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Analysis of Primes in Arithmetical Progressions $7n + k$ up to a Trillion

Research Article

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Abstract: Each prime number fits in one of six unique forms $7n+k$. Each form $7n+k$, for $1 \leq k \leq 6$ forms an arithmetical progression; each containing, as assured by Dirichlet's Theorem, infinite number of primes. This work analyzes occurrences of primes in all of these arithmetical progressions from different angles, both 10 power blockwise as well as absolutely in ranges going as high as $1,000,000,000,000$.

MSC: 11A41, 11N05, 11N25

Keywords: Arithmetical progressions, block-wise distribution, prime, prime density, prime spacing

1. Introduction

As is well known, a prime number is a positive integer greater than 1 having only two positive integral divisors, viz., 1 and itself. They happen to be infinite in number [1].

2. Primes Distributions

Irregularity in the distribution of prime numbers amongst the natural numbers is very clear by their often-quoted two properties: On one side it is strongly conjectured that there are infinitely many twin primes, those successive prime pairs which are very much close, accommodating only one composite number inbetween and on the other side, the proved fact that there are successive primes with as high gaps between them as desired.

One parameter of measurement of prime distribution is the number of primes less than or equal to x. It is denoted by $\pi(x)$.

3. Primes Distributions in Arithmetical Progressions

An arithmetical progression is integer sequence of form $an+b$, with a and b fixed integers and n varying over all non-negative integers. If we take a as a positive integer and let b take values from 0 to $a-1$, then the a number of arithmetical progressions $an + k$, with $0 \leq k < a$ generate and every integer fits in one and only one of them.

For any fixed a, every prime number also occurs in some or other arithmetical progression $an + b$; wherein we are interested in how many of them will be in each such progression. As there are infinitely many primes, for every fixed positive integer

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a, at least one of the progressions can contain infinitely many primes. Dirichlet [2] elegantly proved that every arithmetical progression $an + b$ satisfying the condition $gcd(a, b) = 1$ contains infinitely many primes.

Already in [4], [5], [6], and [7], the symbol $\pi_{a,b}(x)$ is introduced to represent the number of primes in an arithmetical progression $an + b$ that are less than or equal to x.

4. Primes Distributions in Arithmetical Progressions $7n + k$

After dividing any positive integer by a positive integer m, the process of integer division gives one of the integers $0, 1, 2, \cdots, m - 1$ as remainders. Taking $m = 7$ gives remainders in the process of division by 7 to be $0, 1, 2, 3, 4, 5$ and 6. As each positive integer after dividing by 7 yields one and only one amongst these values as remainder, each integer must be of either of the forms $7n + 0 = 7n$ or $7n + 1$ or $7n + 2$ or $7n + 3$ or $7n + 4$ or $7n + 5$ or $7n + 6$, which are arithmetical progressions under consideration here.

First few numbers of the form $7n$ are

$$7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, \cdots$$

Each one of them is perfectly divisible by 7. Except the first member, viz., 7, none of these is prime. This sequence contains only one prime 7 with all its other members being composite numbers. $7n$ when seen as arithmetical progression $7n + 0$, with $gcd(7, 0) = 7 > 1$, is not a candidate for occurrence of any more primes by Dirichlet's Theorem also.

First few numbers of the form $7n + 1$ are

$$1, 8, 15, 22, 29, 36, 43, 50, 57, 64, 71, 78, \cdots$$

This contains infinitely many primes as $gcd(7, 1)$ is 1 as per requirement of Dirichlet's Theorem.

First few numbers of the form $7n + 2$ are

$$2, 9, 16, 23, 30, 37, 44, 51, 58, 65, 72, 79, \cdots$$

This sequence also contains infinitely many primes as $gcd(7, 2)$ is 1 as per requirement of Dirichlet's Theorem.

First few numbers of the form $7n + 3$ are

$$3, 10, 17, 24, 31, 38, 45, 52, 59, 66, 73, 80, \cdots$$

This one also has infinitely many primes in it as $gcd(7, 3)$ is 1 as per need of Dirichlet's Theorem.

First few numbers of the form $7n + 4$ are

$$4, 11, 18, 25, 32, 39, 46, 53, 60, 67, 74, 81, \cdots$$

This sequence also does contain infinitely many primes as $gcd(7, 4)$ is 1 as required by Dirichlet's Theorem.

First few numbers of the form $7n + 5$ are

$$5, 12, 19, 26, 33, 40, 47, 54, 61, 68, 75, 82, \cdots$$

This one also does contain infinitely many primes as $gcd(7, 5)$ is 1 as demanded by Dirichlet's Theorem.

First few numbers of the form $7n + 6$ are

$$6, 13, 20, 27, 34, 41, 48, 55, 62, 69, 76, 83, \cdots$$

This progression also contains infinitely many primes as $gcd(7,6)$ is 1 as necessitated by Dirichlet's Theorem.

Like for some arithmetical progressions given in [8], independent proofs about infinitude of primes in each of these arithmetical progressions can separately be provided.

We present here a comparative analysis of the prime numbers contained in arithmetical progressions $7n+1$, $7n+2$, $7n+3$, $7n+4$, $7n+5$ and $7n+6$.

5. Prime Number Race

For any fixed positive integer a and all integers b, $0 \leq b < a$, all the arithmetical progressions $an+b$ which contain infinitely many primes are compared to decide which one amongst them contains more number of primes. This is term well-known as prime number race [9].

Here we have compared dominance of the number of primes of form $7n+1$, $7n+2$, $7n+3$, $7n+4$, $7n+5$ and $7n+6$ till one trillion, i.e., $1,000,000,000,000$ (10^{12}). The huge prime database could be made available in minimum time by use of best of the algorithms obtained by exhaustive comparisons in [10], [11], [12], [13], [14], [15] with betterment estimates in [16]. Java Programming Language [17], was chosen to implement these best algorithms on electronic computers to analyze big range of primes thoroughly.

Table 1: Number of Primes of form $7n+k$ in First Blocks of 10 Powers.

Sr. No.	Range $1-x$ (1 to x)	Number of Primes of Form					
		$7n+1$ ($\pi_{7,1}(x)$)	$7n+2$ ($\pi_{7,2}(x)$)	$7n+3$ ($\pi_{7,3}(x)$)	$7n+4$ ($\pi_{7,4}(x)$)	$7n+5$ ($\pi_{7,5}(x)$)	$7n+6$ ($\pi_{7,6}(x)$)
1	1-10	0	1	1	0	1	0
2	1-100	3	4	5	3	5	4
3	1-1,000	28	27	30	26	29	27
4	1-10,000	203	203	209	202	211	200
5	1-100,000	1,593	1,584	1,613	1,601	1,604	1,596
6	1-1,000,000	13,063	13,065	13,105	13,069	13,105	13,090
7	1-10,000,000	110,653	110,771	110,815	110,776	110,787	110,776
8	1-100,000,000	960,023	960,114	960,213	960,085	960,379	960,640
9	1-1,000,000,000	8,474,221	8,474,796	8,475,123	8,474,021	8,474,630	8,474,742
10	1-10,000,000,000	75,840,762	75,841,428	75,843,438	75,841,922	75,842,174	75,842,786
11	1-100,000,000,000	686,339,040	686,342,043	686,338,138	686,340,737	686,346,250	686,348,604
12	1-1,000,000,000,000	6,267,973,536	6,267,979,692	6,267,994,788	6,267,984,796	6,267,992,446	6,267,986,759

Since all primes, except 7, are of only of one of these forms, their quantity seems quite averagely distributed. The deviation from respective averages is plotted ahead.

