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Solutions of Diophantine Equations Generated by Generalised Lucas Balancing Sequences

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Abstract: Diophantine Equations with infinitely many solutions find their place in the applications of cryptography. In this paper we generate some quadratic Diophantine Equations in two variables with infinitely many solutions, using Generalised Lucas Balancing Sequences and obtain all the solutions of each these Diophantine Equations expressed in terms of Generalised Lucas Balancing Sequences.

MSC: 94A60, 11T71.

Keywords: Diophantine equation, Lucas Balancing Sequences, Solutions.

1. Introduction

Balancing Numbers B_n and Lucas Balancing Numbers C_n are defined by Behera and Panda [1] as the natural numbers that satisfy the recurrence relations

$$B_{n+1} = 6B_n - B_{n-1}$$
$$C_{n+1} = 6C_n - C_{n-1}$$

with $B_0 = 0$, $B_1 = 1$ and $C_0 = 1$, $C_1 = 3$ where B_n, C_n are the nth Balancing Number and Lucas Balancing Number respectively. Panda and Rout [5] generalized the above recurrence relation to

$$B_{n+1} = pB_n - qB_{n-1}$$
$$C_{n+1} = pC_n - qC_{n-1}$$

with $B_0 = 0$, $B_1 = 1$ and $C_0 = 1$, $C_1 = p/2$ for p, q positive integers.They proved that all properties of Balancing Numbers and Lucas Balancing Number also hold for their generalised sequence when $q = 1$. In this paper using the properties of Generalised Lucas Balancing sequences we generate the quadratic Diophantine Equations $x^2 \pm pxy + y^2 + (\frac{p^2-4}{4}) = 0$ and $x^2 \pm pxy + y^2 \pm (\frac{p^2-4}{4})x = 0$ in two variables and prove that these equations are with infinitely many solutions. We further obtain all the solutions of each of these Diophantine Equations in terms of Lucas Balancing Numbers. Marlewski and

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Zarzycki [7] proved that there exists infinitely many integer solutions (x, y), that are positive, for the Diophantine equation $x^2 - pxy + y^2 + x = 0$ if and only if $p = 3$. Mojtaba Bahramain and Hussan Daghigh [8] proved that for a positive integer p the Diophantine equation $x^2 \pm pxy - y^2 \pm x = 0$ has positive solutions (x, y) that are infinitely many and they expressed these solutions in terms of Fibonacci sequences. A similar approach is adapted for the Diophantine equations mentioned above to express all the infinitely many solutions in each case in terms of Generalised Lucas Balancing Sequences.

2. Properties of Generalised Balancing Sequences and Generalised Lucas Balancing Sequences

In this section we investigate some properties of Generalised Lucas Balancing Sequences. Generalised Balancing Sequences are numbers satisfying the recurrence relation

$$B_{n+1} = pB_n - B_{n-1}$$

with $B_0 = 0$, $B_1 = 1$ for p, a positive integer. The equation $B_{n+1} = pB_n - B_{n-1}$ can be expressed as a matrix equation given as

$$\begin{bmatrix} B_{n+1} \\ B_n \end{bmatrix} = \begin{bmatrix} p & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} B_n \\ B_{n-1} \end{bmatrix}$$

and the matrix $\begin{bmatrix} p & -1 \\ 1 & 0 \end{bmatrix}$ is denoted as Q_{B_p}. Generalised Lucas Balancing Sequences are numbers satisfying the recurrence relation

$$C_{n+1} = pC_n - C_{n-1}$$

with $C_0 = 1$, $C_1 = p/2$ for p, a positive integer. When p is even we get C_n to be an integer sequence. We prove some results on the above Generalised Lucas Balancing Sequence.

Remark 2.1. *The definition of the Generalized Lucas Balancing sequences $C_{n+1} = pC_n - C_{n-1}$ can be extended to all integers.*

Theorem 2.2. *$C_{-n} = C_n$ for all $n \geq 1$.*

Theorem 2.3 ([5]). *$C_{m+n} = C_m C_n + (\frac{p^2-4}{4}) B_m B_n$ and $C_{m-n} = C_m C_n - (\frac{p^2-4}{4}) B_m B_n$ $\forall$ integers m, n.*

Remark 2.4. *We observe that $C_n = (\frac{p}{2}) B_n - B_{n-1} = \frac{B_{n+1} - B_{n-1}}{2}$.*

Definition 2.5. *We define Balancing R-Matrix as*

$$R_B = \begin{bmatrix} \frac{p}{2} & -1 \\ 1 & -\frac{p}{2} \end{bmatrix}$$

Remark 2.6. $R_B Q_{B_p}^n = \begin{bmatrix} C_{n+1} & -C_n \\ C_n & -C_{n-1} \end{bmatrix}$

Theorem 2.7. *$C_n^2 - pC_n C_{n-1} + C_{n-1}^2 + \frac{p^2-4}{4} = 0$ for all integers n.*

Proof. Follows from $det(R_B Q_{B_p}^n) = det\left(\begin{bmatrix} C_{n+1} & -C_n \\ C_n & -C_{n-1} \end{bmatrix}\right)$ □

Theorem 2.8. *For* $S = \begin{bmatrix} \frac{p}{2} & \frac{p^2-4}{4} \\ 1 & \frac{p}{2} \end{bmatrix}$, $S^n = \begin{bmatrix} C_n & \frac{p^2-4}{4}B_n \\ B_n & C_n \end{bmatrix}$

Theorem 2.9. $C_n^2 - \frac{p^2-4}{4}B_n^2 = 1$ *for all integers* n.

