International Journal of Mathematics And its Applications
Volume 4, Issue 2–A (2016), 71–77.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Monopoly Free and Monopoly Cover in Graphs

Research Article

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Abstract: In a graph $G = (V, E)$, a set $M \subseteq V(G)$ is said to be a monopoly set of G if every vertex $v \in V - M$ has at least $\frac{d(v)}{2}$ neighbors in M. The monopoly size $mo(G)$ is the minimum cardinality of a monopoly set among all monopoly sets in G. In this paper, we introduce two type sets associated with a monopoly set, monopoly free and monopoly cover sets. A set $F \subseteq V$ is called a monopoly free of G if F does not contain any monopoly set as a subset. The monopoly free set is maximal if for every $v \notin F$ there exists $A \subseteq F$ such that $A \cup \{v\}$ is a monopoly. The cardinality of a largest maximal monopoly free set, denoted $mof(G)$, and is called a monopoly free size of G. A set $C \subseteq V$ is called a monopoly cover of G if C contains, at least, one vertex from each monopoly set in G. The cardinality of a minimum monopoly cover set in G, denoted by $moc(G)$, and is called a monopoly cover size of G. We investigate the relationship between these three parameters. Exact values of $mof(G)$ and $moc(G)$ for some standard graphs are found. Bounds for $mof(G)$ and $moc(G)$ are obtained. It is shown that $mof(G) + moc(G) = n$ for every graph G of order n. Moreover, it is shown that for every integer $k \geq 3$ there exist a graph G of order $n \geq 6$ such that $mof(G) = moc(G) = mo(G) = k$. finally, we characterize all graphs G with $mof(G) = 0, 1$ and $n - 2$.

MSC: 05C07, 05C69.

Keywords: Monopoly set, Monopoly free set, Monopoly cover set, Monopoly free size, Monopoly cover size.

1. Introduction

By a graph $G = (V, E)$ we mean a simple graph, that is finite, having no loops no multiple and directed edges. As usual, we denote $n = |V|$ and $m = |E|$ to the number of vertices and edges in a graph G, respectively. For a vertex $v \in V$, the open neighborhood, $N(v)$, of v in G is the set of all vertices that are adjacent to v and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The degree , $d(v)$, of a vertex v in G is the number of its neighbors in G. An isolated vertex is a vertex with degree zero and a pendant vertex is a vertex with degree one. We denote by $\Delta(G)$ and $\delta(G)$ to maximum and minimum degree among the vertices of G, respectively. As usual, $\overline{G}$ denote to the complement graph of G, for a subset $S \subseteq V$, $\bar(S) = V - S$, $\lceil x \rceil$ denote to the smallest integer number that greater than or equals to x and $\lfloor x \rfloor$ to the greatest integer number that smaller than or equals to x. A set $I \subseteq V$ is independent if no two vertices in I are adjacent. The independent sets of maximum cardinality are called maximum independent sets. The number of vertices in a maximum independent set in a graph G is the independence number (or vertex independence number) of G and is denoted by $\alpha(G)$. For more terminologies and notations in graph theory, we refer the reader to the books [3].

A set $M \subseteq V(G)$ is called a monopoly set of G if for every vertex $v \in V(G) - M$ has at least $\frac{d(v)}{2}$ neighbors in M. The monopoly size of G is the smallest cardinality of a monopoly set in G, denoted by $mo(G)$. The concept of monopoly in the

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graph was introduced by Khoshkhak K. et al. [5]. Some properties of monopoly in graphs have studied in [7], Other types of monopoly in graphs have been subsequently proposed, for example, see [8–10]. In particular, a monopoly in graphs is a dynamic monopoly (dynamos) that, when colored black at a certain time step, will cause the entire graph to be colored black in the next time step under an irreversible majority conversion process. Dynamos were first introduced by Peleg D. [11]. For more details in monopoly and dynamos in graphs, we refer to [1, 2, 4, 6, 12].

A set $F \subseteq V(G)$ is called a monopoly free set of G if F does not contain any monopoly set as a subset, i.e., for every monopoly set M in G, $M - F \neq \phi$. The monopoly free set of G is maximal if for every $v \notin F$ there exists a subset $A \subseteq F$ such that $A \cup \{v\}$ is a monopoly set in G. A maximum monopoly free set is a maximal monopoly free set of largest cardinality. The cardinality of the maximum monopoly free set in G is denoted by $mof(G)$ and is called a monopoly free size of G. For simplicity of notion, we will refer to a monopoly free set of G as a mof-set. A set $C \subseteq V$ is called a monopoly cover of G if C contains at least one vertex from each monopoly set in G, i.e., for every monopoly set M in G, $M \cap C \neq \phi$. The monopoly cover set of G is minimal if no proper subset of C is a monopoly cover. A minimum monopoly cover set is a minimal monopoly cover set of smallest cardinality. The cardinality of a minimum monopoly cover set in G is denoted by $moc(G)$ and is called a monopoly cover size of G. once again, we will refer to a monopoly cover set of G as a moc-set.