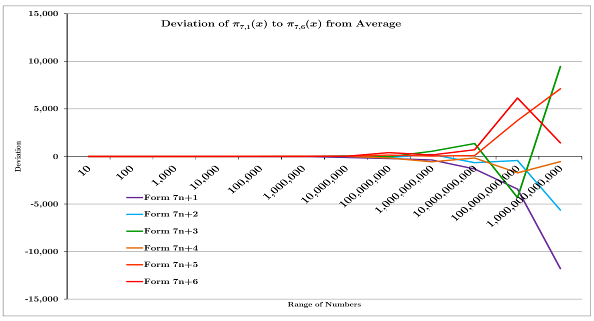

Figure 1: Deviation of $\pi_{7,k}(x)$ from Average

The number of primes of the form $7n+5$ is seen always above the average; while $7n+1$ is below the average up to 10^{12} in almost all discrete blocks of 10 powers. This trend might have changed in between and is a subject matter of future explorations.

6. Block-wise Distribution of Primes

Lack of the formula for all primes together with their infinitude makes them mysterious. We continue with the plain approach of considering all primes up to limit of one trillion (10^{12}) and dividing this complete number range under consideration in blocks of powers of 10 each :

$$1-10,\ 11-20,\ 21-30,\ 31-40,\ \cdots$$
$$1-100,\ 101-200,\ 201-300,\ 301-400,\ \cdots$$
$$1-1000,\ 1001-2000,\ 2001-3000,\ 3001-4000,\ \cdots$$
$$\vdots$$

A detail analysis is performed on various fronts and owing to our range limit of 10^{12}, there are 10^{12-i} number of blocks of 10^i size for each $1 \leq i \leq 12$.

6.1. The First and the Last Primes in the First Blocks of 10 Powers

The search of the first and the last prime number in every first block of each 10 power till 10^{12} is performed. Once available, the first prime of first power of 10 continues for all higher sized blocks.

Table 2: First Primes of form $7n+k$ in First Blocks of 10 Powers

Sr. No.	Blocks of Size	First Prime in the First Block					
		Form $7n+1$	Form $7n+2$	Form $7n+3$	Form $7n+4$	Form $7n+5$	Form $7n+6$
1	10	Not Found	2	3	Not Found	5	Not Found
2	100 and all 10^n for $n \geq 2$	29	2	3	11	5	13

The largest prime numbers of various forms in first blocks of 10 powers are as follows.

Table 3: Last Primes of form $7n+k$ in First Blocks of 10 Powers

Sr. No.	Blocks of Size	Last Prime in the First Block					
		Form $7n+1$	Form $7n+2$	Form $7n+3$	Form $7n+4$	Form $7n+5$	Form $7n+6$
1	10	Not Found	2	3	Not Found	5	Not Found
2	100	71	79	73	67	89	97
3	1,000	967	947	997	991	971	937
4	10,000	9,941	9,949	9,929	9,923	9,973	9,967
5	100,000	99,989	99,871	99,991	99,971	99,923	99,833
6	1,000,000	999,979	999,959	999,953	999,961	999,983	999,907
7	10,000,000	9,999,991	9,999,971	9,999,937	9,999,973	9,999,883	9,999,877
8	100,000,000	99,999,971	99,999,839	99,999,959	99,999,827	99,999,989	99,999,941
9	1,000,000,000	999,999,883	999,999,751	999,999,353	999,999,893	999,999,929	999,999,937
10	10,000,000,000	9,999,999,787	9,999,999,851	9,999,999,943	9,999,999,881	9,999,999,833	9,999,999,967
11	100,000,000,000	99,999,999,947	99,999,999,871	99,999,999,977	99,999,999,943	99,999,999,769	99,999,999,833
12	1,000,000,000,000	999,999,999,937	999,999,999,959	999,999,999,673	999,999,999,989	999,999,999,899	999,999,999,767

Once found, the first primes for specific forms in all the first blocks continue to be same for higher sizes. The deviation of the last primes of these forms in the first blocks has following trend.

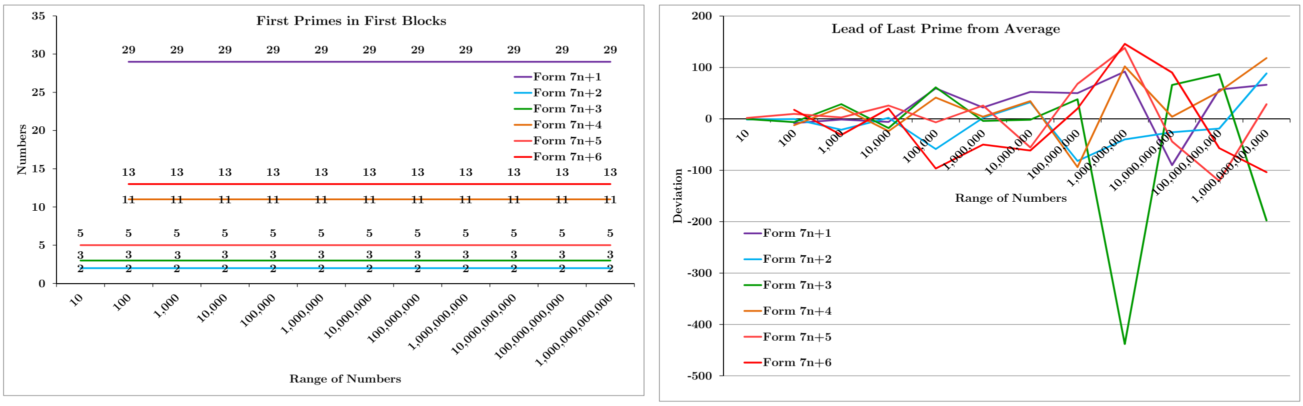

Figure 2: First & Last Primes of form $7n + k$ in First Blocks of 10 Powers.

6.2. Minimum Number of Primes in Blocks of 10 Powers

For all blocks of each first 10 power ranging from 10^1 to 10^{12}, the minimum number of primes of each form coming in each such block has been determined and the block-wise deviation of minimum number of primes of different forms found there from respective averages is as it appears in the figure ahead.

Table 4: Minimum Number of Primes of form $7n + k$ in Blocks of 10 Powers

Sr. No.	Blocks of Size	Minimum Number of Primes in Blocks					
		Form $7n + 1$	Form $7n + 2$	Form $7n + 3$	Form $7n + 4$	Form $7n + 5$	Form $7n + 6$
1	10	0	0	0	0	0	0
2	100	0	0	0	0	0	0
3	1,000	0	0	0	0	0	0
4	10,000	29	29	28	28	28	28
5	100,000	506	513	513	508	505	501
6	1,000,000	5,766	5,771	5,798	5,790	5,802	5,789
7	10,000,000	59,675	59,817	59,736	59,676	59,788	59,843
8	100,000,000	602,206	602,328	602,250	601,897	602,304	602,097
9	1,000,000,000	6,030,499	6,030,924	6,031,408	6,030,567	6,031,413	6,030,063
10	10,000,000,000	60,325,839	60,326,188	60,332,353	60,330,357	60,333,294	60,329,390
11	100,000,000,000	604,306,097	604,328,075	604,326,144	604,316,906	604,317,490	604,329,720
12	1,000,000,000,000	6,267,973,536	6,267,979,692	6,267,994,788	6,267,984,796	6,267,992,446	6,267,986,759

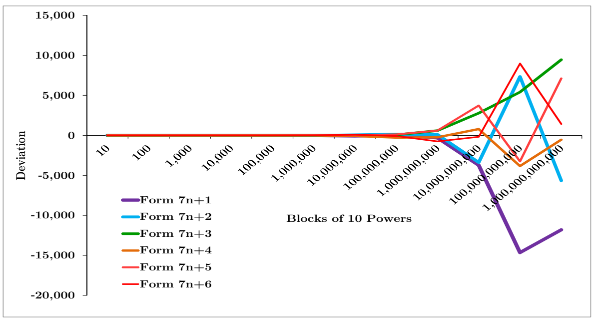

Figure 3: Deviation in Minimum Number of Primes of form $7n + k$ in Blocks of 10 Powers from Average

The first 10 power blocks till one trillion with minimum number of primes of these six forms in them are determined.