Proof. Follows from $det(S^n) = 1$. □

3. Some Relations of Generalised Balancing Sequences and Convergents of a Continued Fraction with Respect to Positive Integer p

For any positive integer D , if $\sqrt{D}$ can be written as continued fraction that is infinite and simple, given as

(i). $\sqrt{D} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 ..}}$ then it is denoted as $\sqrt{D} = [a_0, a_1, a_2, \ldots]$.

(ii). $\sqrt{D} = a_0 - \frac{1}{a_1 - \frac{1}{a_2 - ..}}$ then it is denoted as $\sqrt{D} = (a_0, a_1, a_2, \ldots)$.

For a non-negative integer n, the nth convergent of the continued fraction $[a_0, a_1, a_2, ...]$ is the real number $[a_0, a_1, a_2, ...a_{n-1}] = h_n/k_n$. The convergent in this case satisfy the recurrence relations give as

$$h_{-1} = 0, h_0 = a_0; h_{n+1} = a_{n+1}h_n + h_{n-1}, n \geq 0$$

$$k_{-1} = 0, k_0 = 1; k_{n+1} = a_{n+1}k_n + k_{n-1}, n \geq 0.$$

Similarly for n, non-negative integer, the $\mathrm{n^{th}}$ convergents of the continued fraction $(a_0, a_1, a_2, \ldots)$ is the real number $(a_0, a_1, a_2, ...a_{n-1}) = h_n/k_n$.The convergents in this case satisfy the recurrence relations give as

$$h_{-1} = 0, h_0 = a_0; h_{n+1} = a_{n+1}h_n - h_{n-1}, n \geq 0$$

$$k_{-1} = 0, k_0 = 1; k_{n+1} = a_{n+1}k_n - k_{n-1}, n \geq 0.$$

Let $p \geq 2$ be any integer then, $p^2 - 4$ is a real number and the infinite simple continued fraction of $\sqrt{p^2-4}$ is given as

$$\sqrt{p^2-4} = (p, \overline{(p+1)/2, 2, (p+1)/2, 2p}), \quad \text{when } p \text{ is odd}$$

$$\sqrt{p^2-4} = (p, \overline{p/2, 2p}), \quad \text{when } p \text{ is even.}$$

In the following theorems the convergents of $\sqrt{p^2-4}$ are expressed in terms of the Generalised Balancing sequence B_n.

Theorem 3.1. *For an odd positive integer* p*, if the nth convergent of the continued fraction* $\sqrt{p^2-4}$ *is* h_n/k_n *then for every non-negative integer* n *the following holds*

(i). $h_{8n} = B_{6n+2} - B_{6n}$ *and* $k_{8n} = B_{6n+1}$

(ii). $h_{8n+1} = 1/2[B_{6n+3} + B_{6n+2} - B_{6n+1} - B_{6n}]$ *and* $k_{8n+1} = 1/2[B_{6n+2} + B_{6n+1}]$

(iii). $h_{8n+2} = B_{6n+3} - B_{6n+1}$ *and* $k_{8n+2} = B_{6n+2}$

(iv). $h_{8n+3} = 1/2[B_{6n+4} - B_{6n+2}$ *and* $k_{8n+2} = 1/2B_{6n+3}$

(v). $h_{8n+4} = B_{6n+5} - B_{6n+3}$ *and* $k_{8n+4} = B_{6n+4}$

(vi). $h_{8n+5} = 1/2[B_{6n+6} + B_{6n+5} - B_{6n+4} - B_{6n+3}]$ *and* $k_{8n+5} = 1/2[B_{6n+5} + B_{6n+4}]$

(vii). $h_{8n+6} = B_{6n+6} - B_{6n+4}$ *and* $k_{8n+2} = B_{6n+5}$

(viii). $h_{8n+7} = 1/2[B_{6n+7} - B_{6n+5}]$ *and* $k_{8n+7} = 1/2B_{6n+6}$.

Proof. Given p is odd then by the continued fraction we have $\sqrt{p^2-4} = (p, \overline{(p+1)/2, 2, (p+1)/2, 2p})$ we have for $a_0 = p$, $a_1 = p + 1/2$, $a_2 = 2$, $a_3 = p + 1/2$, $a_4 = 2p$, $n \geq 1$.

$$\begin{bmatrix} h_n & k_n \\ h_{n-1} & k_{n-1} \end{bmatrix} = \begin{bmatrix} a_n & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} h_{n-1} & k_{n-1} \\ h_{n-2} & k_{n-2} \end{bmatrix}, n \geq 1.$$

Now taking $A_n = \begin{bmatrix} a_n & -1 \\ 1 & 0 \end{bmatrix}$ and $P_n = \begin{bmatrix} h_n & k_n \\ h_{n-1} & k_{n-1} \end{bmatrix}$ note we have $P_n = A_n P_{n-1}$, $n \geq 1$, and setting $N = A_4A_3A_2A_1$, then by induction we have for any positive integer t,

$$N^t = \begin{bmatrix} B_{3t+1} & -2B_{3t} \\ 1/2B_{3t} & -B_{3t-1} \end{bmatrix}.$$