In this paper, we investigate the relationship between these three parameters $mo(G), mof(G)$ and $moc(G)$. Exact values of $mof(G)$ and $moc(G)$ for some standard graphs are found. Also,bounds for $mof(G)$ and $moc(G)$ are obtained. It is shown that $mof(G) + moc(G) = n$ for every graph G of order n. Moreover, it is shown that for every integer $k \geq 3$ there exist a graph G of order $n \geq 6$ such that $mof(G) = moc(G) = mo(G) = k$. Finally, we characterize all graphs G of order n with $mof(G) = 0, 1$ and $n - 2$. On the other side, $moc(G) = 2, n - 1$ and n.

The following are some fundamental results which will be required for many of our arguments in this paper:

Theorem 1.1 ([5]). *Let G be a graph on n vertices with m edges whose maximum degree is $\Delta(G)$. Then $\frac{2m}{3\Delta(G)} \leq mo(G) \leq \frac{n}{2}$.*

Theorem 1.2 ([7]). *Let G be a graph of order n and minimum degree δ. Then $\frac{\delta}{2} \leq mo(G) \leq n - \frac{\delta+2}{2}$.*

2. Monopoly Free and Monopoly Cover in Graphs

Since every graph G possess a monopoly set it follows that every graph posses a monopoly cover set and if every subset of $V(G)$ contains a monopoly set, then we say $mof(G) = 0$ and $moc(G) = n$. For example, in a complete graph K_3, every subset of $V(K_3)$ is a monopoly set, $mo(K_3) = 1$, then the moc-set of K_3 is $V(K_3)$ and the mof-set of K_3 is ϕ hence $moc(K_3) = n$ and $mof(K_3) = 0$.

Theorem 2.1. *For every graph G, A set $F \subseteq V$ is a mof-set if and only if $\overline{F}$ is a moc-set.*

Proof. In a graph G, a set $F \subseteq V$ is a mof-set of G, if and only if for every monopoly set M in G, $M - F \neq \phi$, if and only if for every monopoly set M in G, $M \cap \overline{F} \neq \phi$, if and only if $\overline{F}$ is a moc-set of G. □

Corollary 2.2. *For every graph G of order n, $mof(G) + moc(G) = n$.*

Proposition 2.3. *For every graph G, if a set $F \subseteq V$ is a maximal mof-set, then for every $v \in V - F$ there exists a subset $M_v \subseteq F$ such that $M_v \cup \{v\}$ is a monopoly set in G.*

Proof. Let F be a maximal mof-set of G. Then by a maximality of F, $F \cup \{v\}$ is a monopoly set of G, for every $v \in V - F$. If $F \cup \{v\}$ is minimal, then $M_v = F$, Otherwise, there exists a proper subset $M_v \subset F$ such that $M_v \cup \{v\}$ is a monopoly set of G. □

Proposition 2.4. *For any graph G, if a set $C \subseteq V$ is a minimal moc-set, then for every $v \in C$ there exists a monopoly set M_v in G such that $M_v \cap C = \{v\}$.*

Proof. Let $C \subseteq V$ be a minimal moc-set of G. Then $C - \{v\}$ is not a moc-set in G, for every $v \in C$, that means there exists a monopoly set M_v in G such that $M_v \cap (C - \{v\}) = \phi$, but by the definition of a moc-set $M_v \cap C \neq \phi$. Hence, $M_v \cap C = \{v\}$. □

In the following result, we will just determining the exact values of $mof(G)$ for some standard graph G and by using Corollary 2.2 any one can easily find the corresponding values of $moc(G)$ for each graph.

Proposition 2.5. *For every $n \geq 2$*

(1). For the path P_n and a star $K_{1,n-1}$ graphs, $mof(P_n) = mof(K_{1,n-1}) = n - 2$.

(2). For the cycle graph C_n, $n \geq 3$, $mof(C_n) = n - 3$.

(3). For the complete graph K_n, $mof(K_n) = \lfloor \frac{n}{2} \rfloor - 1$.

