International Journal of Mathematics And its Applications
Volume 4, Issue 2–A (2016), 31–35.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

E–Colouring of a Graph

Research Article

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Abstract: In this paper we introduce a new colouring of graphs called E–Colouring. This is not a proper colouring in general. We also define E–Chromatic Number and E–Homomorphism of a graph. We prove several related results. In particular we prove that for a given E–chromatic colouring there is a vertex in every colour class which has an equitable neighbour in every other colour class.

MSC: 05C15, 05C69.

Keywords: E–Colouring, E–Chromatic Colouring, E–Chromatic Number, E–Homomorphism, Equitable Independent Set, Maximal Equitable Independent Set, Transversal, Colour Class.

1. Introduction

The concept of an equitable edge was introduced in [2]. An edge uv of a graph is said to be an equitable edge if $|d(u)-d(v)| \leq 1$. Two vertices u and v are said to be equitable adjacent if there is an equitable edge containing u and v. The concept of an equitable independent set was introduced in [5]. A set S of vertices of a graph G is said to be equitable independent if no two vertices of S are equitable adjacent. An equitable independent set S is said to be a maximal independent set [5] if for every vertex v in $V(G) - S$, $S \cup \{v\}$ is not an equitable independent set. We assume that our graphs are finite, simple and undirected.

2. E–Colouring of a Graph

We introduced here a new concept of colouring.

Definition 2.1 (E-Colouring of a Graph)**.** *Let G be a graph. An E-Colouring of the vertices of the graph G is an assignment* $f : V(G) \to \{1, 2, \ldots, k\}$ *such that if u and v are equitable adjacent then the colours* $f(u)$ *and* $f(v)$ *are different. An E-Colouring* $f : V(G) \to \{1, 2, \ldots, k\}$ *is also called a k-E-Colouring.*

Note that an E-Colouring of a graph G need not be a proper colouring of G.

Definition 2.2 (Equitable Independent Set [5])**.** *Let G be a graph and* $S \subseteq V(G)$*, then S is said to be an equitable independent set if no two vertices of S are equitable adjacent. Obviously, every independent set is an equitable independent set but an equitable independent set need not be an independent set.*

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Example 2.3.

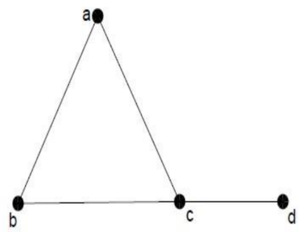

Figure 1

Let $G = (V, E)$ *be a graph.* $V(G) = \{a, b, c, d\}$*. The function* $f : V(G) \to \{1, 2, 3\}$ *defined as* $f(a) = 1$*,* $f(b) = 2$*,* $f(c) = 3$*,* $f(d) = 3$*.*

The above colouring is an E–Colouring of the graph and observe that it is not a proper colouring of the graph.

Remark 2.4. *Let G be a graph and* $f : V(G) \to \{1, 2, \ldots, k\}$ *be a k-E-Colouring of G. We assume that this assignment f of colours is onto* $\forall$ *i let* $C_i = \{v \in V(G)/f(v) = i\}$ *(*$i = 1, 2, \ldots, k$*). Then obviously each* C_i *is non-empty equitable independent subset of G and* $C_i \cap C_j = \phi$ *if* $i \neq j$*. In fact the sets* $C_1, C_2, \ldots, C_k$ *is a partition of V(G) into equitable independent subset of G. These sets* C_i *are called colour classes corresponding to this E–Colouring.*
Conversely if we have a partition of V(G) say $C_1, C_2, \ldots, C_k$ *consisting of equitable independent sets then we have an E–Colouring arising naturally as follows. Define* $f : V(G) \to \{1, 2, \ldots, k\}$ *as* $f(v) = i$ *if* $v \in C_i$ *for some i, for this E–Colouring obviously the sets* $C_1, C_2, \ldots, C_k$ *are the colour classes.*
For example, the colour classes corresponding to the E–Colouring in the above graph are $C_1 = \{a\}$*,* $C_2 = \{b\}$*,* $C_3 = \{c, d\}$*.*

