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Reliability Analysis of Multi-State Series System

Research Article

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Abstract: Reliability Analysis considering Multiple Possible States is known as Multi-State Reliability Analysis. Multi-State System Reliability Models allow both the System and its Components to assume more than two levels of performance. Though Multi-State Reliability Models provide more realistic and more precise representation of Engineering System, they are much more complex and present major difficulties in System definition and performance evaluation. This paper presents a new Systematic Approach for the Reliability Analysis of Multi-State Series System which helps to determine the Expected Throughput and Performance-Free Failure Operation (PFFO) of the System. This Approach is applied to a Mathematical Model dealing with a Particular Population, being categorized into four groups based on their Haemoglobin level and the Health Status of the Population under Study is analyzed.

Keywords: Multi-State System, Series System, Descartes Product, Multi-Series System.

1. Introduction

All Systems are designed to perform their intended tasks in a given environment. Some Systems can perform their tasks with various distinctive levels of efficiency usually referred to as Performance rates. A System that can have a finite number of Performance Rates is called a Multi-State System (MSS). Usually a MSS is composed of elements that in their turn can be Multi-State. Actually, a Binary System is the simplest case of a MSS having two distinctive States (perfect functioning and complete failure).

The basic concepts of MSS Reliability were primarily introduced in the mid of the 1970's by Murchland (1975), El-Neveihi et al. (1978), Barlow and Wu (1978) and Ross (1979). Natvig (1982), Block and Savits (1982) and Hudson and Kapur (1982) extended the results obtained in these works. Since that time MSS Reliability began intensive development. Essential achievements that were attained up to the mid 1980's were reflected in Natvig (1985) and in El-Neveihi and Prochan (1984) where can be found the state of the art in the field of MSS Reliability at this stage. The history of ideas in MSS reliability theory at next stages are found in Lisnianski, Levitin(2003) and Natvig (2007) [8].

Reliability Analysis of Multistate Systems has a long history. The first papers dedicated to this subject appeared as early as 1978 (Barlow & Wu,1978; EI-Neweilhi et al, 1978) Later, several papers with introduction of a new technique for MultiSystem analysis appeared (Ushakov 1986, 1988, 1998) and finally a real burst of research papers on the subject (Lisnianski and Levitin, 2003: levitin et.al., 2003: Levitin, 2004, 2005). [1] Most of the Reliability literature deal with Binary Systems of Binary Components in which only 2 states are functioning and failed. Some recent work by Barlow & Wu [2] EI-Neweilhi,

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Proschan, and Sethuram [4] and Ross [6] treat the general case of more than two states. This idea is quite useful since in many real life situations, Components (and/or) System can be in intermediate states [7].

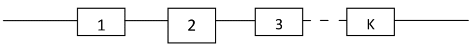
In this Paper, we apply a New Technique for evaluating the Expected Throughput and Probability of Failure Free Operation of Multistate Series System; which is applied to Reliability Models [5]. We study the Health status of a particular Population categorized into different groups based on their haemoglobin level. The paper is organized as follows. In Section 1, we give the Introduction on Multistate Series System, In Section 2, we give the basic definitions and related concepts. In Section 3, we derive the expression for calculating the Expected Throughput of Multi State Series System and in Section 4, we deal with the Application of the proposed model and in Section 5, we draw the conclusion.

2. Basic Definitions and Related Concepts

2.1. Series System

In a Series System, all Components in the System should be operating to maintain the required operation of the System. Thus the failure of any one Component of the System will cause failure of the whole System.

Series configuration



Block diagram of K unit series system

Figure 1

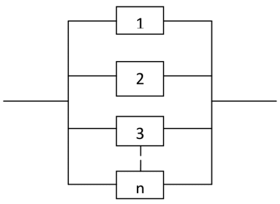
The System Reliability of the Series system is given by

$$R_s = r_1 r_2 \dots r_m = \prod_{i=1}^{m} r_i$$

2.2. Parallel System

In a Parallel System, the System operates if one or more Components operate and the System fails if all components fail. The Parallel n-Components are represented by the following block diagram [8].