Table 5: First Blocks of 10 Powers with Minimum Number of Primes of form $7n+k$

Sr. No.	Blocks of Size	First Block with Minimum Number of Primes					
		Form $7n+1$	Form $7n+2$	Form $7n+3$	Form $7n+4$	Form $7n+5$	Form $7n+6$
1	10	0	10	20	0	20	0
2	100	3,400	4,200	6,400	2,800	3,800	1,100
3	1,000	74,673,000	163,014,000	57,502,000	273,388,000	35,018,000	67,667,000
4	10,000	773,514,760,000	520,793,000,000	801,425,060,000	454,847,640,000	724,483,020,000	461,343,210,000
5	100,000	846,428,800,000	944,598,200,000	866,675,100,000	936,768,900,000	942,228,800,000	975,758,200,000
6	1,000,000	999,652,000,000	992,313,000,000	993,219,000,000	906,367,000,000	965,097,000,000	926,912,000,000
7	10,000,000	975,060,000,000	998,020,000,000	970,760,000,000	987,710,000,000	999,270,000,000	997,370,000,000
8	100,000,000	999,000,000,000	980,600,000,000	996,400,000,000	975,100,000,000	971,600,000,000	997,300,000,000
9	1,000,000,000	997,000,000,000	998,000,000,000	990,000,000,000	999,000,000,000	995,000,000,000	997,000,000,000
10	10,000,000,000	990,000,000,000	990,000,000,000	990,000,000,000	990,000,000,000	990,000,000,000	990,000,000,000
11	100,000,000,000	900,000,000,000	900,000,000,000	900,000,000,000	900,000,000,000	900,000,000,000	900,000,000,000

The last such blocks in our range of one trillion with minimum number of primes of these different forms in them are also determined.

Table 6: Last Blocks of 10 Powers with Minimum Number of Primes of form $7n+k$

Sr. No.	Blocks of Size	Last Block with Minimum Number of Primes					
		Form $7n+1$	Form $7n+2$	Form $7n+3$	Form $7n+4$	Form $7n+5$	Form $7n+6$
1	10	999,999,999,990	999,999,999,990	999,999,999,990	999,999,999,990	999,999,999,990	999,999,999,990
2	100	999,999,999,800	999,999,999,600	999,999,999,900	999,999,999,700	999,999,999,900	999,999,999,900
3	1,000	999,998,814,000	999,999,666,000	999,999,041,000	999,999,401,000	999,999,339,000	999,999,899,000
4	10,000	773,514,760,000	563,068,490,000	953,603,980,000	454,847,640,000	724,483,020,000	952,710,080,000
5	100,000	846,428,800,000	944,598,200,000	866,675,100,000	936,768,900,000	942,228,800,000	975,758,200,000
6	1,000,000	999,652,000,000	992,313,000,000	993,219,000,000	906,367,000,000	965,097,000,000	926,912,000,000
7	10,000,000	975,060,000,000	998,020,000,000	970,760,000,000	987,710,000,000	999,270,000,000	997,370,000,000
8	100,000,000	999,000,000,000	980,600,000,000	996,400,000,000	975,100,000,000	971,600,000,000	997,300,000,000
9	1,000,000,000	997,000,000,000	998,000,000,000	990,000,000,000	999,000,000,000	995,000,000,000	997,000,000,000
10	10,000,000,000	990,000,000,000	990,000,000,000	990,000,000,000	990,000,000,000	990,000,000,000	990,000,000,000
11	100,000,000,000	900,000,000,000	900,000,000,000	900,000,000,000	900,000,000,000	900,000,000,000	900,000,000,000

The comparative trend deserves graphical representation.

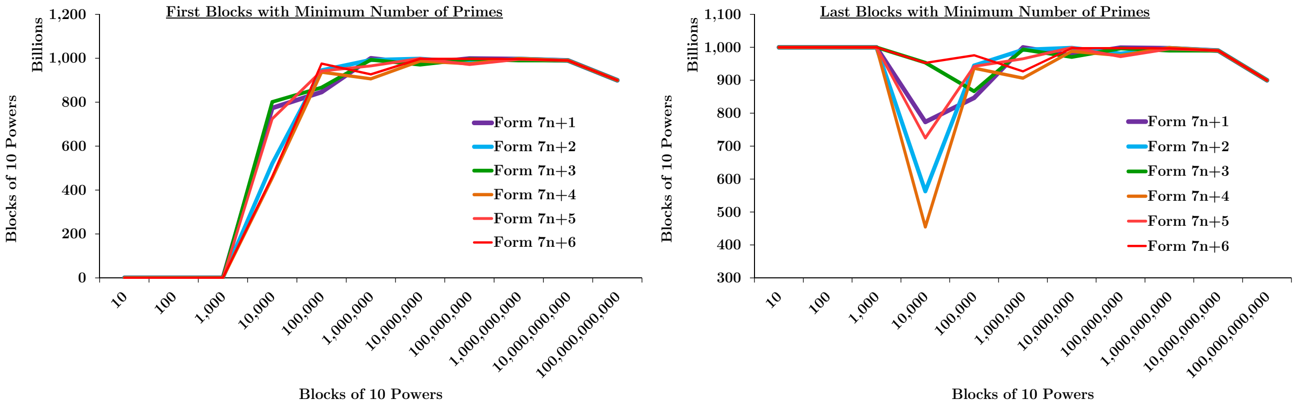

Figure 4: First & Last Blocks of 10 Powers with Minimum Number of Primes of form $7n+k$.

The determination of frequency of blocks of minimum occurrences of primes of $7n + k$ forms is now due.

Table 7: Frequency of Minimum Number of Primes of form $7n + k$ in Blocks of 10 Powers

Sr. No.	Blocks of Size	No. of Times Minimum No. of Primes Occurring in Blocks					
		Form $7n+1$	Form $7n+2$	Form $7n+3$	Form $7n+4$	Form $7n+5$	Form $7n+6$
1	10	93,732,026,464	93,732,020,308	93,732,005,212	93,732,015,204	93,732,007,554	93,732,013,241
2	100	5,035,939,997	5,035,927,212	5,035,927,620	5,035,915,029	5,035,890,322	5,035,901,175
3	1,000	743,646	742,535	744,827	744,553	743,138	742,918
4	10,000	1	2	2	1	1	5
5	100,000 & all 10^n for $n \geq 5$	1	1	1	1	1	1

The block-wise deviation of frequency of occurrence of minimum number of primes from corresponding averages is plotted.

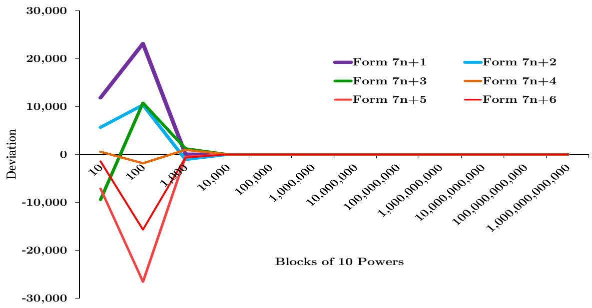

Figure 5: Deviation in Frequency of Minimum Number of Primes in Blocks from Average

6.3. Maximum Number of Primes in Blocks of 10 Powers

Now all blocks of each 10 power ranging from 10^1 to 10^12 are analyzed for the maximum number of primes of various forms $7n + k$ in each of them.