Definition 2.5 (E–Chromatic Number and E–Chromatic Colouring)**.** *Let G be a graph. The smallest positive integer k such that G admits a k–E-Colouring is called the E–Chromatic Number of the graph. Such a k–E–Colouring is called the E–Chromatic Colouring of a graph.*

Notation 2.6. *The E–Chromatic number of a graph is denoted by* $\chi_E(G)$ *or E–ch(G).*

Example 2.7. *Consider the graph*

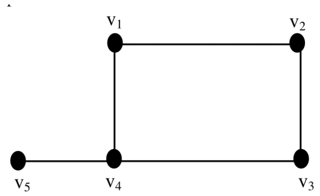

Figure 2

If we assign colour i to v_i *(*$i = 1, 2, 3, 4$*) and colour 4 to* v_5 *then this colouring is an E-Chromatic Colouring of G and note that E-Chromatic number of this graph = 4.*

Note that this colouring is not a proper colouring of the graph. Also it is observed that there is a proper colouring of G which is an E–Chromatic Colouring. This colouring is as follows.
Assign colour i to c_i $(i = 1, 2, 3, 4)$ and assign colour -1 to v_5.

Definition 2.8 (Homomorphism of Graphs [6])**.** *Let G and H be two graphs. A function* $f : V(G) \to V(H)$ *is said to be a homomorphism from G to H if whenever u and v are adjacent vertices of G , f(u) and f(v) are adjacent vertices of H.*

Definition 2.9 (E-Homomorphism of Graphs)**.** *Let G and H be two graphs. A function $f : V(G) \to V(H)$ is said to be an E-homomorphism from G to H if whenever u and v are equitable adjacent vertices of G, f(u) and f(v) are adjacent vertices of H.*

Example 2.10. *Consider the following two graphs G and H. Graph G has the vertex set $\{v_1, v_2, v_3, v_4\}$ which is as shown below.*

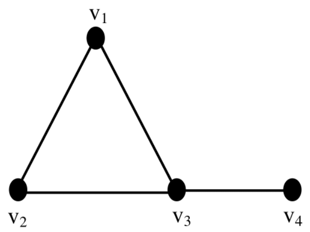

Figure 3

The vertex set of the graph H is $\{v_1, v_2, v_3, v_4\}$ and it is as shown below.

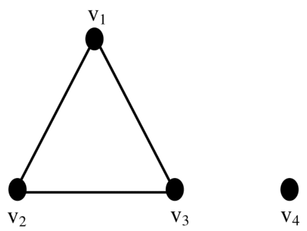

Figure 4

Define the function $f : V(G) \to V(H)$ as follows: $f(v_i) = v_i$ $(i = 1, 2, 3, 4)$. It is obvious that f is an E–Homomorphism from G to H. Let $g = f^{-1}$ then it is obvious that $g : V(H) \to V(G)$ is also an E–Homomorphism from H to G.

Remark 2.11. *Suppose f is an E-homomorphism from G to H which is onto. Let $y \in V(H)$ and consider $S = f^{-1}(y)$. Suppose x_1, $x_2 \in S$ and suppose they are equitable adjacent. Since f is an E-homomorphism, $f(x_1)$ and $f(x_2)$ are adjacent and therefore they are distinct. However $f(x_1) = f(x_2) = y$. This is a contradiction. Therefore S must be an equitable independent set. Now, we will see that an n–E–Colouring of a graph G gives rise to an E-homomorphism from $G \to K_n$.*

Remark 2.12. *Suppose f is an E-homomorphism from G to H and G is regular or bi–regular. Therefore any two vertices of G are adjacent iff they are equitable adjacent. Therefore f is a homomorphism from G to H.*