Parallel configuration



n units Parallel System

Figure 2

For Constant Failure Rates of Parallel units,

$$R_p(t) = 1 - (1 - r_1(t))(1 - r_2(t)) \dots (1 - r_n(t))$$

Where $R_p(t)$ is the Parallel System Reliability at time t. $r_i(t)$is the Reliability of the i^{th} component

$$R_p(t) = 1 - \prod_{i=1}^{n} (1 - r_i(t)).$$

2.3. Descartes Product of Two Sets

	(p_{11}, a_{11})	(p_{12}, a_{12})	$\dots$	(p_{15}, a_{15})
(p_{21}, a_{21})	(P_{11}, A_{11})	(P_{21}, A_{21})	$\dots$	(P_{51}, A_{51})
(p_{22}, a_{22})	(P_{12}, A_{12})	(P_{22}, A_{22})	$\dots$	(P_{25}, A_{25})
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
(p_{27}, a_{27})	(P_{17}, A_{17})	(P_{27}, A_{27})	$\dots$	(P_{27}, A_{27})

We present the first polynomial $\varphi_1(z)$ as a set of pairs $\{(p_{11}, a_{11}), (p_{12}, a_{12}), \dots, (p_{15}, a_{15})\}$ and the second polynomial $\varphi_2(z)$ as the set of pairs $\{(p_{21}, a_{21}), (p_{22}, a_{22}), \dots, (p_{25}, a_{25})\}$, where p'_{jk}s are corresponding co-efficient and a'_{jk}s are corresponding powers of polynomials in unfolded form. Here P_{jk} is found as $P_{jk} = P_{j1} \times P_{2k}$ and A_{jk} is found as $A_{jk} = a_{j1} + a_{2k}$. In this case, we can keep a polynomial from of specific type : powers of product of two terms, say Z_a and Z_b, will be presented by some transforms over power of individual terms, namely,

$$p_a z^a \underset{f}{\otimes} p_b z^b = p_a p_b z^{f(a,b)}$$

where f is an arbitrary given function. Assume that unit k is characterized by following discrete distribution of its operational parameter

$$X_k : P\{X_k = x_{kj}\} = p_{kj}.$$

Then we can characterize the distribution of the operational parameter of unit k with the following vector of pairs.

$$\begin{aligned} Q_k &= \{(p_{k1}, x_{k1}), (p_{k2}, x_{k2}), \dots, (p_{ks(k)}, x_{ks(k)})\} \\ &= \{(p_{kj}, x_{kj}),\ 1 \le j \le s(k)\}, \end{aligned}$$

where s(k) is number of different values of random variable X_k. Interaction of operational parameters of two units X_k and X_i can be written as

$$\begin{aligned} Q_K \underset{f}{\otimes} Q_i &= \{(p_{kj}, x_{kj}),\ 1 \le j \le s(k)\} \underset{f}{\otimes} \{(p_{il}, x_{il}),\ 1 \le l \le s(i)\} \\ &= (p_{kj} \times p_{il}; f(x_{kj}, x_{il}),\ j = 1, 2, \dots, s(k),\ l = 1, 2 \dots, s(i). \end{aligned}$$

Interaction of Operational parameters of N-units can be written as

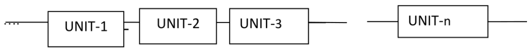
$$\underset{f}{\otimes}(Q_1, \dots, Q_K, \dots, Q_N) = \left(\prod_{i=1}^{N} P_{i,m(i)}, f(x_{1,m(1)} \dots x_{N,m(N)}) \right),\ i = 1, 2, \dots, n.$$

for all combinations of $m(i)$ where $1 \le m(i) \le s(i)$.