Table 8: Maximum Number of Primes of form $7n + k$ in Blocks of 10 Powers

Sr. No.	Blocks of Size	Maximum Number of Primes in Blocks					
		Form $7n+1$	Form $7n+2$	Form $7n+3$	Form $7n+4$	Form $7n+5$	Form $7n+6$
1	10	1	1	1	1	1	1
2	100	5	5	5	5	5	5
3	1,000	28	27	30	26	29	27
4	10,000	203	203	209	202	211	200
5	100,000	1,593	1,584	1,613	1,601	1,604	1,596
6	1,000,000	13,063	13,065	13,105	13,069	13,105	13,090
7	10,000,000	110,653	110,771	110,815	110,776	110,787	110,776
8	100,000,000	960,023	960,114	960,213	960,085	960,379	960,640
9	1,000,000,000	8,474,221	8,474,796	8,475,123	8,474,021	8,474,630	8,474,742
10	10,000,000,000	75,840,762	75,841,428	75,843,438	75,841,922	75,842,174	75,842,786
11	100,000,000,000	686,339,040	686,342,043	686,338,138	686,340,737	686,346,250	686,348,604
12	1,000,000,000,000	6,267,973,536	6,267,979,692	6,267,994,788	6,267,984,796	6,267,992,446	6,267,986,759

Analyzing deviation from average, it is found that the block-wise maximality of primes of form $7n + 1$ has always lagged behind and that of $7n + 5$ has been above the average of all of these.

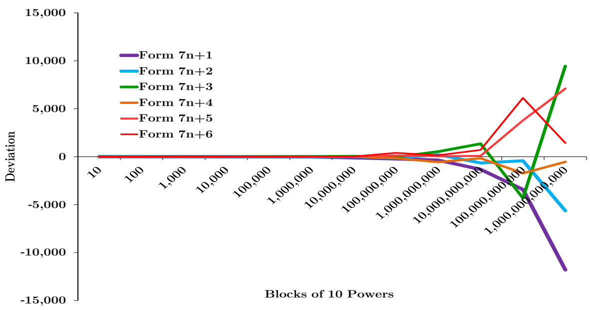

Figure 6: Deviation in Maximum Number of Primes of form $7n + k$ in Blocks of 10 Powers from Average

The first 10 power blocks till one trillion with maximum number of primes of these six forms in them are also determined.

Table 9: First Blocks of 10 Powers with Maximum Number of Primes of form $7n + k$

Sr. No.	Blocks of Size	First Block with Maximum Number of Primes					
		Form $7n+1$	Form $7n+2$	Form $7n+3$	Form $7n+4$	Form $7n+5$	Form $7n+6$
1	10	20	0	0	10	0	10
2	100	530,200	21,100	0	100	0	1,693,600
3	1,000	0	0	0	0	0	0
4	10,000	0	0	0	0	0	0
5	100,000	0	0	0	0	0	0
6	1,000,000	0	0	0	0	0	0
7	10,000,000	0	0	0	0	0	0
8	100,000,000	0	0	0	0	0	0
9	1,000,000,000	0	0	0	0	0	0
10	10,000,000,000	0	0	0	0	0	0
11	100,000,000,000	0	0	0	0	0	0

The last such blocks in our range of one trillion with maximum number of primes of these different forms in them are given ahead.

Table 10: Last Blocks of 10 Powers with Maximum Number of Primes of form $7n + k$

Sr. No.	Blocks of Size	Last Block with Maximum Number of Primes					
		Form $7n+1$	Form $7n+2$	Form $7n+3$	Form $7n+4$	Form $7n+5$	Form $7n+6$
1	10	999,999,999,930	999,999,999,950	999,999,999,670	999,999,999,980	999,999,999,890	999,999,999,760
2	100	999,993,651,700	999,997,532,200	999,998,422,300	999,977,974,900	999,988,728,100	999,973,496,500
3	1,000	0	0	0	0	0	0
4	10,000	0	0	0	0	0	0
5	100,000	0	0	0	0	0	0
6	1,000,000	0	0	0	0	0	0
7	10,000,000	0	0	0	0	0	0
8	100,000,000	0	0	0	0	0	0
9	1,000,000,000	0	0	0	0	0	0
10	10,000,000,000	0	0	0	0	0	0
11	100,000,000,000	0	0	0	0	0	0

As in general, the prime density has a decreasing trend with increasing range of numbers, it is natural that for larger block sizes, the first as well as the last occurrences of maximum number of primes in them start in the block 0, which is the very first block.

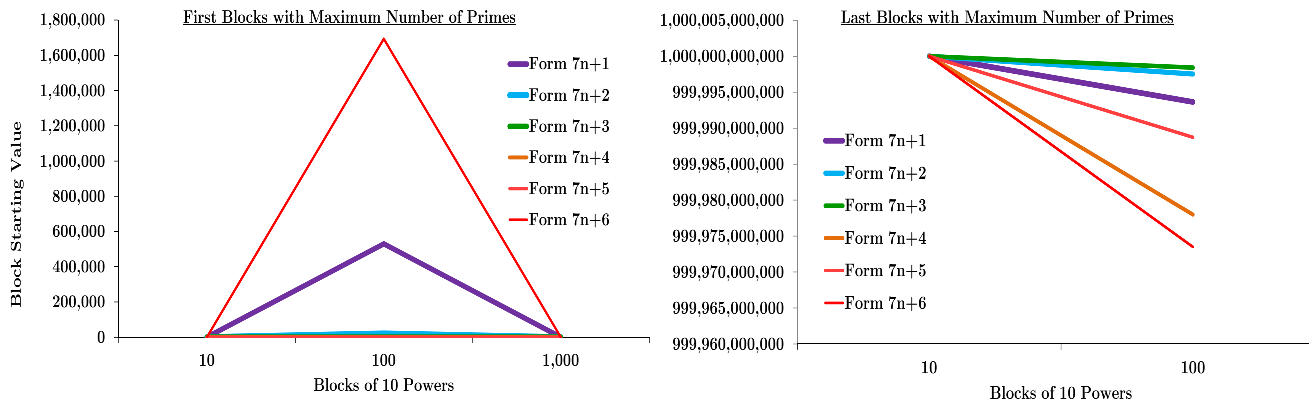

Figure 7: First & Last Blocks of 10 Powers with Maximum Number of Primes of form $7n + k$.

Well-known decrease in the prime density assures that the maximum number of primes cannot occur as frequently, at least for higher ranges.

Table 11: Frequency of Maximum Number of Primes of form $7n + k$ in Blocks of 10 Powers

Sr. No.	Blocks of Size	No. of Times Maximum No. of Primes Occurring in Blocks					
		Form $7n + 1$	Form $7n + 2$	Form $7n + 3$	Form $7n + 4$	Form $7n + 5$	Form $7n + 6$
1	10	6,267,973,536	6,267,979,692	6,267,994,788	6,267,984,796	6,267,992,446	6,267,986,759
2	100	77,946	77,531	78,016	77,890	77,366	77,557
3	1,000 & all 10^n for $n \geq 3$	1	1	1	1	1	1

Now the block-wise deviation of frequency of occurrence of maximum number of primes from corresponding averages is plotted.