Theorem 2.13. *Let G be a graph and f is an n–E–Colouring of a graph G, then f can be regarded as an E- homomorphism from $G \to K_n$.*

Proof. Let $f : V(G) \to \{1, 2, \ldots, n\}$ be an n–E–Colouring of a graph G. Consider the complete graph K_n whose vertices are $\{1, 2, \ldots, n\}$. If we regard f as a function from V(G) $\to$ V(K_n) then if u and v are equitable adjacent vertices of G then f(u) and f(v) are distinct members of the set {1, 2, ..., n}. This means that f(u) and f(v) are adjacent vertices of K_n. Thus, f is an E-homomorphism from $G \to K_n$. □

Remark 2.14. *Considering the above theorem we can say that the E–Chromatic number of a graph G is the smallest positive integer n such that there is an E-homomorphism from $G \to K_n$.*

Remark 2.15. *Let G be a graph and suppose $f : V(G) \to \{1, 2, \ldots, n\}$ is a chromatic colouring of the graph. (i.e. $\chi(G) = n$) then obviously f is an E–Colouring of the graph. Therefore the E–Chromatic number of $G \leq n =$ Chromatic number of G.*

Notation 2.16. *Suppose G and H are graphs. If there is an E- homomorphism from G to H then we will write* $G\overset{E}{\to}H$*. If there is no E-homomorphism from G to H then we denote it as* $G\overset{E}{\nrightarrow}H$*. If there is an E-homomorphism from G to H and If there is an E- homomorphism from H to G, we denote this by* $G\overset{E}{\leftrightarrows}H$*.*

Proposition 2.17. *Let G and H be two graphs such that* $G\overset{E}{\to}H$ *then* $\chi_E(G) \leq \chi(H)$*.*

Proof. Suppose $\chi(H) = n$ and let $f : V(H) \to \{1, 2, \ldots, n\}$ be a chromatic colouring of H. f can be regarded as a homomorphism from $H \to K_n$. Let g be an E–homomorphism from G to H. Now, $f \circ g : V(G) \to V(K_n)$ is an E–homomorphism from G to K_n. Therefore $\chi_E(G) \leq n = \chi(H)$. □

Let G be a graph and $f : V(G) \to \{1, 2, \ldots, n\}$ be an E-Chromatic Colouring of G. Let $C_1, C_2, \ldots, C_n$ be the colour classes of G corresponding to this E–Colouring. We have the following theorem

Theorem 2.18. *In every colour class C there is a vertex v such that v has an equitable neighbour in every other colour class.*

Proof. Suppose there is a colour class in which no such vertex exists. For simplicity we may assume that this colour class is C_1. $\forall\ v \in C_1$ there is a colour class say C_j in which v has no equitable neighbour. (This j depends on vertex v) Now, we obtain a new E–Colouring of G as follows. Assign colour j to v if $v \in C_1$ and v has no equitable neighbour in C_j. Repeat this $\forall$ v in C_1. Thus, now we have an E–Colouring of G consisting of colours $\{2, 3, \ldots, n\}$. This is a contradiction because the given colouring $f : V(G) \to \{1, 2, \ldots, n\}$ is an E-Chromatic Colouring. Therefore the statement of the theorem holds. □

Now, we prove a necessary condition under which the chromatic number of a graph is same as the E–Chromatic number of a graph.

