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One Point Union of $\alpha-$graceful Graphs

Research Article

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Abstract: The notation of α-labeling of a graph is the natural generalization of graceful labeling and it helps produce new graceful graphs by some graph operations. In this paper, we have proved that one point union of two α-graceful graphs is α-graceful under some conditions, one point union of an α-graceful graph and a graceful graph is graceful. We have also proved that the consecutive one point union of a finite number of α-graceful graphs with a graceful graph by merging one vertex under some conditions is also a graceful graph. These results help enlarge the class of graceful graphs.

MSC: 05C78.

Keywords: Graceful graph, $\alpha-$graceful graph, one point union of two graphs.

1. Introduction

Most graph labeling methods trace their origin α-labeling as a graceful labeling with an additional property that there exit a non-negative integer $k(0 \leq k < |E(G)|)$ such that for each edge $e = (x, y) \in E(G)$, $\min\{f(x), f(y)\} \leq k < \max\{(f(x), f(y)\}$ in a graph G. Here f is a graceful labeling for a graph G and it becomes α- labeling if it satisfies above property. A graph G which admits an α-labeling, we call α-graceful graph. An α-graceful graph is necessarily bipartite. This was introduced by Rosa [6]. Kaneria and Jariya [3] defined smooth graceful labeling and semi smooth graceful labeling. In [5] Kaneria, Meera and M. Khoda have proved that smooth graceful labeling, semi smooth graceful labeling are equivalent with α-graceful labeling. Kaneria, Viradia and Makadia [4] have proved that the path union of a semi smooth graceful graph, star of a semi smooth graceful graph and cycle of a semi smooth graceful graph are graceful. For a comprehensive bibliography of papers on graph labeling are given in Gallian [2]. In this paper, all graphs are finite, simple and undirected and we consider G as a finite simple, undirected graph with $|V(G)| = p$ vertices and $|E(G)| = q$ edges. For all terminology and standard notation we follow by Harary [1]. The present paper is focused to discuss one point union of some graphs and its α-graceful labeling as well as graceful labeling. One point of union of two graphs G and H is the graph $G \cup H$ obtain by merging one vertex of G with one vertex of H as a common vertex of $G \cup H$.

2. Main Results

Theorem 2.1. *Let G_1, G_2 be α-graceful graphs and $|E(G_1)| = q_1, |E(G_2)| = q_2$. Let f_1, f_2 be α-graceful labeling (α-labeling) for G_1 and G_2 respectively. Let k_1 $(0 \leq k1 < q_1)$, k_2 $(0 \leq k_2 < q_2)$ be two non-negative integers such that for every $e_i =$*

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$(x_i, y_i) \in E(G_i)$,$min\{f_i(x_i), f_i(y_i)\} \leq k_i < max\{(f_i(x_i), f_i(y_i)\}$, $\forall\ i = 1, 2$. Then one point union of G_1 and G_2 by merging vertex $v \in V(G_1)$ with $f_1(v) = k_1$ and vertex $w \in V(G_2)$ with $f_2(w) = 0$ is also an α-graceful graph.

Proof. Take, $V_1^{(i)} = \{u \in V(G_i)/f_i(u) \leq k_i\}$ and $V_2^{(i)} = V(G_i) - V_1^{(i)}$, $\forall\ \ i = 1, 2$. Define a vertex labeling function $h : V(G_1 \cup G_2) \longrightarrow \{0, 1, \dots, q_1 + q_2\}$ as follows.