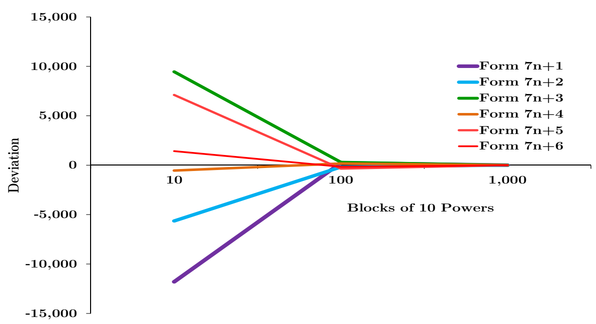

Figure 8: Deviation in Frequency of Maximum Number of Primes in Blocks from Average

7. Spacings Between Primes of form $7n + k$ in Blocks of 10 Powers

7.1. Minimum Spacing Between Primes in Blocks of 10 Powers

Omitting prime-free blocks and all blocks of smaller size 10, the minimum spacing between primes of all forms $7n+1$, $7n+2$, $7n+3$, $7n+4$, $7n+5$ and $7n+6$ in the blocks of increasing power of 10 from 100 onwards are determined to be 14. This is the first even integral multiple of 7. Clearly as for larger block sizes, the minimum spacing value cannot increase, it remains same afterwards for all blocks of all higher powers of 10 in all ranges, virtually till infinity!

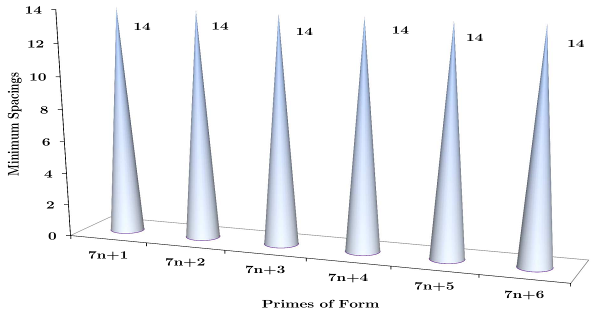

Figure 9: Minimum Block Spacings between Primes of form $7n + k$

Except the block-size of 10, the first and the last primes in the 10 power blocks with minimum block spacings are as follows.

Table 12: First Starters of Minimum Block Spacings between Primes of form $7n + k$ in Blocks of 10^n

Sr. No.	Blocks of Size	First Prime with Respective Minimum Block Spacing					
		Form $7n+1$	Form $7n+2$	Form $7n+3$	Form $7n+4$	Form $7n+5$	Form $7n+6$
1	10	Not Found	Not Found	Not Found	Not Found	Not Found	Not Found
2	100 & all 10^n for $n \geq 2$	29	23	3	53	5	83

Withing specific limit like that of ours of 1 trillion, the last primes in the 10 power blocks with minimum block spacings are also more or less uniform.

Table 13: Last Starters of Minimum Block Spacings between Primes of form $7n + k$ in Blocks of 10^n

Sr. No.	Blocks of Size	Last Prime with Respective Minimum Block Spacing					
		Form $7n+1$	Form $7n+2$	Form $7n+3$	Form $7n+4$	Form $7n+5$	Form $7n+6$
1	10	Not Found	Not Found	Not Found	Not Found	Not Found	Not Found
2	100	999,999,993,749	999,999,994,457	999,999,998,567	999,999,999,863	999,999,998,513	999,999,995,483
3	1,000 & all 10^n for $n \geq 3$	999,999,993,749	999,999,999,287	999,999,998,567	999,999,999,863	999,999,998,513	999,999,996,197

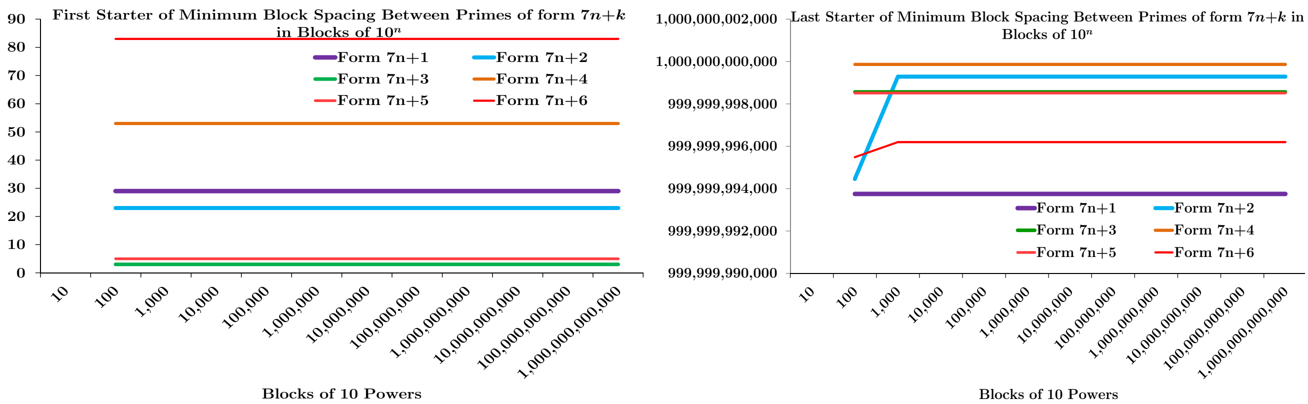

Figure 10: First & Last Starters of Minimum Block Spacings between Primes of form $7n + k$ in Blocks of 10^n

The number of times this minimum block spacing occurs between primes of all forms $7n + 1$, $7n + 2$, $7n + 3$, $7n + 4$, $7n + 5$ and $7n + 6$ is noteworthy.

Table 14: Frequency of Minimum Block Spacings between Primes of form $7n + k$

Sr. No.	Blocks of Size	Number of Minimum Block Spacings Occurring for Primes					
		Form $7n+1$	Form $7n+2$	Form $7n+3$	Form $7n+4$	Form $7n+5$	Form $7n+6$
1	10	0	0	0	0	0	0
2	100	311,770,604	311,758,174	311,752,528	311,745,077	311,747,223	311,754,644
3	1,000	367,877,711	367,864,098	367,868,966	367,867,484	367,869,338	367,862,706
4	10,000	373,488,315	373,474,760	373,481,415	373,479,744	373,478,126	373,474,324
5	100,000	374,049,446	374,035,656	374,043,124	374,039,376	374,038,041	374,035,184
6	1,000,000	374,105,552	374,091,587	374,099,147	374,095,467	374,094,007	374,091,658
7	10,000,000	374,111,200	374,097,095	374,104,803	374,101,058	374,099,619	374,097,291
8	100,000,000	374,111,746	374,097,669	374,105,359	374,101,621	374,100,173	374,097,869
9	1,000,000,000	374,111,789	374,097,735	374,105,414	374,101,672	374,100,238	374,097,920
10	10,000,000,000	374,111,795	374,097,735	374,105,423	374,101,679	374,100,249	374,097,926
11	100,000,000,000	374,111,796	374,097,735	374,105,423	374,101,680	374,100,252	374,097,926
12	1,000,000,000,000	374,111,796	374,097,735	374,105,423	374,101,680	374,100,252	374,097,926

We see increase in the occurrence of minimum spacings for prime numbers of all forms. The cause of this is that as we increase block size, some primes with desired spacing coming at the crossing of smaller sized blocks find themselves inside larger blocks, which raises their count. Then this rate of increase gradually decreases owing to fading prime frequency.