(4). For the complete bipartite graph $K_{r,s}$, $2 \leq r \leq s$,

$$mof(K_{r,s}) = \begin{cases} s + \frac{r}{2} - 2, & \text{if } r \text{ is even;} \\ s + \lfloor \frac{r}{2} \rfloor - 1, & \text{if } r \text{ is odd.} \end{cases}$$

(5). For the wheel graph $W_{1,n-1}$, $n \geq 4$, $mof(W_{1,n-1}) = n - 3$.

(6). For the complete multipartite graph $K_{n_1,n_2,...,n_k}$, $2 \leq n_1 \leq n_2 \leq ... \leq n_k$ and $k \geq 3$

$$mof(K_{n_1,n_2,...,n_k}) = \lfloor \frac{n + n_k}{2} \rfloor - 2.$$

Where $n = n_1 + n_2 ... + n_k$.

(7). For the totaly disconnected graph $\overline{K_n}$, $mof(\overline{K_n}) = 0$.

(8). For every connected graph G, $mof(G \cup \overline{K_p}) = mof(G) + p$.

Proof. We left the proof of this Theorem because it is consequences of the next results. □

By the definition of the monopoly set of a graph G, every graph G posses a monopoly set. Furthermore, for every nontrivial graph G the set $V - \{v\}$ is a monopoly of G for every $v \in V(G)$. this gives the following bounds for $mof(G)$ and $moc(G)$ in terms the order of G.

Proposition 2.6. *For any graph G of order n,*

$$0 \leq mof(G) \leq n - 2,$$

$$2 \leq moc(G) \leq n.$$

The bounds are sharp, K_2 attending the lower bound for $mof(G)$ (The upper bound for $moc(G)$) and P_n attending the upper bound for $mof(G)$ (the lower bound for $moc(G)$).

Theorem 2.7. *Let G be a graph of order $n \geq 2$ and size $m \geq 1$. Then $mof(G) = (n-2) - \lfloor \frac{\delta}{2} \rfloor$.*

Proof. Let $v \in V(G)$ be a vertex with $d(v) = \delta$ and let a set $S \subset N(v)$ with $|S| = \lfloor \frac{\delta}{2} \rfloor + 1$. Now, let $M = V - (S \cup \{v\})$. Then M does not contain any monopoly set, because $v \notin M$ and $|N(v) \cap M| < \frac{d(v)}{2}$. Hence, M is a mof-set in G. Therefore,

$$\begin{aligned} mof(G) \geq |M| &= |V - (S \cup \{v\})| \\ &= n - (|S| + 1) = n - (\lfloor \frac{\delta}{2} \rfloor + 1 + 1) \\ &= (n-2) - \lfloor \frac{\delta}{2} \rfloor. \end{aligned}$$

On the other hand, let $v \in V(G)$ be a vertex with $d(v) = \delta$ and let a set $S \subset N(v)$ with $|S| = \lceil \frac{\delta}{2} \rceil$. Now, let $D = \{v\} \cup (N(v) - S)$. Then $V - D$ is a monopoly set of G (For details of the proof, see the proof of Theorem 1.2), that means any subset M of $V(G)$ with $|M| \geq |V - D| = n - |D| = n - (1 + \lceil \frac{\delta}{2} \rceil) = (n-1) - \lceil \frac{\delta}{2} \rceil$ is a monopoly set of G. Thus, $mof(G) \leq (n-2) - \lfloor \frac{\delta}{2} \rfloor$. Therefore, $mof(G) = (n-2) - \lfloor \frac{\delta}{2} \rfloor$. □

Corollary 2.8. *Let G be a graph of order $n \geq 2$ and size $m \geq 1$. Then $moc(G) = \lfloor \frac{\delta}{2} \rfloor + 2$.*

The following results are immediate consequences of Corollary 2.2, Observation 2.6 and Theorem 2.7.

Corollary 2.9. *Let G be a graph of order $n \geq 2$ and size $m \geq 1$. Then*

(a). $mof(G) \geq \lfloor \frac{n}{2} \rfloor - 1$.

(b). $moc(G) \leq \lceil \frac{n}{2} \rceil + 1$.

(c). $mof(K_n) \leq mof(G) \leq mof(P_n)$.

(d). $moc(P_n) \leq moc(G) \leq moc(K_n)$.