$$h/V_1^{(1)} = f_1/V_1^{(1)}$$

$$h/V(G_2) = f_2/V(G_2) + k_1 \ \text{ and}$$

$$h/V_2^{(1)} = f_1/V_2^{(1)} + q_2.$$

Since f_1 and f_2 both are injective maps and for the vertex $v \in V(G_1) \cap V(G_2)$, $h(v) = f_1(v) = k_1 = f_2(v) + k_1$, by definition of h, it is also an injective map.
Since $f_2^* = h^*/E(G_2)$ for any $e_1 = (x_1,\ y_1) \in E(G_1)$, $\min\{f_1(x_1), f_1(y_1)\} \leq k_1 < \max\{(f_1(x_1), f_1(y_1)\}$, $f_1^*(e_1) = |f(x_1) - f(y_1)| = |h(x_1) - h(y_1)| - q_2$ (i.e. $f_1^* + q_2 = h^*/E(G_1)$) and f_1^*, f_2^* both are bijective maps, h is also a bijective map. It is obvious that for each $e = (a, b) \in E(G_1 \cup G_2)$, $h(a) = f_1(a) + q_2$ or $h(a) = f_2(a) + k_1$ both are greater than $k_1 + k_2$ and $h(b) = f_1(b)$ or $f_2(b) + k_1$ both are less than or equal to $k_1 + k_2$ (assuming $f_i(a) - f_i(b)$ is positive, $\forall\ \ i = 1, 2$). Hence, we find $k = k_1 + k_2$ $(0 \leq k < q_1 + q_2)$, a non-negative integer such that for each $e = (a, b) \in E(G_1 \cup G_2)$, $h(b) \leq k < h(a)$ with above assumption.
Thus, one point union of G_1 and G_2 by merging vertex $v \in V(G_1)$ with $f_1(v) = k_1$ and vertex $w \in V(G_2)$ with $f_2(w) = 0$ is an α-graceful graph. ☐

Theorem 2.2. *Let G_1 be an α-graceful graph, G_2 be a graceful graph and $|E(G_1)| = q_1$, $|E(G_2)| = q_2$. Let f_1 be an α-graceful labeling for G_1 and f_2 be a graceful labeling for G_2. Let k_1 $(0 \leq k_1 < q_1)$ be a non-negative integer such that for every $e_1 = (x_1, y_1) \in E(G_1)$, $min\{f_1(x), f_1(y)\} \leq k_1 < \max\{(f_1(x), f_1(y)\}$. Then one point union of G_1 and G_2 by merging vertex $v \in V(G_1)$ with $f_1(v) = k_1$ and vertex $w \in V(G_{2)}$ with $f_2(w) = 0$ is also a graceful graph.*

Proof. Take, $V_1 = \{u \in V(G_1)/f_1(u) \leq k_1\}$ and $V_2 = V(G_1) - V_1$. Define labeling function $h : V(G_1 \cup G_2) \longrightarrow \{0, 1 \dots, q_1 + q_2\}$ as follows.

$$h/V_1 = f_1/V_1$$

$$h/V(G_2) = f_2/V(G_2) + k_1 \quad \text{and}$$

$$h/V_2 = f_1/V_2 + q_2.$$

Since f_1 and f_2 both are injective maps and for the vertex $v \in V(G_1) \cap V(G_2)$, $h(v) = f_1(v) = k_1 = f_2(v) + k_1$, by definition of h, it is also an injective map.
Since $f_2^* = h^*$, $f_1^* + q_2 = h^*$ both are bijective maps, h^* is also a bijective map. Thus, $h : V(G_1 \cup G_2) \longrightarrow \{0, 1 \dots, q_1 + q_2\}$ is graceful labeling. Thus, one point union of G_1 and G_2 by merging vertex $v \in V(G_1)$ with $f_1(v) = k_1$ and vertex $w \in V(G_{2)}$ with $f_2(w) = 0$ is a graceful graph. ☐

Theorem 2.3. *Let $G_1, G_2, \dots, G_l$ be α-graceful graphs, and $|E(G_i)| = q_i$, $\forall\ i = 1, 2, \dots, l$. Let f_i be an α-graceful labeling for G_i, $\forall\ i = 1, 2, \dots, l$. Let k_i $(0 \leq k_i < q_i)$, $i = 1, 2, \dots, l$ be a non-negative integer such that for every $e_i = (x_i, y_i)$ $\in E(G_i)$, $min\{f_i(x), f_i(y)\} \leq k_i < max\{(f_i(x_i), f_i(y_i)\}$, $\forall\ i = 1, 2, \dots, l$. Let H be a graceful graph with a graceful labeling function $f_{l+1} : V(H) \longrightarrow \{0, 1, \dots, q_{l+1}\}$, where $q_{l+1} = |E(H)|$. Then the consecutive one point union of $G_1, G_2, \dots, G_l$ and H by merging vertex $v_i \in V(G_i)$ with $f_i(v_i) = k_i$ and vertex $w_i \in V(G_{i+1)}$ with $f_{i+1}(w_i) = 0, \forall\ i = 1, 2, \dots, l$ (assuming $G_{l+1} = H$) is also a graceful graph.*