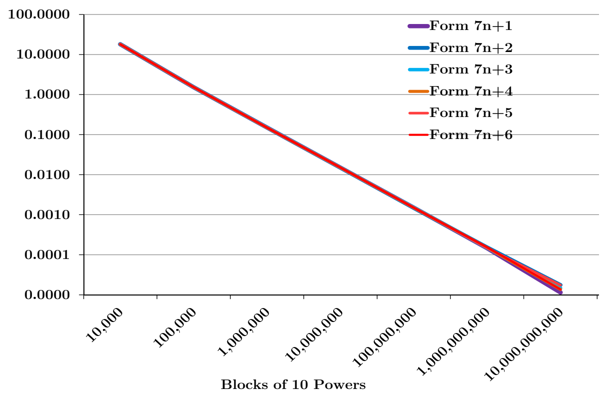

Figure 11: % Increase in Occurrences of Minimum Block Spacings between Primes of form $7n + k$ in Blocks of 10^n

7.2. Maximum Spacing Between Primes in Blocks of 10 Powers

Unlike the minimum spacing between primes in blocks of 10 powers, the maximum spacing in them shows a little increasing trend with increase in the block size.

Table 15: Maximum Block Spacing between Primes of form $7n + k$

Sr. No.	Blocks of Size	Maximum Spacing Between Primes					
		Form $7n+1$	Form $7n+2$	Form $7n+3$	Form $7n+4$	Form $7n+5$	Form $7n+6$
1	10	Not Found	Not Found	Not Found	Not Found	Not Found	Not Found
2	100	98	98	98	98	98	98
3	1,000	994	994	994	994	994	994
4	10,000	3,318	3,038	3,066	3,122	3,108	3,024
5	100,000 & all 10^n for $n \geq 5$	3,318	3,038	3,066	3,122	3,108	3,150

Till our ceiling of 10^12, a trend of increase and settling is observed as shown in the figure.

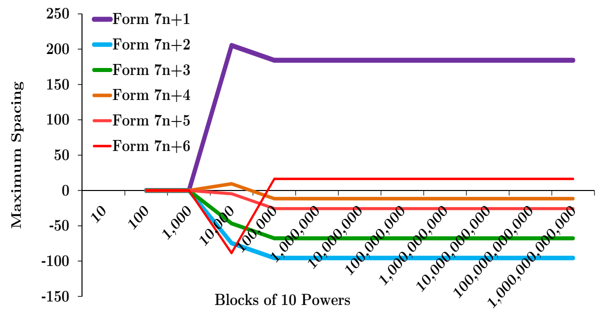

Figure 12: Deviation of Maximum Block Spacings between Primes of form $7n+k$ from Average

The first & the last prime numbers of forms $7n+k$ with these maximum block spacings within different blocks are as follows.

Table 16: First Starters of Maximum Block Spacings between Primes of form $7n+k$ in Blocks of 10^n

Sr. No.	Blocks of Size	First Prime with Respective Maximum Block Spacing					
		Form $7n+1$	Form $7n+2$	Form $7n+3$	Form $7n+4$	Form $7n+5$	Form $7n+6$
1	10	Not Found	Not Found	Not Found	Not Found	Not Found	Not Found
2	100	53,201	15,101	29,501	1,901	224,201	1,301
3	1,000	745,831,003	1,152,838,003	4,575,760,003	3,352,978,003	464,770,003	6,175,768,003
4	10,000	857,234,220,221	548,658,004,421	517,425,410,561	714,877,644,191	134,572,683,299	153,787,932,689
5	100,000 & all 10^n for $n \geq 5$	857,234,220,221	548,658,004,421	517,425,410,561	714,877,644,191	134,572,683,299	502,607,208,767

Table 17: Last Starters of Maximum Block Spacings between Primes of form $7n+k$ in Blocks of 10^n

Sr. No.	Blocks of Size	Last Prime with Respective Maximum Block Spacing					
		Form $7n+1$	Form $7n+2$	Form $7n+3$	Form $7n+4$	Form $7n+5$	Form $7n+6$
1	10	Not Found	Not Found	Not Found	Not Found	Not Found	Not Found
2	100	999,999,937,301	999,999,884,501	999,999,982,901	999,999,984,701	999,999,957,101	999,999,990,401
3	1,000	999,840,340,003	999,192,706,003	999,921,916,003	999,608,182,003	999,898,933,003	998,669,725,003
4	10,000	857,234,220,221	548,658,004,421	517,425,410,561	714,877,644,191	893,062,422,523	153,787,932,689
5	100,000 & all 10^n for $n \geq 5$	857,234,220,221	548,658,004,421	517,425,410,561	714,877,644,191	893,062,422,523	502,607,208,767

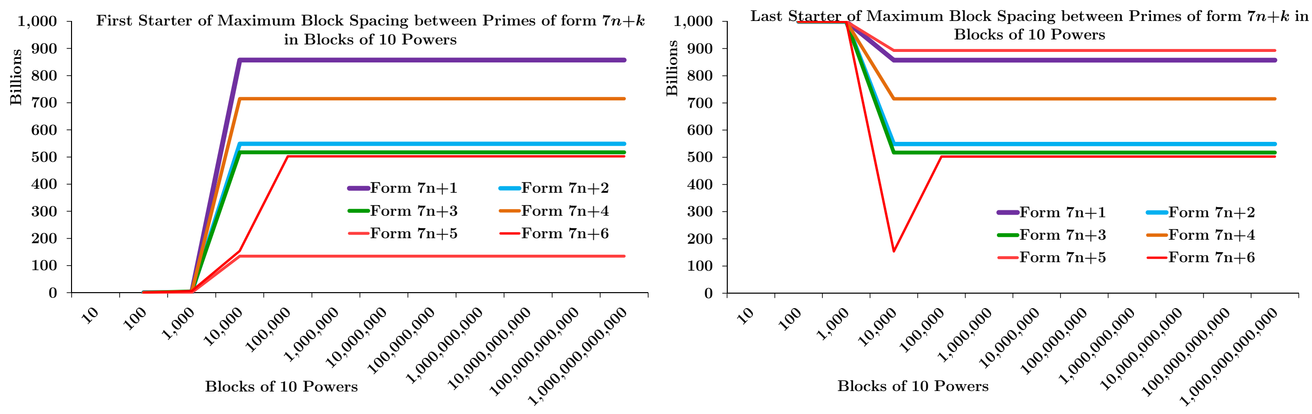

Figure 13: First & Last Starters of Maximum Block Spacings between Primes of form $7n+k$ in Blocks of 10^n

Table 18: Frequency of Maximum Block Spacings between Primes of form $7n + k$

Sr. No.	Blocks of Size	Number of Maximum Block Spacings Occurring for Primes					
		Form $7n + 1$	Form $7n + 2$	Form $7n + 3$	Form $7n + 4$	Form $7n + 5$	Form $7n + 6$
1	10	0	0	0	0	0	0
2	100	8,927,987	8,929,783	8,932,889	8,928,289	8,931,033	8,929,226
3	1,000	1,486	1,607	1,536	1,564	1,644	1,556
4	10,000 & all 10^n for $n \geq 4$	1	1	1	1	2	1

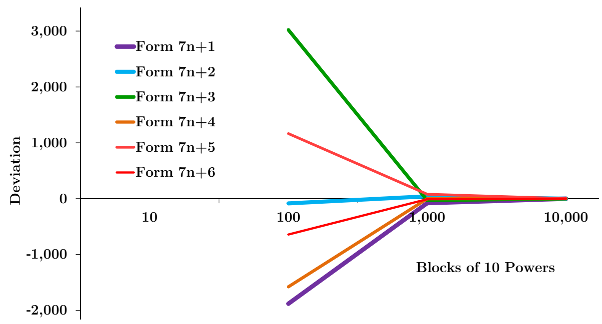

Figure 14: Deviation of Number of Occurrences of Maximum Block Spacings between Primes of form $7n + k$ from Average

8. Units Place and Tens Place Digits in Primes of form $7n + k$

Only 6 different digits are possible in units place of primes. The number of primes of form $7n + k$ with these different digits in units have been determined till 1 trillion. So ignoring these special cases 2 & 3, deviations from average are also drawn.

