$ and $v_{n-1}$ in numbers of $t$ and $m$ respectively to obtain graph $G$. To define vertex labeling function $f : V(G) \to \{0, 1, 2, \ldots, n, n+1, \ldots, t+m+n\}$, we take following cases. We take following cases.

**Case 1:** $t \geq \left[\frac{n}{2}\right]$

**Subcase 1.1:** $n \equiv 0\ (mod\ 2)$

$$f(v_i) = \begin{cases} m+t+n-\left(\frac{i-1}{2}\right) & ;\ if\ \ i = 1,3,\ldots,(n-1) \\ t+\left(\frac{i-2}{2}\right) & ;\ if\ \ i = 2,4,\ldots,(n-2) \\ m+t+\frac{n}{2} & ;\ if\ \ i = n \\ i-(n+1) & ;\ if\ \ i = n+1, n+2, \ldots, t+n \\ i-\frac{n}{2}-2 & ;\ if\ \ i = n+t+1, n+t+2, \ldots, t+m-\frac{n}{2}+2 \\ i-\frac{n}{2}-1 & ;\ if\ \ i = t+m-\frac{n}{2}+3, \ldots, m+t+n \end{cases}$$

**Subcase 1.2:** $n \equiv 1\ (mod\ 2)$

$$f(v_i) = \begin{cases} m+n+t-\left(\frac{i-1}{2}\right) & ;\ if\ \ i = 1,3,\ldots,n \\ t+\left(\frac{i-2}{2}\right) & ;\ if\ \ i = 2,4,\ldots,(n-1) \\ i-(n+1) & ;\ if\ \ i = n+1, n+2, \ldots, n+t \\ i-\left[\frac{n}{2}\right]-2 & ;\ if\ \ i = t+n+1, t+n+2, \ldots, t+n+\left[\frac{n}{2}\right]-1 \\ i-\left[\frac{n}{2}\right]-1 & ;\ if\ \ i = t+n+\left[\frac{n}{2}\right], \ldots, m+t+n \end{cases}$$

**Case 2:** $m \geq \left[\frac{n}{2}\right]$

**Subcase 2.1:** $n \equiv 0\ (mod\ 2)$

$$f(v_i) = \begin{cases} m+t+\frac{n}{2}+\left(\frac{i+1}{2}\right) & ;\ if\ \ i = 1,3,\ldots,(n-1) \\ m+\frac{n}{2}-\left(\frac{i+2}{2}\right) & ;\ if\ \ i = 2,4,\ldots,(n-2) \\ m+t+\frac{n}{2} & ;\ if\ \ i = n \\ i-(t+n+1) & ;\ if\ \ i = t+n+1, t+n+2, \ldots, m+t+n \\ m-\frac{n}{2}-2+i & ;\ if\ \ i = n+1, n+2, \ldots, t+\frac{n}{2}+2 \\ m-\frac{n}{2}-1+i & ;\ if\ \ i = t+\frac{n}{2}+3,\ t+\frac{n}{2}+4, \ldots, t+n \end{cases}$$

**Subcase 2.2:** $n \equiv 1\ (mod\ 2)$

$$f(v_i) = \begin{cases} m+\left[\frac{n}{2}\right]-\left(\frac{i+1}{2}\right) & ;\ if\ \ i = 1,3,\ldots,(n-2) \\ m+t+\left[\frac{n}{2}\right]+\left(\frac{i+2}{2}\right) & ;\ if\ \ i = 2,4,\ldots,(n-1) \\ m+t+\left[\frac{n}{2}\right]+1 & ;\ if\ \ i = n \\ i-(t+n+1) & ;\ if\ \ i = t+n+1, t+n+2, \ldots, m+t+n \\ m+(i-n+1) & ;\ if\ \ i = n+1, n+2, \ldots, n+\left[\frac{n}{2}\right]-1 \\ m-n+i+2 & ;\ if\ \ i = n+\left[\frac{n}{2}\right], n+\left[\frac{n}{2}\right]-1, \ldots, t+n \end{cases}$$

By defined pattern of function $f$, it can be observe that $f$ is one-one, as there is no repeated vertex label. Now we shall prove $f^*$ is a bijection. First of all we compute range of $f^*$. i.e. $f^*(E(G))$.

**Case 1:** $t \geq \left[\frac{n}{2}\right]$

**Subcase 1.1:** $n \equiv 0\ (mod\ 2)$. Observe that,

$$\{f^*(v_1 v_i)/i = n+1, n+2, \ldots, n+t\} = \{m+n+1, m+n+2, \ldots, m+n+t\},$$

$$f^*(v_1 v_2) = m+n,\ f^*(v_2 v_3) = m+n-1, \ldots, f^*(v_{n-2} v_{n-1}) = m+3, (v_{n-1} v_n) = 1,\ f^*(v_1 v_n) = \frac{n}{2}\ \ and$$

$$\{f^*(v_{n-1}v_i)/i = t+n+1, t+n+2, \dots, t+m+n\} = \{2, 3, \dots, \frac{n}{2}-1, \frac{n}{2}+1, \dots, m+2\}$$

$$i.e.\ \ f^*(E(G)) = \{1, 2, 3, \dots, m+t+n\}$$

Hence, $f^*$ is onto in this case.