We have

$$\begin{aligned} P_{4n} &= A_{4n}P_{4n-1} \\ &= A_{4n}A_{4n-1}P_{4n-2} \\ &= A_{4n}A_{4n-1}A_{4n-2}P_{4n-3} \\ &= A_{4n}A_{4n-1}A_{4n-2}A_{4n-3}P_{4n-4} \\ &= NP_{4n-4} \end{aligned}$$

therefore we have $P_{8n} = N^{2n}P_0$

$$P_{8n} = \begin{bmatrix} B_{6n+1} & -2B_{6n} \\ 1/2B_{6n} & -B_{6n-1} \end{bmatrix} \begin{bmatrix} p & 1 \\ 1 & 0 \end{bmatrix}$$

Now by definition of P_n as

$$P_{8n} = \begin{bmatrix} h_{8n} & k_{8n} \\ h_{8n-1} & k_{8n-1} \end{bmatrix}$$

we have $h_{8n} = pB_{6n+1} - 2B_{6n} = B_{6n+2} - B_{6n}$ and $k_{8n} = B_{6n+1}$. Similarly others can be proven. □

Theorem 3.2. *For an even positive integer p, if the nth convergent of the continued fraction* $\sqrt{p^2-4}$ *is* h_n/k_n *then for all non-negative integer n the following hold*

(i). $h_{2n} = B_{2n+2} - B_{2n}$ *and* $k_{2n} = B_{2n+1}$

(ii). $h_{2n+1} = 1/2[B_{2n+3} - B_{2n+1}]$ *and* $k_{2n+1} = 1/2B_{2n+2}$

Proof. Given p is even. We have the continued fraction for $\sqrt{p^2-4} = (p, \overline{p/2, 2p})$ with $a_0 = p, a_1 = p/2, a_2 = 2p$, then note $P_n = A_nP_{n-1}$ $\forall\, n \geq 1$ with $A_{2n} = \begin{bmatrix} 2p & -1 \\ 1 & 1 \end{bmatrix}$ and $A_{2n-1} = \begin{bmatrix} p/2 & -1 \\ 1 & 0 \end{bmatrix}$. We have $P_{2n} = A_{2n}P_{2n-1} = A_{2n}A_{2n-1}P_{2n-2}$. Now setting $M = A_2A_1$ by induction we get for all positive integer t,

$$M^t = \begin{bmatrix} B_{2t+1} & -2B_{2t} \\ 1/2B_{2t} & -B_{2t-1} \end{bmatrix}.$$

we have

$$\begin{aligned} P_{2n} &= MP_{2n-2} \\ &= M^n P_0 \\ P_n &= \begin{bmatrix} B_{2n+1} & -2B_{2n} \\ 1/2B_{2n} & -B_{2n-1} \end{bmatrix} \begin{bmatrix} p & 1 \\ 1 & 0 \end{bmatrix} \\ \Rightarrow h_{2n} &= pB_{2n+1} - 2B_{2n} = B_{2n+2} - B_{2n} \ \ and \ k_{2n} = B_{2n+1} \end{aligned}$$

□

4. Solutions of Diophantine Equations Generated by Generalized Lucas Balancing Sequences

In this section we first generate some Diophantine Equations from the property of Generalized Lucas Balancing Sequences. The property $C_n^2 - pC_nC_{n-1} + C_{n-1}^2 + \frac{p^2-4}{4} = 0$ of Generalized Lucas Balancing Sequences given in theorem(2.6) generates the polynomial $x^2 - pxy + y^2 + (\frac{p^2-4}{4}) = 0$ and $x^2 + pxy + y^2 + (\frac{p^2-4}{4}) = 0$ with $\big(C_n, C_{n-1}\big)$ and $\big(C_n, -C_{n-1}\big)$ as solutions respectively .Further extending the property,$C_n^2 - pC_nC_{n-1} + C_{n-1}^2 + \frac{p^2-4}{4} = 0$ it generates the Diophantine equations $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$, $x^2 + pxy + y^2 + (\frac{p^2-4}{4})x = 0$, $x^2 - pxy + y^2 - (\frac{p^2-4}{4})x = 0$ and $x^2 + pxy + y^2 - (\frac{p^2-4}{4})x = 0$ with (C_n^2, C_nC_{n-1}) , $(C_n^2, -C_nC_{n-1})$, $(-C_n^2, -C_nC_{n-1})$ and $(-C_n^2, C_nC_{n-1})$ respectively. In section 4.1 we investigate the solutions of each of the Diophantine Equations $x^2 \pm pxy + y^2 + (\frac{p^2-4}{4}) = 0$ in terms of Generalised Lucas Balancing Sequences. In section 4.2 we investigate the solutions of each of the Diophantine Equations $x^2 \pm pxy + y^2 \pm (\frac{p^2-4}{4})x = 0$ in terms of Generalised Lucas Balancing Sequences.