Proof. Take, $V_1^{(i)} = \{u \in V(G_i)/f_i(u) \leq k_i\}$ and $V_2^{(i)} = V(G_i) - V_1^{(i)}$, $\forall \ i = 1, 2, \ldots, l$. Define a labeling function $h_i : V(\bigcup_{j=1}^{i+1} G_j) \longrightarrow \{0, 1 \ldots, \sum_{j=1}^{i+1} q_j\}$, $\forall \ i = 1, 2, \ldots, l-1$ as follows.

$$\begin{aligned} h_i(w) &= h_{i-1}(w), \text{ when } h_{i-1}(w) \leq \sum_{j=1}^{i} k_j \\ &= h_{i-1}(w) + q_{i+1}, \text{ when } h_{i-1}(w) > \sum_{j=1}^{i} k_j, \ \forall \ w \in V(\bigcup_{j=1}^{i} G_j) \\ h_i(w') &= f_{i+1}(w') + \sum_{j=1}^{i} k_j, \ \forall \ w' \in V(G_{i+1}), \text{ by assuming } h_0 = f_1. \end{aligned}$$

By applying Theorem 2.1, h_i $(i = 1, 2, \ldots, l-1)$ are graceful labeling for $\bigcup_{j=1}^{i+1} G_j$. Moreover, for each $e \in (a, b) \in E(\bigcup_{j=1}^{i+1} G)$, $h_i(a)$ = either $h_{i-1}(a) + q_{i+1}$ or $f_{i+1}(a) + \sum_{j=1}^{i+1} k_j$ these both are greater than $\sum_{j=1}^{i+1}$ and $h_i(b)$ = either $h_{i-1}(b)$ or $f_{i+1}(b) + \sum_{j=1}^{i} k_j$ these both are less than or equal to $\sum_{j=1}^{i+1} k_j$ (assuming $f_{i+1}(a) - f_{i+1}(b)$, $f_1(a) - f_1(b)$ are positive and $h_0 = f_1$) for each $i = 1, 2, \ldots, l-1$.
Thus, $\sum_{j=1}^{i+1} k_j$ is a non-negative integer and for each $e = (a, b) \in E(\bigcup_{j=1}^{i+1} G_j)$, $h_i(b) \leq \sum_{j=1}^{i+1} k_j < h_i(a)$, $\forall \ i = 1, 2, \ldots, l-1$.
Therefore, for each $i = 1, 2, \ldots, l-1$, $h_i : V(\bigcup_{j=1}^{i+1} G_j) \longrightarrow \{0, 1, \ldots, \sum_{j=1}^{i+1} q_j\}$ is α- graceful labeling for the graph $\bigcup_{j=1}^{i+1} G_j$.
Now define $h_l : V(\bigcup_{j=1}^{l} G_j \cup H) \rightarrow \{0, 1 \ldots, \sum_{j=1}^{l+1} q_j\}$ as follows.

$$\begin{aligned} h_l(w) &= h_{l-1}(w) \text{ when } h_{l-1}(w) \leq \sum_{j=1}^{l} k_j \\ &= h_{l-1}(w) + q_{l+1}, \text{ when } h_{l-1}(w) > \sum_{j=1}^{l} k_j \ \forall \ w \in V(\bigcup_{j=1}^{l} G_j); \\ h_l(w') &= f_{l+1}(w') + \sum_{j=1}^{l} k_j \ \forall \ w' \in V(H). \end{aligned}$$

By applying Theorem 2.2, h_l is a graceful labeling for $\bigcup_{j=1}^{l} G_j \cup H$. So, it is a graceful graph. □

Illustration 2.4. *One point union of C_8, $K_{4,3}$ and $K_{3,4} \cup P_7$ is a graceful graph.*

Here, take $V(C_8) = \{v_1, v_2, \ldots, v_8\}$, $V(K_{4,3}) = \{u_1, u_2, u_3, u_4, w_1, w_2, w_3\}$. Define a α-labeling function $f_1 : V(C_8) \longrightarrow \{0, 1 \ldots, 8\}$ as follows.

$$\begin{aligned} f_1(V_i) &= 9 - i, \ \forall \ i = 1, 3, 5, 7 \\ &= \frac{i-2}{2}, \ \forall \ i = 2, 4 \\ &= \frac{i}{2}, \ \forall \ i = 6, 8 \text{ is an } \alpha - \text{labeling for } C_8 \text{ and } K_1 = 4. \end{aligned}$$

Define a α-labeling function $f_2 : V(K_{4,3}) \longrightarrow \{0, 1 \ldots, 12\}$ as follows.

$$\begin{aligned} f_2(u_i) &= i - 1, \ \forall \ i = 1, 2, 3, 4 \\ &= 12 - 4(j-1), \ \forall \ j = 1, 2, 3 \end{aligned}$$

is also an α-labeling for $K_{4,3}$ and $k_2 = 3$. $K_{3,4} \cup P_7$ is a graceful graph with graceful labeling function f_3 given in following Figure 1.