Corollary 2.10. *For every tree T with at least two vertices, $mof(T) = n - 2$,$moc(T) = 2$.*

Theorem 2.11. *Let G be a graph with at least one edge. Then $mof(G) \geq \alpha(G) - 1$. The bound is sharp, the star graph $K_{1,n}$ achieve it.*

Proof. Let G be a graph with size $m \geq 1$ and let $I \subseteq V$ be a maximum independent set of G. Then $I - \{v\}$, for every $v \in I$ does not contain any monopoly set. Hence, $mof(G) \geq |I - \{v\}| = \alpha(G) - 1$. □

Let $\beta(G)$ be the covering number of a graph G. Then we have the following result.

Theorem 2.12. *Let G be a graph with at least one edge. Then $moc(G) \leq \beta(G) + 1$.*

Proof. Let G be a graph of order n and size $m \geq 1$. Then by well-known result $\alpha(G) + \beta(G) = n$, Corollary 2.2 and Theorem 2.11, we have

$$\begin{aligned} moc(G) &= n - mof(G) \\ &\leq n - (\alpha(G) - 1) \\ &\leq (n - \alpha(G)) + 1 \\ &\leq \beta(G) + 1. \end{aligned}$$

□

The following results give the relationship between the parameters $mof(G)$, $moc(G)$ and $mo(G)$ of graphs.

Theorem 2.13. *For every graph G with at least one edge,*

(1). $mof(G) \geq mo(G) - 1$,

(2). $moc(G) \leq mo(G) + 2$.

Proof.

(1). Let M be a minimum monopoly set of a graph G. Then $M - \{v\}$ dose not contain any monopoly set. Hence, $mof(G) \geq |M - \{v\}| = mo(G) - 1$.

(2). By Corollary 2.8, $moc(G) \leq \frac{\delta}{2} + 2$ and by Theorem 1.2, $mo(G) \geq \frac{\delta}{2}$. Then $moc(G) \leq mo(G) + 2$.

□

The bounds in Theorem 2.13 are sharp, for example, the complete graph K_n for every n, is attending the lower bound for $mof(G)$ and also, K_n, for n is odd, is attending the lower bound for $moc(G)$.

Theorem 2.14. *For every graph G with at least one edge,*

(1). $mof(G) \geq (n-2) - mo(G)$,

(2). $moc(G) \leq (n+1) - mo(G)$.

Corollary 2.15. *For every integer $a \geq 1$, there exists a graph G with $mo(G) = a$, $mof(G) = a - 1$ and $moc(G) = a + 1$.*

Proof. The result is true for each integer $a \geq 1$ since the complete graph K_{2a} have the desired properties. □

At the extreme, it is possible for all three of these parameters to have the same prescribed value.

Theorem 2.16. *For every integer $t \geq 3$, there exists a graph G of order $n \geq 6$ such that $mof(G) = moc(G) = mo(G) = t$.*

Proof. We have already noted in Corollary 2.2, that $mof(G) + moc(G) = n$ for every graph G with order n. Then $mof(G) = moc(G)$ if and only if $mof(G) = moc(G) = \frac{n}{2}$, but Theorem 1.1 is stating that, $mo(G) \leq \frac{n}{2}$ for every graph G. Therefore, the result is true, if and only if for every $n \geq 6$, n is even and $t = \frac{n}{2}$. For $t = 3$ and $t = 4$, the result is holding since C_6 and $K_{4,4}$ have the desired properties. Hence, we may assume that $t \geq 5$ (resp. $n \geq 10$). Here For $t \geq 5$, let $v_1, v_2, ..., v_n$, where $n = 2t$, be a vertex set of K_n. Then let G be the graph obtained from K_n by deleting three edges incident to a common vertex $v \in V(K_n)$ (namely, the edges vv_1, vv_2 and vv_3). Hence, $\delta(G) = d_G(v) = n - 4$. Now, we will show that $mof(G) = moc(G) = mo(G) = \frac{n}{2} = t$. Firstly, for $mof(G)$, by Theorem 2.7,

$$mof(G) \geq (n-2) - \frac{n-4}{2} = \frac{n}{2}. \tag{1}$$

On the other hand, let M be any subset of $V(G)$ with $|M| = \frac{n}{2} + 1$. If $v \in M$, then for every other vertex $u \in V(G) - M$

$$|N(u) \cap M| \geq \frac{n}{2} > \frac{d_G(u)}{2}.$$

Hence, M is a monopoly set of G. Now, if $v \notin M$ and v_1, v_2 and $v_3 \in M$, then