Table 19: Number of primes of form $7n + k$ with different units place digits till one trillion

Sr. No.	Digit in Units Place	Number of Primes of form					
		$7n + 1$	$7n + 2$	$7n + 3$	$7n + 4$	$7n + 5$	$7n + 6$
1	1	1,566,997,409	1,566,981,018	1,567,001,317	1,566,998,104	1,566,990,811	1,566,992,321
2	2	0	1	0	0	0	0
3	3	1,567,001,995	1,567,007,149	1,566,990,378	1,566,987,259	1,566,998,652	1,566,994,472
4	5	0	0	0	0	1	0
5	7	1,566,992,306	1,567,000,080	1,567,002,239	1,567,006,872	1,566,999,313	1,566,996,189
6	9	1,566,981,826	1,566,991,444	1,567,000,855	1,566,992,561	1,567,003,669	1,567,003,777

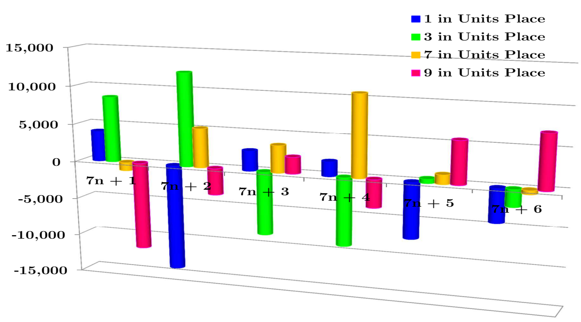

Figure 15: Deviation of Units Place Digits of Primes of form $7n + k$ from Average

Now follow the figures for tens and units place digits together. There are 42 different cases of tens and units places that can occur in any prime.

Table 20: Number of Primes of form $7n + k$ with Different Tens & Units Place Digits till One Trillion

Sr. No.	Digit in Tens and Units Place	Number of Primes of form					
		$7n+1$	$7n+2$	$7n+3$	$7n+4$	$7n+5$	$7n+6$
1	01	156,702,292	156,697,199	156,700,578	156,703,304	156,696,787	156,701,064
2	02	0	1	0	0	0	0
3	03	156,700,698	156,701,521	156,704,979	156,691,981	156,703,091	156,696,773
4	05	0	0	0	0	1	0
5	07	156,698,448	156,704,401	156,696,450	156,703,614	156,694,252	156,704,358
6	09	156,702,103	156,703,796	156,702,044	156,691,676	156,699,541	156,698,877
7	11	156,692,276	156,694,120	156,703,195	156,699,595	156,700,392	156,702,053
8	13	156,699,832	156,704,660	156,702,759	156,693,750	156,701,714	156,697,989
9	17	156,700,053	156,698,640	156,698,813	156,695,335	156,697,824	156,698,640
10	19	156,700,861	156,702,034	156,707,296	156,704,612	156,706,132	156,703,632
11	21	156,705,922	156,695,597	156,695,592	156,701,742	156,704,762	156,703,836
12	23	156,698,341	156,706,166	156,697,267	156,701,379	156,703,329	156,698,631
13	27	156,704,475	156,696,777	156,702,101	156,694,190	156,709,797	156,700,032
14	29	156,701,667	156,697,484	156,707,327	156,701,454	156,696,749	156,692,748
15	31	156,703,334	156,702,020	156,696,689	156,705,296	156,699,537	156,694,420
16	33	156,701,189	156,703,550	156,690,122	156,695,821	156,703,173	156,703,779
17	37	156,694,366	156,701,152	156,702,013	156,700,827	156,703,832	156,696,646
18	39	156,700,994	156,701,187	156,704,908	156,698,215	156,695,338	156,694,721
19	41	156,701,067	156,695,501	156,700,714	156,695,393	156,703,449	156,693,882
20	43	156,699,420	156,696,674	156,698,656	156,708,981	156,695,774	156,698,088
21	47	156,702,611	156,698,396	156,697,739	156,700,264	156,696,364	156,702,358
22	49	156,692,813	156,696,796	156,702,386	156,704,177	156,703,458	156,701,146
23	51	156,699,937	156,705,292	156,699,031	156,697,973	156,701,847	156,700,800
24	53	156,695,952	156,702,555	156,699,935	156,696,521	156,702,686	156,697,938
25	57	156,704,579	156,697,893	156,702,719	156,696,672	156,700,972	156,690,160
26	59	156,704,905	156,694,384	156,694,825	156,698,414	156,702,763	156,704,231
27	61	156,698,957	156,702,466	156,698,139	156,696,058	156,702,101	156,698,389
28	63	156,700,544	156,698,456	156,695,565	156,698,162	156,699,688	156,702,951
29	67	156,690,748	156,700,900	156,704,728	156,703,222	156,699,884	156,703,875
30	69	156,690,167	156,694,076	156,697,099	156,691,254	156,696,892	156,702,956
31	71	156,700,920	156,694,839	156,702,347	156,702,880	156,700,186	156,695,317
32	73	156,704,245	156,695,968	156,696,467	156,700,805	156,699,539	156,699,923
33	77	156,697,568	156,698,745	156,698,072	156,703,626	156,698,830	156,699,802
34	79	156,695,923	156,698,460	156,695,749	156,698,312	156,700,376	156,700,006
35	81	156,692,984	156,694,139	156,704,605	156,693,689	156,696,291	156,698,295
36	83	156,702,438	156,698,757	156,703,759	156,691,391	156,697,499	156,698,056
37	87	156,697,472	156,696,545	156,703,998	156,701,087	156,698,907	156,701,045
38	89	156,694,476	156,697,752	156,698,944	156,700,117	156,700,555	156,710,164
39	91	156,699,720	156,699,845	156,700,427	156,702,174	156,685,459	156,704,265
40	93	156,699,336	156,698,842	156,700,869	156,708,468	156,692,159	156,700,344
41	97	156,701,986	156,706,631	156,695,606	156,708,035	156,698,651	156,699,273
42	99	156,697,917	156,705,475	156,690,277	156,704,330	156,701,865	156,695,296

Out of these 42 cases, the cases of 02 and 05 are exceptional as they occur only once. So, while studying averages, these special cases are always to be kept aside having odd man out roles.

Following deviation from average is seen for occurrences of other last two digits in range of $1 - 10^{12}$ for primes of form $7n+k$.