**Subcase 1.2:** $n \equiv 1\ (mod\ 2)$. Observe that,

$$\{f^*(v_1v_i)/i = n+1, n+2, \dots, n+t\} = \{m+n+1, m+n+2, \dots, m+n+t\},$$

$$f^*(v_1v_2) = m+n, f^*(v_2v_3) = m+n-1, \dots, f^*(v_{n-1}v_n) = m+2, f^*(v_1v_n) = \left[\frac{n}{2}\right]\ \ and$$

$$\{f^*(v_{n-1}v_i)/i = t+n+1, \dots, t+m+n\} = \{1, 2, \dots, \left[\frac{n}{2}\right]-1, \left[\frac{n}{2}\right]+1, \dots, m+1\}$$

$$i.e.\ \ f^*(E(G)) = \{1, 2, 3, \dots, m+t+n\}$$

Hence, $f^*$ is onto in this case.

**Case 2:** $m \geq \left[\frac{n}{2}\right]$

**Subcase 2.1:** $n \equiv 0\ (mod\ 2)$. Observe that,

$$\{f^*(v_1v_i)/i = n+1, n+2, \dots, n+t\} = \{2, 3, \dots, \frac{n}{2}-1, \frac{n}{2}+1, \dots, t+2\},$$

$$f^*(v_1v_2) = t+3,\ f^*(v_2v_3) = t+4, \dots, f^*(v_{n-2}v_{n-1}) = t+n,\ f^*(v_{n-1}v_n) = \frac{n}{2}, f^*(v_1v_n) = 1\ \ and$$

$$\{f^*(v_{n-1}v_i)/i = t+n+1, t+n+2, \dots, t+m+n\} = \{t+n-1, t+n, \dots, m+t+n\}$$

$$i.e.\ \ f^*(E(G)) = \{1, 2, 3, \dots, m+t+n\}$$

Hence, $f^*$ is onto in this case.

**Subcase 2.2:** $n \equiv 1\ (mod\ 2)$. Observe that,

$$\{f^*(v_1v_i)/i = n+1, n+2, \dots, n+t\} = \{1, 2, \dots, \left[\frac{n}{2}\right]-1, \left[\frac{n}{2}\right]+1, \dots, t+1\}, f^*(v_1v_2) = t+3$$

$$f^*(v_2v_3) = t+4, \dots, f^*(v_{n-2}v_{n-1}) = t+n,\ f^*(v_1v_n) = t+2,\ f^*(v_{n-1}v_n) = \left[\frac{n}{2}\right]\ \ and$$

$$\{f^*(v_{n-1}v_i)/i = t+n+1, t+n+2, \dots, m+t+n\} = \{t+n+1, t+n+2, \dots, m+t+n\}$$

$$i.e.\ \ f^*(E(G)) = \{1, 2, 3, \dots, m+t+n\}$$

Hence, $f^*$ is onto in this case.

Thus, we proved that $f^*$ is an onto map in each cases. Further domain of $f^*$ and range of $f^*$ have same cardinality, gives $f^*$ is one-one. Therefore, $f^*$ is bijective. Thus, $f$ is graceful labeling for $G$. By taking $k = m+t+\frac{n}{2}$ when $n \equiv 0\ (mod\ 2)$, it can be observe that $G$ is bipartite graph and for any $uv \in E(G)$, $\min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}$ and hence, $f$ is $\alpha$-labeling when $n \equiv 0\ (mod\ 2)$. Therefore $G$ is graceful graph when $n \equiv 1\ (mod\ 2)$ and it is $\alpha$-graceful graph when $n \equiv 0\ (mod\ 2)$. $\square$

**Illustration 2.6.** *Graph $G$ with $n = 10$, $t = 2$ and $m = 5$ and its $\alpha$-graceful labeling shown in Figure 3.*

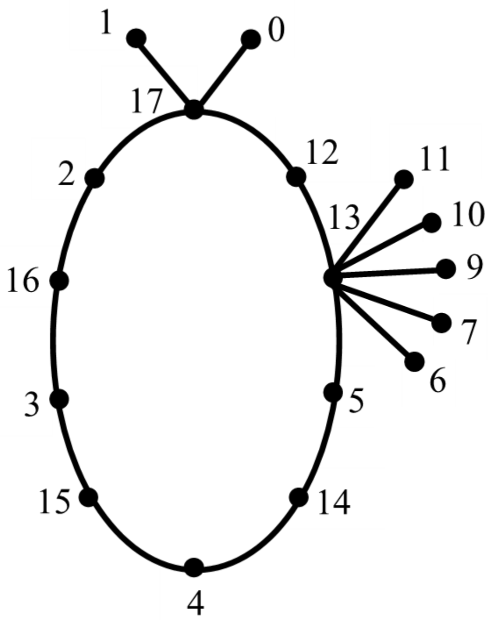


**Figure 3.** **$\alpha$-graceful labeling for graph $G$ [here $k = m + t + \frac{n}{2} = 12$]**

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