4.1. Solution of Diophantine Equation $x^2 \pm pxy + y^2 + (\frac{p^2-4}{4}) = 0$

In this section we show that the Diophantine equations $x^2 \pm pxy + y^2 + (\frac{p^2-4}{4}) = 0$ are solvable in integers for all positive p and obtain the solutions in terms of Generalized Lucas Balancing Sequence.

4.1.1. Existence of Solutions

Theorem 4.1. *For all non negative integer n, the following pairs satisfy the equation* $x^2 - pxy + y^2 + (\frac{p^2-4}{4}) = 0$

$$\begin{aligned} &\big(C_n, C_{n-1}\big) \\ &\big(C_{n-1}, C_n\big) \\ &\big(-C_n, -C_{n-1}\big) \\ &\big(-C_{n-1}, -C_n\big). \end{aligned}$$

Proof. Follows from Theorem 2.6. □

Each of the four formulas of solutions of $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$ define a class of solution. Now we prove that these four classes of solutions are only solutions.

4.1.2. The Four Classes of Solutions for each of the Diophantine Equations $x^2 \pm pxy + y^2 + (\frac{p^2-4}{4}) = 0$

In this section we solve the Diophantine Equation $x^2 - pxy + y^2 + (\frac{p^2-4}{4}) = 0$ in terms of Generalized Lucas-Balancing Sequence.We recall some properties of convergents in the following theorems.

Theorem 4.2 ([10]). *If the integer M satisfies $|M| < \sqrt{D}$ then any positive integer solution (s,t) of $x^2 - Dy^2 = M$ with $gcd(s,t) = 1$ satisfies $s = h_n$, $t = k_n$ where the n^{th} convergent of the infinite simple continued fraction, $\sqrt{D} = (a_0, a_1, a_2, \dots)$ is h_n/k_n for n a positive integer.*

Theorem 4.3 ([10]). *Let the infinite simple continued fraction of $\sqrt{D}$ be $(a_0, a_1, a_2, ...)$ and suppose that m_n and q_n are two sequences given by*

$$\begin{aligned} m_0 &= 0 \\ q_0 &= 1 \\ m_{n+1} &= a_n q_n + m_n \\ q_{n+1} &= (D - m_{n+1}^2)/q_n. \end{aligned}$$

Then

(a). m_n and q_n are integers for any positive integers n.

(b). $h_n^2 - Dk_n^2 = (-1)^{n+1} q_{n+1}$ for any integer $n \geq -1$.

Theorem 4.4. *(C_n, C_{n-1}) is a solution of the equation $x^2 - pxy + y^2 + (\frac{p^2-4}{4}) = 0$ when p is odd.*

Proof. Consider p to be an odd positive integer. Let (x,y) be any solution of $x^2 - pxy + y^2 + (\frac{p^2-4}{4}) = 0$ then there exists c, e, positive integers such that (x,y)=(c,e). Then on substituting (c,e) we have

$$c^2 - pce + e^2 + \frac{p^2-4}{4} = 0$$

This equation has integer solutions if and only if

$$\begin{aligned} \triangle &= p^2e^2 - 4(e^2 + \frac{p^2-4}{4}) \\ &= e^2(p^2-4) - (p^2-4) \ \text{ is a square} \end{aligned}$$

Therefore there is an integer t satisfying

$$\begin{aligned} \triangle = t^2 &= (p^2-4)(e^2-1) \\ t^2 - (p^2-4)(e^2-1) &= 0 \\ \frac{t^2}{p^2-4} - (e^2-1) &= 0 \\ \frac{t^2}{p^2-4} + 1 - (e^2-1) &= 1 \\ 4\left(\frac{t^2}{p^2-4} + 1\right) - 4(e^2-1) &= 4 \\ 4\left(\frac{t^2}{p^2-4} + 1\right) - 4\frac{(e^2-1)}{(p^2-4)}(p^2-4) &= 4 \\ (t')^2 - (p^2-4)(e')^2 &= 4 \end{aligned}$$

then we obtain

$$c = \frac{pe \pm t}{2},$$

by solving for (t', e') from the equation $(t')^2 - (p^2-4)(e')^2 = 0$. Now considering the continued fraction of $\sqrt{p^2-4}$ given as $\sqrt{p^2-4} = \big(p, \overline{(p+1)/2, 2, (p+1)/2, 2p}\big)$ with

$$\begin{aligned} a_0 &= p, \\ a_{4n-3} &= \frac{(p+1)}{2}, \\ a_{4n-2} &= 2, \\ a_{4n-1} &= \frac{p+1}{2}, \\ a_{4n} &= 2p, \end{aligned}$$

for $n \geq 0$. Now by the above theorem we have the periodic sequence give as

$$\{(-1)^{n+1} q_{n+1}\}_{n=-1}^{\infty} = \{1, \overline{4, p+2, 4, 1}\}.$$

Now assuming (t, e) is a positive solution of $t^2 - (p^2-4)e^2 = 4$ we have $(t, e) = (h_n, k_n)$ for some positive integers n by Theorem 3.3 and as $h_n^2 - Dk_n^2 = (-1)^{n+1} q_{n+1}$ by Theorem 3.4, we have by periodicity we have