3. Multistateseries System

Consider a Simple Multistate Series System consisting of n-different units, where each unit has different states and the Expected Throughput of Multi State Series System changes randomly due to external and internal causes. The Fig .3.below represents a Multi State System Configuration [7].

MultiStates Series Configuration



Block diagram of n- unit series system

Figure 3

First unit:

$$p_{11} = Pr\{v = X_1\} = x_1$$

$$p_{12} = Pr\{v = X_2\} = x_2$$

$$p_{13} = Pr\{v = X_3\} = x_3$$

$$p_{14} = Pr\{v = X_4\} = x_4$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$p_{1n} = Pr\{v = X_n\} = x_n$$

Second unit:

$$p_{21} = Pr\{v = X_1\} = y_1$$

$$p_{22} = Pr\{v = X_2\} = y_2$$

$$p_{23} = Pr\{v = X_3\} = y_3$$

$$p_{24} = Pr\{v = X_4\} = y_4$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$p_{2n} = Pr\{v = X_n\} = y_n$$

where $X_1, X_2, \ldots, X_n$ denotes different states $X_i > X_j$ for $j > i$. The Entire System is characterized by minimum value of its units' through puts, that is, we have to use the operator $\underset{min}{\otimes}$ because

$$f^{(v)}_{SERIES}(x_k, x_j) = \min(x_k, x_j)$$

Let us consider the following recurrence procedure. Consider the interaction of parameters of units 1 and 2. Take a Descartes product in the form of the following table.

		Unit-1				
		$State:1$ (x_1, X_1)	$State:2$ (x_2, X_2)	$State:3$ (x_3, X_3)	$\ldots$	$State:n$ (x_n, X_n)
Unit-2	$State:1$ (y_1, X_1)	$\underset{\min(X_1,X_1)=X_1}{(x_1)(y_1)} = x_1y_1$	$\underset{\min(X_2,X_1)=X_2}{(x_2)(y_1)} = x_2y_1$	$\underset{\min(X_3,X_1)=X_3}{(x_3)(y_1)} = x_3y_1$	$\ldots$	(0, 0)
	$State:2$ (y_2, X_2)	$\underset{\min(X_2,X_1)=X_2}{(x_1)(y_2)} = x_1y_2$	$\underset{\min(X_2,X_2)=X_2}{(x_2)(y_2)} = x_2y_2$	$\underset{\min(X_3,X_2)=X_3}{(x_3)(y_2)} = x_3y_2$	$\ldots$	(0, 0)
	$State:3$ (y_3, X_3)	$\underset{\min(X_3,X_1)=X_3}{(x_1)(y_3)} = x_1y_3$	$\underset{\min(X_3,X_2)=X_3}{(x_2)(y_3)} = x_2y_3$	$\underset{\min(X_3,X_3)=X_3}{(x_3)(y_3)} = x_3y_3$	$\ldots$	(0, 0)
	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
	$State:n$ (y_n, X_n)	(0, 0)	(0, 0)	(0, 0)	$\ldots$	(0, 0)

Third unit:

$$p_{31} = Pr\{v = X_1\} = z_1$$

$$p_{32} = Pr\{v = X_2\} = z_2$$

$$p_{33} = Pr\{v = X_3\} = z_3$$

$$p_{34} = Pr\{v = X_4\} = z_4$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$p_{3n} = Pr\{v = X_n\} = z_n$$

Here and below the sum of all probabilities is not exactly equal to 1 due to rounding of results of multiplication of corresponding probabilities. The new equivalent unit has to be combined with the third unit