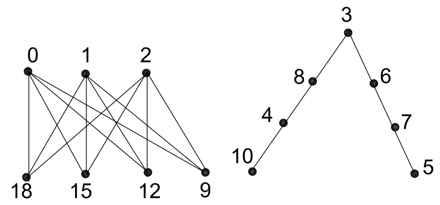

Figure 1. **graceful labeling function f_3 for $K_{3,4} \cup P_7$.**

It is obvious that $h_1 : V(C_8 \cup K_{4,3}) \longrightarrow \{0, 1, \ldots, 20\}$ by merging the vertex v_8 with the vertex u_1 as $f_1(v_8) = k_1 = 4$ and $f_2(u_1) = 0$ defined by

$$\begin{aligned} h_1(u_i) &= f_l(v_i) + 12 = 21 - i, \ \ \forall \ \ i = 1,3,5,7 \\ &= f_l(v_i) \ \ \forall \ \ i = 2,4,6,8 \\ h_1(u_i) &= f_2(u_i) + 4 = i + 3 \ \ \forall \ \ i = 1,2,3,4 \\ h_1(w_j) &= f_2(w_j) + 4 = 16 - 4(j-1) \ \ \forall \ \ j = 1,2,3; \end{aligned}$$

is an α-graceful labeling for $C_8 \cup K_{4,3}$ with the non-negative integer $k_1 + k_2 = 7$. $h_2 : V(C_8 \cup K_{4,3} \cup K_{3,4} \cup P_7) \longrightarrow \{0, 1, \ldots, 38\}$ by merging the vertex u_4 with vertex v of $K_{3,4} \cup P_7$ where $f_3(v) = 0$, as $h_1(u_4) = 7$ defined by

$$\begin{aligned} h_2(v_i) &= h_l(v_i) + 18 = 39 - i, \ \ \forall \ \ i = 1,3,5,7 \\ &= \ \ h_l(v_i) = f_1(v_i) \ \ \forall \ \ i = 2,4,6,8 \\ h_2(u_i) &= \ \ h_1(u_i) = f_2(u_i) + 4 \ \ \forall \ \ i = 1,2,3,4 \\ h_2(w_j) &= h_1(w_j) = f_2(w_j) + 4 \ \ \forall \ \ j = 1,2,3 \\ h_2(w') &= f_3(w^{'}) + 7, \ \ \forall \ \ w' \in V(K_{3,4} \cup P_7); \end{aligned}$$

is a graceful labeling for required graph $C_8 \cup K_{4,3} \cup (K_{3,4} \cup P_7)$ by merging the vertex v_8 with the vertex u_1 and the vertex u_4 with the vertex of $K_{3,4} \cup P_7$ whose vertex label is 0 under f_3. Such α-graceful labeling for $C_8 \cup K_{4,3}$ and graceful labeling for the required graph $C_8 \cup K_{4,3} \cup (K_{3,4} \cup P_7)$ are shown in following figure - $2, 3$ respectively.

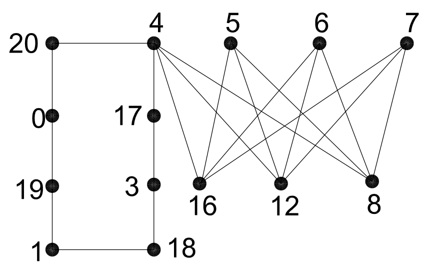

Figure 2. **α-graceful labeling of $C_8 \cup K_{4,3}$ [here $k = 7$]**

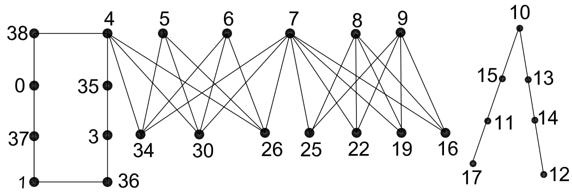

Figure 3. **Graceful labeling for one point union of graphs $C_8, K_{4,3}$ and $(K_{3,4} \cup P_7)$.**

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