$$\begin{aligned} h_{8n}^2 - (p^2-4)k_{8n}^2 &= (-1)^{8n+1} q_{8n+1} = 4 \\ h_{8n+2}^2 - (p^2-4)k_{8n+2}^2 &= (-1)^{8n+3} q_{4n+1} = 4 \\ h_{8n+3}^2 - (p^2-4)k_{8n+3}^2 &= (-1)^{8n+4} q_{8n+4} = 1 \\ h_{8n+4}^2 - (p^2-4)k_{8n+4}^2 &= (-1)^{8n+1} q_{8n+5} = 4 \\ h_{8n+6}^2 - (p^2-4)k_{8n+6}^2 &= (-1)^{8n+7} q_{8n+7} = 4 \\ h_{8n+7}^2 - (p^2-4)k_{8n+7}^2 &= (-1)^{8n+1} q_{8n+8} = 1 \end{aligned}$$

for all $n \geq 0$. Therefore all the solutions (t', e') of $(t')^2 - (p^2-4)(e')^2 = 4$ are

$$\begin{aligned} (t', e') &= (h_{8n}, k_{8n}) \\ &= (2h_{8n+3}, 2k_{8n+3}) \\ &= (h_{8n+6}, k_{8n+6}); \quad n \geq 0 \end{aligned}$$

Case 1: when $t' = h_{8n}$ and $e' = k_{8n}$

$$\begin{aligned} (e')^2 &= (k_{8n})^2 \\ 4\frac{(e^2-1)}{(p^2-4)} &= (B_{6n+1})^2 \\ e^2 - 1 &= \frac{(p^2-4)}{4} B_{6n+1}^2 \\ e^2 &= \frac{(p^2-4)}{4} B_{6n+1}^2 + 1 = C_{6n+1}^2 \\ e &= C_{6n+1} \end{aligned}$$

Now

$$(t')^2 = (h_{8n})^2$$

$$
\begin{aligned}
4\left(\frac{t^2}{p^2-4}+1\right) &= (B_{6n+2}-B_{6n})^2 = 4C_{6n+1}^2 \\
\frac{t^2}{p^2-4} &= C_{6n+1}^2 - 1 \\
t^2 &= \left(C_{6n+1}^2-1\right)\left(p^2-4\right) \\
t^2 &= \frac{\left(p^2-4\right)^2}{4}B_{6n+1}^2
\end{aligned}
$$

Case 2: when $t' = h_{8n+2}$ and $e' = k_{8n+2}$

$$
\begin{aligned}
(e')^2 &= (k_{8n+2})^2 \\
4\frac{(e^2-1)}{(p^2-4)} &= (B_{6n+2})^2 \\
e^2 - 1 &= \frac{(p^2-4)}{4}B_{6n+2}^2 \\
e^2 &= \frac{(p^2-4)}{4}B_{6n+2}^2 + 1 = C_{6n+2}^2 \\
e &= C_{6n+2}
\end{aligned}
$$

Now

$$
\begin{aligned}
(t')^2 &= (h_{8n})^2 \\
4\left(\frac{t^2}{p^2-4}+1\right) &= \left(B_{6n+3}-B_{6n+1}\right)^2 = 4C_{6n+2}^2 \\
\frac{t^2}{p^2-4} &= C_{6n+2}^2 - 1 \\
t^2 &= \left(C_{6n+2}^2-1\right)\left(p^2-4\right) \\
t^2 &= \frac{\left(p^2-4\right)^2}{4}B_{6n+2}^2
\end{aligned}
$$

Now for $c = \frac{pe+t}{2}$ we get $c = C_{6n+3}$. The solutions $(c, e) = (C_{6n+3}, C_{6n+2})$.

Case 3: when $t' = 2h_{8n+3}$ and $e' = 2k_{8n+3}$. We get the solutions $(c, e) = (C_{6n+4}, C_{6n+3})$.

Case 4: when $t' = h_{8n+4}$ and $e' = k_{8n+4}$. We get the solutions $(c, e) = (C_{6n+5}, C_{6n+4})$.

Case 5: when $t' = h_{8n+6}$ and $e' = k_{8n+6}$. We get the solutions $(c, e) = (C_{6n+6}, C_{6n+5})$.

Case 6: when $t' = 2h_{8n+7}$ and $e' = 2k_{8n+7}$. We get the solutions $(c, e) = (C_{6n+7}, C_{6n+6})$.

And finally as $(x, y) = (c, e)$ we obtain

$$
\begin{aligned}
(x,y) &= \left(C_{6n+2}, C_{6n+1}\right) \\
(x,y) &= \left(C_{6n+3}, C_{6n+2}\right) \\
(x,y) &= \left(C_{6n+4}, C_{6n+3}\right) \\
(x,y) &= \left(C_{6n+5}, C_{6n+4}\right) \\
(x,y) &= \left(C_{6n+6}, C_{6n+5}\right) \\
(x,y) &= \left(C_{6n+7}, C_{6n+6}\right)
\end{aligned}
$$

and therefore

$$(x,y) = \left(C_{2n}, C_{2n-1}\right); n \geq 1$$

or

$$(x,y) = \big(C_{2n+1}, C_{2n}\big); n \geq 1$$

Therefore $(x,y) = \big(C_n, C_{n-1}\big)$ is a solution of $x^2 - pxy + y^2 + (\frac{p^2-4}{4}) = 0$ when p is odd for $k > 2$ and also for $k = 1$. □