		Equivalent unit				
		$State:1$ (z_1, X_1)	$State:2$ (z_2, X_2)	$State:3$ (z_3, X_3)	$\dots$	$State:n$ (z_n, X_n)
Unit-2	$State:1$ (y_1, X_1)	$\underset{\min(X_1,X_1)=X_1}{(y_1)(z_1)} = y_1z_1$	$\underset{\min(X_2,X_1)=X_2}{(y_1)(z_2)} = y_1z_2$	$\underset{\min(X_3,X_1)=X_3}{(y_1)(z_3)} = y_1z_3$	$\dots$	(0, 0)
	$State:2$ (y_2, X_2)	$\underset{\min(X_2,X_1)=X_2}{(y_2)(z_1)} = y_2z_1$	$\underset{\min(X_2,X_2)=X_2}{(y_2)(z_2)} = y_2z_2$	$\underset{\min(X_3,X_2)=X_3}{(y_2)(z_3)} = y_2z_3$	$\dots$	(0, 0)
	$State:3$ (y_3, X_3)	$\underset{\min(X_3,X_1)=X_3}{(y_3)(z_1)} = y_3z_1$	$\underset{\min(X_3,X_2)=X_3}{(y_3)(z_2)} = y_3z_2$	$\underset{\min(X_3,X_3)=X_3}{(y_3)(z_3)} = y_3z_3$	$\dots$	(0, 0)
	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
	$State:n$ (y_n, X_n)	(0, 0)	(0, 0)	(0, 0)	$\dots$	(0, 0)

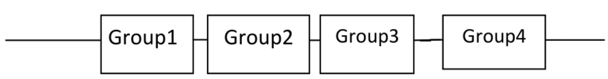
These results allow calculating the expected of the level E[v].

$$\begin{aligned}
E[v] &= z_1y_1X_1 + [(z_1.y_1) + (z_2.y_1) + (z_2.y_2)]\,X_2 + [(z_1)(y_1) + (z_2)(y_2) + (z_3)(y_3) + (z_3)(y_2) + (z_3)(y_1)]\,X_3 + \dots \\
&+ [(z_1)(y_1) + (z_2)(y_2) + (z_3)(y_3) + (z_3)(y_2) + (z_3)(y_1) + \dots + (z_n)(y_{n-j})]\,X_n \\
&= [z_1y_1]\,X_1 + \left[z_1y_1 + \sum_{i=1}^{2} z_2y_i\right] X_2 + \left[\sum_{i=1}^{3} z_iy_3 + \sum_{j=1}^{2} z_2y_j\right] X_3 + \left[\sum_{i=1}^{4} z_4y_i + \sum_{i=1}^{3} z_iy_3\right] X_4 + \dots \\
&+ \left[\sum_{i=1}^{n-2} z_{n-2}y_i + \sum_{j=1}^{n-3} z_jy_{n-1}\right] X_{n-2} + \left[\sum_{i=1}^{n-1} z_{n-1}y_i + \sum_{j=1}^{n-2} z_jy_{n-2}\right] X_{n-1} + \left[\sum_{i=1}^{n} z_ny_i + \sum_{j=1}^{n-1} z_jy_{n-1}\right] X_n
\end{aligned}$$

4. The Example for a SSM (Series system model)

We analyze the problem below to check the health status of the Population under study. Consider a Simple example of Multistate Series System Consisting of four different groups of people, where each group has people with four states of different levels of haemoglobin in their blood. The Hb level varies due to various internal & external causes.

Series configuration



Block diagram of 4-unit series system
Figure 4

Considered structure of group of peoples HB level.

First unit:

$$p_{11} = Pr\{Hb = X_1\} = 0.7$$

$$p_{12} = Pr\{Hb = X_2\} = 0.2$$

$$p_{13} = Pr\{Hb = X_3\} = 0.1$$

$$p_{14} = Pr\{Hb = X_4\} = 0$$

Second unit:

$$p_{21} = Pr\{Hb = X_1\} = 0.8$$

$$p_{22} = Pr\{Hb = X_2\} = 0.15$$

$$p_{23} = Pr\{Hb = X_3\} = 0.05$$

$$p_{24} = Pr\{Hb = X_4\} = 0$$

X_1, X_2, X_3, X_4. High, medium, low, very low values and $X_i > X_j$ of $i < j$. Third unit has the same distribution of the second one.