Theorem 2.19. *Let G be a graph. If Chromatic Number of G = E–Chromatic Number of G then for every chromatic colouring of G and for every colour class C of this chromatic colouring there is a vertex v in C such that v has an equitable neighbour in every other colour class of this colouring.*

Proof. Suppose there is a chromatic colouring f of G which fails to satisfy the condition. Then there is a colour class C of f such that $\forall\ v \in C$ there is a colour class C' of f such that $C' \neq C$ and v has no equitable neighbour in C'. We obtain a new E–Colouring f' by assigning the colour of C' to v, $\forall\ v \in C$. Then the E–Colouring f' uses one less colour than f. Which is a contradiction. Therefore the statement of the theorem holds. □

Definition 2.20 (Maximal Equitable Independent Set [5])**.** *Let G be a graph and S be an equitable independent subset of G. Then S is said to be maximal equitable independent set if for every vertex v in V(G)–S,* $S \cup \{v\}$ *is not an equitable independent set.*

Example 2.21. *Consider the graph G which has the vertex set* $\{v_1, v_2, v_3, v_4\}$ *which is as shown below.*

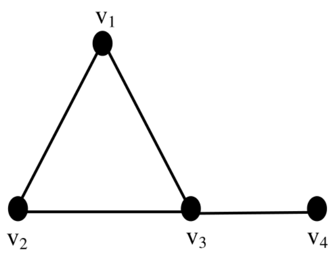

Figure 5

Obviously $S = \{v_3, v_4\}$ is a maximal equitable independent set. There are other maximal equitable independent set also. For example $T = \{v_2, v_4\}$ is also a maximal equitable independent set.

Theorem 2.22. *Let G be a graph and S be an equitable independent set which is not a maximal equitable independent set then there is an E–Chromatic Colouring of G such that S is not a colour class in this E–Chromatic Colouring.*

Proof. If S is not a colour class in every E–Chromatic Colouring then the theorem is proved. Suppose there is an E–Chromatic Colouring f of G such that S is a colour class in this E–Chromatic Colouring. Since S is not a maximal equitable independent set, therefore ∃ a vertex x which is not in S and x is not equitable adjacent with every vertex of S. Now, x lies in some colour class S' such that $S' \neq S$. Suppose $S' = \{x\}$ then by Theorem 2.18 x is equitable adjacent with some vertex of every colour class. In particular, x is equitable adjacent with some vertex of S also. Which is not true. Thus, S' contains some other element y also. i.e. S' has at least two vertices. Now, we obtain a new E–Colouring as follows.
Assign the colour of S to the vertex x and do not change colours of any other vertex. Obviously, this colouring is an E–Colouring and uses the same number of colours as f. We call this colouring as f'. Obviously S is not a colour class in this colouring f'. Thus the theorem is proved. □

Definition 2.23 (Transversal)**.** *Let S be a non-empty set and $\{S_1, S_2, \ldots, S_k\}$ be a family of nonempty subsets of S. A subset T of S is said to be a transversal of this family if $T \cap S_i \neq \phi$, $\forall\ i \in \{1, 2, \ldots, k\}$.*

Theorem 2.24. *Let G be a graph and S be an equitable independent subset of G. Suppose f is an E–Chromatic Colouring of G. Then either V(G)–S is a transversal of the colour classes of f or there is an E–Chromatic Colouring f' of G such that S is a colour class in this E–Colouring.*

Proof. Let f be an E–Chromatic Colouring of G. Suppose $V(G) - S = T$ is not a transversal for this E–Colouring f of G. Therefore, T does not contain any vertex of a particular colour of this colouring. This means that for a particular colour of this colouring the corresponding colour class C does not intersect T. Therefore $C \subseteq S$. If $C = S$ then S is a colour class for this colouring f itself. Suppose $C \subset S$. Also $y \in S - C$, then the colour of y is different from the colour of C and also $\{y\}$ cannot be a colour class for this colouring because otherwise y would be equitable adjacent to some vertex of C (by Theorem 2.18). Which is a contradiction as S is an equitable independent set.
Therefore there is a vertex y' outside S which has the same colour as y. Thus we have proved that for every y in S–C there is a vertex y' such that y' is not in S and y' has the same colour as y. Now we define a new colouring f' as follows. Assign the same colour as that of C to all vertices of S and do not change the colours of vertices which are outside of S. Then this is an E–Chromatic Colouring and S is a colour class in this colouring. □

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