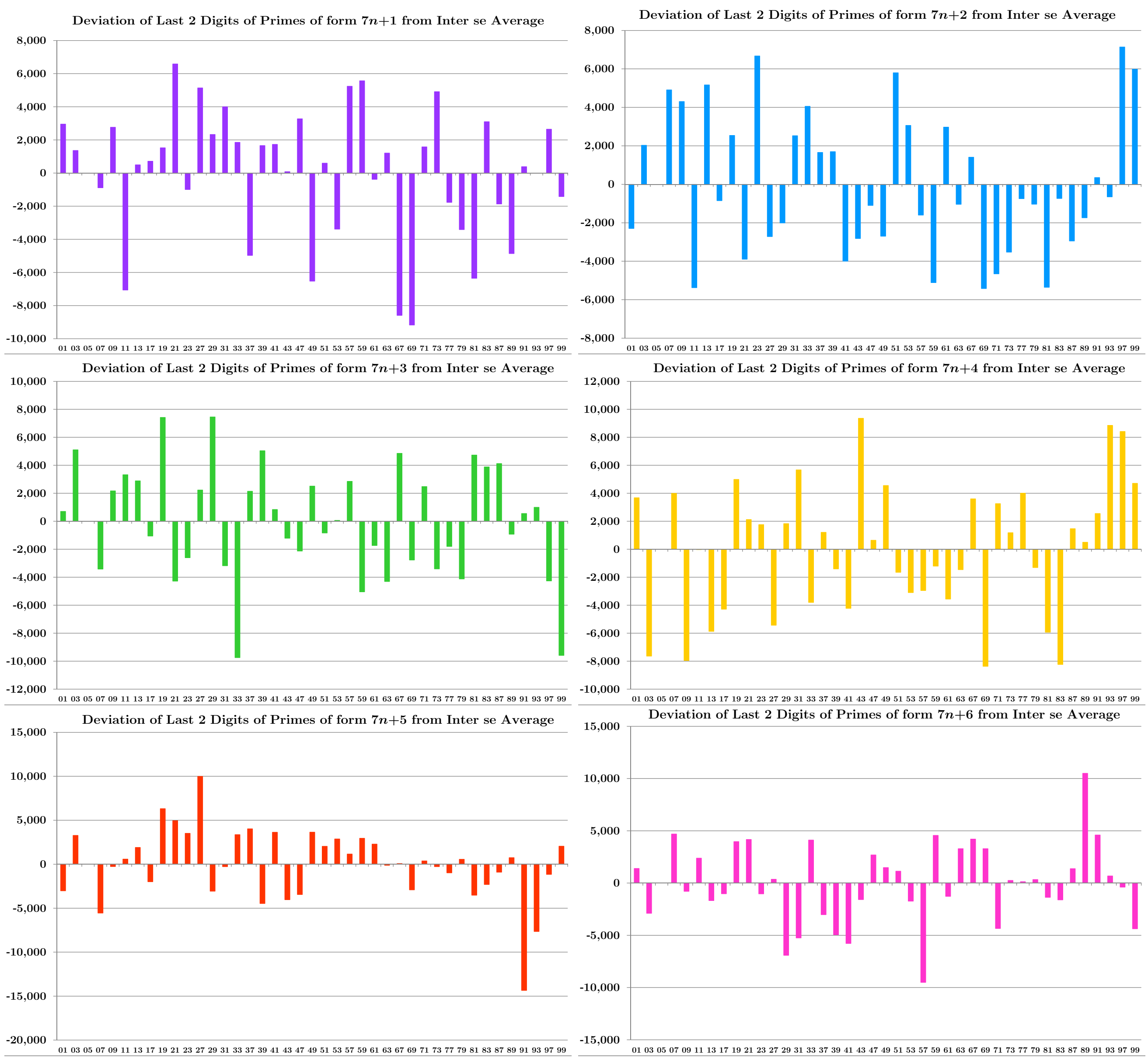

Figure 16: Deviation of Last 2 Digits of Primes of form $7n + k$ from Inter se Average

9. Analysis of Successive Primes of form $7n + k$

When two successive primes are of same form $7n + k$, the case becomes interesting. Their analysis is graphically presented.

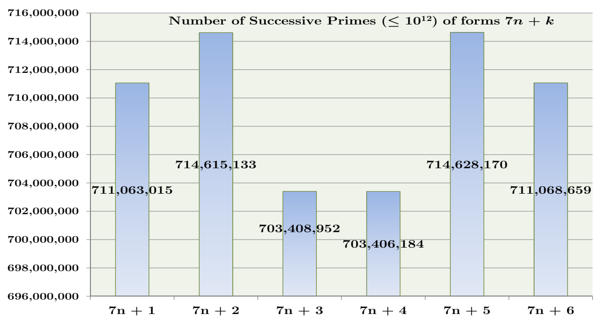

Figure 17: Number of Successive Primes of form $7n + k$ till 10^12

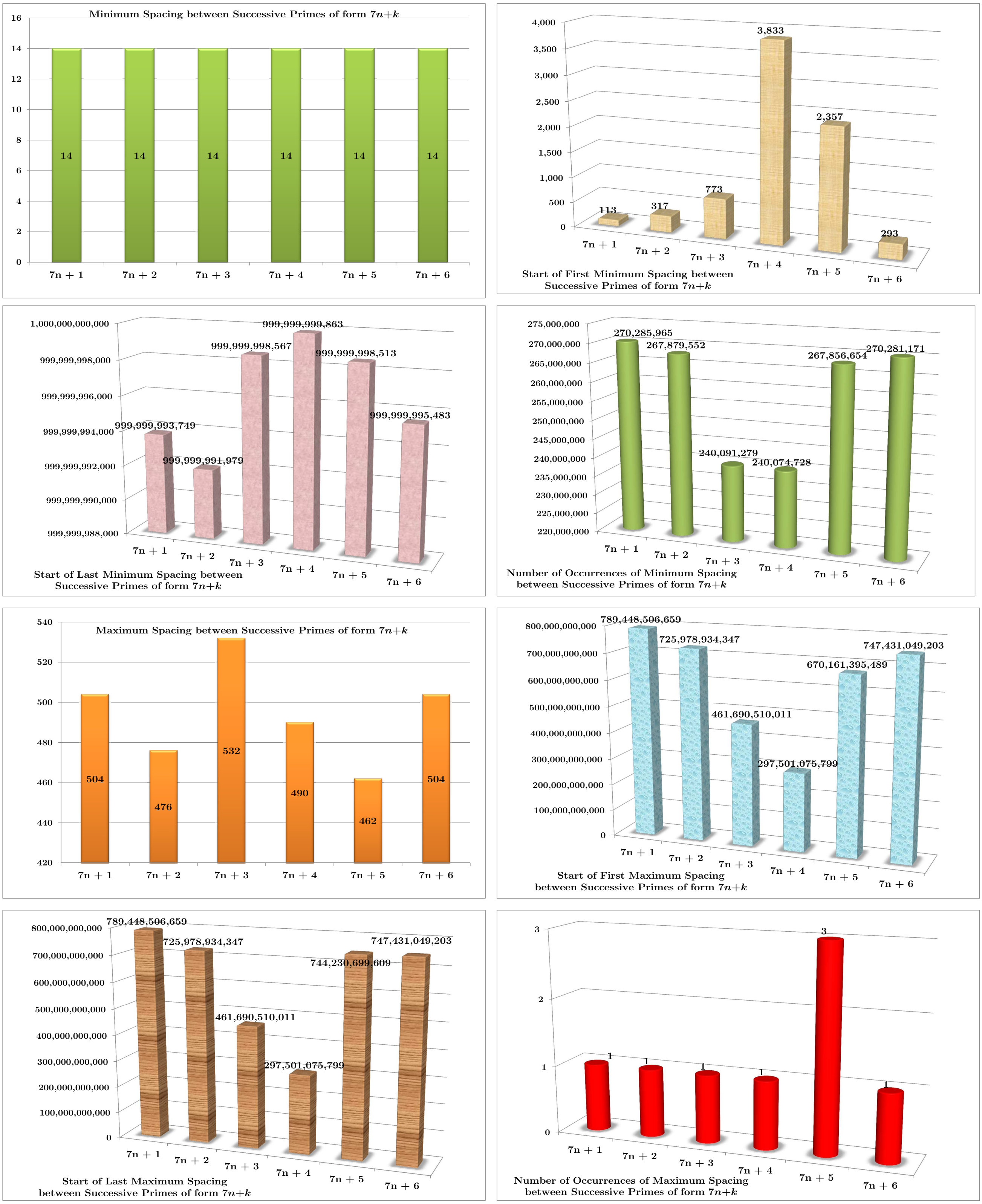

Figure 18: Successive Occurrence of Primes of form $7n + k$ till $10^{1}2$

Since long consistent efforts are been taken to study random distribution of primes. This work is an addition to that with respect to a specific linear pattern of $7n + k$. The availability of more and more rigorous analysis like this will help give a deeper insight into understanding of prime distribution.

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