Theorem 4.5. *(C_n, C_{n-1}) is a solutions of the equation $x^2 - pxy + y^2 + (\frac{p^2-4}{4}) = 0$ when p is even.*

Proof. Consider p to be an even positive integer. Let (x,y) be any solution of $x^2 - pxy + y^2 + (\frac{p^2-4}{4}) = 0$ then there exists c, e, positive integers such that (x,y)=(c,e). Then on substituting (c,e) we have

$$c^2 - pce + e^2 + \frac{p^2-4}{4} = 0$$

This equation has integer solutions if and only if

$$\begin{aligned}\triangle &= p^2e^2 - 4(e^2 + \frac{p^2-4}{4})\\ &= e^2(p^2-4) - (p^2-4) \text{ is a square}\end{aligned}$$

Therefore there is an integer t satisfying

$$\begin{aligned}\triangle = t^2 &= (p^2-4)(e^2-1)\\ t^2 - (p^2-4)(e^2-1) &= 0\\ \frac{t^2}{p^2-4} - (e^2-1) &= 0\\ \frac{t^2}{p^2-4} + 1 - (e^2-1) &= 1\\ 4\Big(\frac{t^2}{p^2-4} + 1\Big) - 4(e^2-1) &= 4\\ 4\Big(\frac{t^2}{p^2-4} + 1\Big) - 4\frac{(e^2-1)}{(p^2-4)}(p^2-4) &= 4\\ (t')^2 - (p^2-4)(e')^2 &= 4\end{aligned}$$

then we obtain

$$c = \frac{pe \pm t}{2},$$

by solving for (t', e') from the equation $(t')^2 - (p^2-4)(e')^2 = 0$. As p be an even positive integer we have $\sqrt{p^2-4} = \big(p, \overline{\frac{p}{2}, 2p}\big)$. Let $a_0 = p$, $a_{2n+1} = \frac{p}{2}$, $a_{2n+2} = 2p$ $\forall\, n \geq 0$. We have by periodic sequence

$$\{(-1)^{n+1}q_{n+1}\}_{n=-1}^{\infty} = \{\overline{1,4}\}$$

and

$$h_{2n}^2 - (p^2-4)k_{2n}^2 = 4 \ \forall n \geq 0.$$

Moreover in this case all solutions of

$$(t')^2 - (p^2-4)(e')^2 = 4$$

are

Case 1:

$$(t', e') = (h_{2n}, k_{2n})$$

$$(c, e) = (\frac{pt + e}{2}, e)$$

But from the theorem if n is even then

$$c = C_{2n+2}$$
$$e = C_{2n+1}$$
$$(x, y) = (c, e) = \big(C_{2n+2}, C_{2n+1}\big).$$

Case 2:

$$(t', e') = (h_{2n+1}, k_{2n+1})$$
$$(c, e) = (\frac{pt + e}{2}, e)$$

But from the theorem if n is even then

$$c = C_{2n+3}$$
$$e = C_{2n+2}$$
$$(x, y) = (c, e) = \big(C_{2n+3}, C_{2n+2}\big).$$

Therefore the solution of $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$ is $(x, y) = \big(C_{2n}, C_{2n-1}\big)$ or $(x, y) = \big(C_{2n+1}, C_{2n}\big)$ ie $(x, y) = \big(C_n, C_{n-1}\big)$ is a solution for $x^2 - pxy + y^2 + (\frac{p^2-4}{4}) = 0$ for a positive even integer p, $k > 2$ and also for $k = 2$. ☐

Remark 4.6. *For the solution (x, y) of the equation $x^2 - pxy + y^2 + (\frac{p^2-4}{4}) = 0$ note $(-x, y)$ satisfies the equation $x^2 - pxy + y^2 + (\frac{p^2-4}{4}) = 0$. Therefore by repeating the above arguments we get for all non negative integer n, the four classes of solution of the equation $x^2 - pxy + y^2 + (\frac{p^2-4}{4}) = 0$ are*

$$\big(-C_n, C_{n-1}\big)$$
$$\big(C_{n-1}, -C_n\big)$$
$$\big(C_n, -C_{n-1}\big)$$
$$\big(-C_{n-1}, C_n\big).$$

4.2. Solution of Diophantine Equation $x^2 \pm pxy + y^2 \pm (\frac{p^2-4}{4})x = 0$

In this section we show that the Diophantine equations $x^2 \pm pxy + y^2 \pm (\frac{p^2-4}{4})x = 0$ are solvable in integers for all positive p and obtain the solutions in terms of Generalized Lucas Balancing Sequence.