The entire hemoglobin characterized minimum current value of its units throughput that is will use the operation $\underset{min}{\otimes}$ because

$$f^{(Hb)}_{series}(Hb_{x_k}, Hb_{x_j}) = \min(x_k, x_j)$$

		Unit-1			
		$State:1$ (0.7, 18)	$State:2$ (0.2, 17)	$State:3$ (0.1, 16)	$State:4$ (0, 0)
Unit-2	$State:1$ (0.8, 18)	$\underset{\min(18,18)=18}{(0.7)(0.8)} = 0.56$	$\underset{\min(18,17)=17}{(0.2)(0.8)} = 0.16$	$\underset{\min(18,16)=16}{(0.1)(0.8)} = 0.08$	(0, 0)
	$State:2$ (0.15, 17)	$\underset{\min(18,17)=17}{(0.7)(0.15)} = 0.105$	$\underset{\min(17,17)=17}{(0.2)(0.15)} = 0.03$	$\underset{\min(17,16)=16}{(0.1)(0.15)} = 0.015$	(0, 0)
	$State:3$ (0.05, 16)	$\underset{\min(18,16)=16}{(0.7)(0.05)} = 0.035$	$\underset{\min(17,16)=16}{(0.2)(0.05)} = 0.01$	$\underset{\min(16,16)=16}{(0.1)(0.05)} = 0.005$	(0, 0)
	$State:4$ (0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)

Third unit:

$$p_{31} = Pr\{Hb = 18\} = 0.56$$

$$p_{32} = Pr\{Hb = 17\} = 0.105 + 0.03 + 0.16 = 0.295$$

$$p_{33} = Pr\{Hb = 16\} = 0.035 + 0.01 + 0.005 + 0.015 + 0.08 = 0.145$$

$$p_{34} = Pr\{Hb = 14\} = 0$$

Here and below the sum of all probabilities is not exactly equal to 1 due to rounding of results of multiplication of corresponding probabilities. The new equivalent unit has to be combined with the third unit

		Equivalent Unit			
		State : 1 (0.56, 18)	*State* : 2 (0.295, 17)	*State* : 3 (0.145, 16)	*State* : 4 (0, 0)
Unit-2	*State* : 1 (0.8, 18)	(0.56)(0.8) = 0.448 min(18,18)=18	(0.295)(0.8) = 0.236 min(18,17)=17	(0.145)(0.8) = 0.116 min(18,16)=16	(0, 0)
	State : 2 (0.15, 17)	(0.56)(0.15) = 0.084 min(17,18)=17	(0.295)(0.15) = 0.04425 min(17,17)=17	(0.145)(0.15) = 0.02175 min(17,16)=16	(0, 0)
	State : 3 (0.005, 16)	(0.56)(0.05) = 0.028 min(16,18)=16	(0.295)(0.005) = 0.01475 min(16,17)=16	(0.145)(0.005) = 0.00725 min(16,16)=16	(0, 0)
	State : 4 (0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)

These results allow calculating the expected value of the hemoglobin level E[Hb].

$$\begin{aligned} E[Hb] &= 0.448(18) + [0.084 + 0.04425 + 0.236](17) + [0.028 + 0.01475 + 0.00725 + 0.02175 + 0.116](16) \\ &= 17.26025 \end{aligned}$$

PFFO of the system is obtained by some chosen criteria of failure. For instance, if a failure criterion is $V < 17$ then PFFO is equal to

$$\begin{aligned} P\{V > 17\} &= 0.448 + 0.084 + 0.04425 + 0.236 \\ &= 0.81225 \end{aligned}$$

81.225 % of the population are having good range of haemoglobin level.

5. Conclusion

This Paper offers ideas for using Reliability Principles to determine the Health States of the Population and take measures to improve Reliability. Applying the lessons from Reliability Engineering to Health Care System, holds the promise of moving our Blood System to new levels of Hb Consistency and Quality.

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