4.2.1. Existence of Solutions

Lemma 4.7. *If the solution of $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$ is (x, y) then $(x, px - y)$ and $(py - x - \frac{p^2-4}{4}, y)$ are also solutions of the equation.*

Proof. Follows by the simple verification. ☐

Theorem 4.8. *For all non negative integer n, the following pairs satisfy the equation $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$*

$$\big(C_{2n}^2, C_{2n}C_{2n-1}\big)$$
$$\big(C_{2n}^2, C_{2n}C_{2n+1}\big)$$
$$\big(C_{2n+1}^2, C_{2n}C_{2n+1}\big)$$
$$\big(C_{2n+1}^2, C_{2n+1}C_{2n+2}\big).$$

Proof. First note $\big(C_{2n}^2, C_{2n}C_{2n-1}\big)$ satisfies $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$. On substituting $\big(C_{2n}^2, C_{2n}C_{2n-1}\big)$ in $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x$, we get

$$\begin{aligned}(C_{2n}^2)^2 - pC_{2n}^3C_{2n-1} + C_{2n}^2C_{2n-1}^2 + (\frac{p^2-4}{4})C_{2n}^2 &= C_{2n}^2\big[C_{2n}^2 - pC_{2n}C_{2n-1} + C_{2n-1}^2 + (\frac{p^2-4}{4})\big] \\ &= C_{2n}^2\big[C_{2n}^2 - C_{2n-1}\big(pC_{2n} - C_{2n-1}\big) + (\frac{p^2-4}{4})\big] \\ &= C_{2n}^2\big[C_{2n}^2 - C_{2n-1}C_{2n+1} + (\frac{p^2-4}{4})\big] \\ &= C_{2n}^2.0 \\ &= 0\end{aligned}$$

Therefore $(x, y) = \big(C_{2n}^2, C_{2n}C_{2n-1}\big)$ is a solution of $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$. Now by above lemma note $(x, px - y)$ satisfies the Diophantine equation $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$ and we have

$$\begin{aligned}(x, px - y) &= \big(C_{2n}^2, pC_{2n}^2 - C_{2n}c_{2n-1}\big) \\ &= \big(C_{2n}^2, C_{2n}[pC_{2n} - C_{2n-1}]\big) \\ &= \big(C_{2n}^2, C_{2n}C_{2n+1}\big)\end{aligned}$$

Therefore $(x, y) = \big(C_{2n}^2, C_{2n}C_{2n+1}\big)$ is a solution of $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$. Now for $(x, y) = \big(C_{2n}^2, C_{2n}C_{2n+1}\big)$ as it is a solution of $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$, again by above lemma $(py - x - \frac{p^2-4}{4}, y)$ is also a solution of the Diophantine equation $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$ and we have $(py - x - \frac{p^2-4}{4}, y) = \big(C_{2n+1}^2, C_{2n}C_{2n+1}\big)$. Therefore $(x, y) = \big(C_{2n+1}^2, C_{2n}C_{2n+1}\big)$ is a solution of $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$.
Similarly, as $\big(C_{2n+1}^2, C_{2n}C_{2n+1}\big)$ satisfies $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$ then $(x, px - y)$ is also a solution of the Diophantine equation $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$ and we have $(x, px - y) = \big(C_{2n+1}^2, C_{2n+1}C_{2n+2}\big)$. Therefore $(x, y) = \big(C_{2n+1}^2, C_{2n+1}C_{2n+2}\big)$ is a solution of $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$. □

Each of the four formulas for solution of $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$, as in the above theorem, defines a class of solutions. We prove that these four classes of solutions are the only solutions for the Diophantine equation $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$ in the following section.

4.2.2. The Four Classes of Solutions for each of the Diophantine Equations $x^2 \pm pxy + y^2 \pm (\frac{p^2-4}{4})x = 0$

In this section, we prove that the four classes of solutions obtained in the above are the only solutions of the Diophantine equation $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$.

Theorem 4.9. *If positive integers p, x and y satisfy the equations $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$ then there exists c, e such that $(x, y) = (c^2, ce)$ with $gcd(c, e) = 1$, where c and e are positive integers.*

Proof. Similar to Theorem (1) in [7]. □

Theorem 4.10. *For an odd positive integer p, every positive solution of $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$ is of the form $\left(C_{2n}^2, C_{2n-1}C_{2n}\right)$.*

Proof. Consider p to be an odd positive integer. Let (x, y) be any solution of $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$ which is positive then by Theorem 4.4 above note that there exists c and e, positive integers such that $(x, y) = (c^2, ce)$ with $gcd(c, e) = 1$. Then on substituting (c^2, ce) we have

$$c^4 - pc^3e + c^2e^2 + (\frac{p^2-4}{4})c^2 = 0$$
$$c^2 - pce + e^2 + (\frac{p^2-4}{4}) = 0$$

We have $(c, e) = \left(C_{2n}, C_{2n-1}\right)$. Therefore $(x, y) = (c^2, ce) = \left(C_{2n}^2, C_{2n-1}C_{2n}\right)$. □

Theorem 4.11. *For an even positive integer p, every positive solution (x, y) of $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$ is of the form $\left(C_{2n}^2, C_{2n-1}C_{2n}\right)$.*

Proof. Consider p to be an even positive integer. Let (x, y) be any solution of $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$ which is positive then by Theorem 4.5 above note that there exists c and e, positive integers such that $(x, y) = (c^2, ce)$ with $gcd(c, e) = 1$. Then on substituting (c^2, ce) we have

$$c^4 - pc^3e + c^2e^2 + (\frac{p^2-4}{4})c^2 = 0$$
$$c^2 - pce + e^2 + (\frac{p^2-4}{4}) = 0$$

We have $(c, e) = \left(C_{2n}, C_{2n-1}\right)$. Therefore for a positive even integer p, every positive solution of $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$ is of the form $(x, y) = (c^2, ce) = \left(C_{2n}^2, C_{2n-1}C_{2n}\right)$. □

Theorem 4.12. *For a positive integer p, all the solutions of $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$ are*

(1). $\left(C_{2n}^2, C_{2n-1}C_{2n}\right)$

(2). $\left(C_{2n}^2, C_{2n}C_{2n+1}\right)$

(3). $\left(C_{2n+1}^2, C_{2n}C_{2n+1}\right)$

(4). $\left(C_{2n+1}^2, C_{2n+1}C_{2n+2}\right)$ *for all integers $n \geq 0$.*

Proof. Let p be any integer and (x, y) be any solution of $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$. Then (x, y) has to be $(0, 0)$, $((\frac{p^2-4}{4}), 0)$, positive solution or non-positive solution. If (x, y) is a solution that is non-positive then it is of the form

(i). $x > 0, y < 0$ or

(ii). $x < 0, y < 0$ or

(iii). $x < 0, y > 0$.

Case (i): If the solution (x, y) is as in (i) with $x > 0$ and $y < 0$; note by taking $x' = x$ and $y' = -y$ we have (x', y') is a solution of the $x^2 + pxy + y^2 + (\frac{p^2-4}{4})x = 0$ that is positive.

Case (ii): If (x, y) is as in (ii) with $x < 0$ and $y < 0$; then by taking $x' = -x$ and $y' = -y$ we have (x', y') is a solution of $x^2 - pxy + y^2 - (\frac{p^2-4}{4})x = 0$ that is positive.

Case (iii): If (x, y) is as in (iii) with $x < 0$ and $y > 0$; then by taking $x' = -x$ and $y' = y$ we have (x', y') is a solution of $x^2 + pxy + y^2 - (\frac{p^2-4}{4})x = 0$ that is positive.

Therefore the solution (x, y), that are positive, are the only solution of $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$. Now note if (x, y) is any positive solutions then by Theorem 3.7 and Theorem 3.2 we get that

(1). $\left(C_{2n}^2, C_{2n-1}C_{2n}\right)$

(2). $\left(C_{2n}^2, C_{2n}C_{2n+1}\right)$

(3). $\left(C_{2n+1}^2, C_{2n}C_{2n+1}\right)$

(4). $\left(C_{2n+1}^2, C_{2n+1}C_{2n+2}\right)$ are all the solutions of $x^2 - pxy + y^2 + (\frac{p^2-4}{4})x = 0$.

□

The above theorem in general classifies all the solutions of the equations $x^2 \pm pxy + y^2 \pm (\frac{p^2-4}{4})x = 0$ as shown in the following theorems.

Theorem 4.13. *For a positive integer p all the solutions of $x^2 + pxy + y^2 + (\frac{p^2-4}{4})x = 0$ are*

(1). $\left(C_{2n}^2, -C_{2n-1}C_{2n}\right)$

(2). $\left(C_{2n}^2, -C_{2n}C_{2n+1}\right)$

(3). $\left(C_{2n+1}^2, -C_{2n}C_{2n+1}\right)$

(4). $\left(C_{2n+1}^2, -C_{2n+1}C_{2n+2}\right)$ *for all integers $n \geq 0$.*

Theorem 4.14. *For a positive integer p all the solutions of $x^2 - pxy + y^2 - (\frac{p^2-4}{4})x = 0$ are*

(1). $\left(-C_{2n}^2, -C_{2n-1}C_{2n}\right)$

(2). $\left(-C_{2n}^2, -C_{2n}C_{2n+1}\right)$

(3). $\left(-C_{2n+1}^2, -C_{2n}C_{2n+1}\right)$

(4). $\left(-C_{2n+1}^2, -C_{2n+1}C_{2n+2}\right)$ *for all integers $n \geq 0$.*

Theorem 4.15. *For a positive integer p all the solutions of $x^2 + pxy + y^2 - (\frac{p^2-4}{4})x = 0$ are*

(1). $\left(-C_{2n}^2, C_{2n-1}C_{2n}\right)$

(2). $\left(-C_{2n}^2, C_{2n}C_{2n+1}\right)$

(3). $\left(-C_{2n+1}^2, C_{2n}C_{2n+1}\right)$

(4). $\left(-C_{2n+1}^2, C_{2n+1}C_{2n+2}\right)$ *for all integers $n \geq 0$.*

Remark 4.16. *It is observed that for any positive solution (x, y) of any of the equations $x^2 \pm pxy + y^2 \pm (\frac{p^2-4}{4})y = 0$, the interchanged pair (y, x) is a positive solution of the corresponding equations $x^2 \pm pxy + y^2 \pm (\frac{p^2-4}{4})x = 0$ and vice versa. Hence by the above arguments the solutions of $x^2 \pm pxy + y^2 \pm (\frac{p^2-4}{4})y = 0$ also can be obtained in terms of Generalised Lucas Balancing Sequences.*

5. Conclusion

In this paper we discussed some properties of the Generalised Lucas Balancing Sequences $C_{n+1} = pC_n - C_{n-1}$. We considered the Diophantine Equations $x^2 \pm pxy + y^2 \pm (\frac{p^2-4}{4})x = 0$ that are generated by the property of Generalised Lucas Balancing Sequences and obtained all the solutions of each of the equation expressed in terms of Generalised Lucas Balancing Sequences.

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