$$|N(v) \cap M| = \frac{n}{2} + 1 - 3 = \frac{n-4}{2} = \frac{d_G(v)}{2}.$$

Hence, also, M is a monopoly set of G. Therefore

$$mof(G) \leq \frac{n}{2}. \tag{2}$$

From equations 1 and 2, we obtain $mof(G) = \frac{n}{2}$. Then, for 4moc(G), by Corollary 2.2, $moc(G) = \frac{n}{2}$. Finally, for $mo(G)$, let M be any subset of $V(G)$ with $|M| < \frac{n}{2}$. Since G has $n-4$ vertex with degree $n-1$ and since $n-4 > \frac{n}{2}$ for every $n \geq 10$, it follows that there must exists at least a vertex $u \in V(G) - M$ with $d(u) = n-1$. Then

$$\frac{d(u)}{2} = \frac{n-1}{2} > \frac{n}{2} - 1 \geq |M| \geq |N(u) \cap M|.$$

Hence, M is not a monopoly set of G. That means, $mo(G) \geq \frac{n}{2}$, but by Theorem 1.1 $mo(G) \leq \frac{n}{2}$. Therefore, $mo(G) = \frac{n}{2}$, this completes the proof. □

Corollary 2.17. *Let G be a graph of order $n \geq 6$ and size $m \geq 1$. Then $mof(G) = moc(G)$, if and only if n is even number and $\delta(G) = n-4$ or $n-3$.*

The following result is an immediate consequence of Observation 2.6 and Corollary 2.9.

Theorem 2.18. *Let G be a graph with at least one edge. Then $mof(G) + 2 \leq n \leq 2mof(G) + 3$.*

Corollary 2.19. *For every positive integer k, there exists only finitely many connected graphs G with $mof(G) = k$.*

The following result characterizes all graphs G of order $n \geq 1$ for which $mof(G) \in \{0, 1, n-2\}$.

Theorem 2.20. *Let G be a graph of order $n \geq 1$ and size $m \geq 0$. Then*

(a). $mof(G) = 0$, if and only if $G = \overline{K_n}$, $G = K_2$ or $G = K_3$.

(b). $mof(G) = 1$, if and only if $G = K_1 \cup K_p$, $p = 2, 3$, $G = P_3$, $G = C_4$, $G = K_4$, $G = K_4 - e$, for every edge $e \in E(K_4)$ and $G = K_5$.

(c). $mof(G) = n-2$, if and only if G has a pendant vertex.

Proof. Let G be a graph of order n and size m.

(a). If $mof(G) = 0$, then the maximum mof-set of G is ϕ that means a set $\{v\}$ is a monopoly set of G for every $v \in V(G)$. Hence, by the definition of a monopoly set $d(v) \leq 2$ for every $v \in V(G)$ and $d(v) = d(u)$ for every v and $u \in V(G)$. Therefore, $G = \overline{K_n}$ for every n or $G = K_n$ for $n \leq 3$.
Conversely, clear.

(b). If $mof(G) = 1$, then by part (a), G has at least one edge and by Theorem 2.13, $3 \leq n \leq 5$. we consider the following cases:

Case 1 If $n = 3$, then by Observation 2.6, $0 \leq mof(G) \leq 1$, but once again by part (a), $mof(G) = 0$ if and only if $G = K_3$. Hence, $mof(G) = 1$ for $G = K_1 \cup K_2$ and $G = P_3$.

Case 2 If $n = 4$, then by part (a) and Observation 2.6, $1 \leq mof(G) \leq 2$. Hence, if G is connected, then by Theorem 2.7, $mof(G) = 1$ if and only if the minimum degree of G is greater than or equal 2. i.e., $G \in \{C_4, K_4, K_4 - e\}$ for some edge $e \in E(K_4)$, if G is disconnected, then $mof(G) = 1$ if and only if $G = K_1 \cup K_3$. Otherwise, $mof(G) = 2$

Case 3 If $n = 5$, then by Theorem 2.7, $mof(G) = 1$ if and only if $\delta(G) = 4$, i.e., $G = K_5$. Otherwise, $mof(G) \geq 2$.

(c). The result is immediate consequences of Theorem 2.7.

□

Corollary 2.21. *Let G be a graph of order $n \geq 1$ and size $m \geq 0$. Then*

(a). $moc(G) = n$, if and only if $G = \overline{K_n}$, $G = K_2$ or $G = K_3$.

(b). $moc(G) = n-1$, if and only if $G = K_1 \cup K_p$, $p = 2, 3$, $G = P_3$, $G = C_4$, $G = K_4$, $G = K_4 - e$, for every edge $e \in E(K_4)$ and $G = K_5$.

(c). $moc(G) = 2$, if and only if G has a pendant vertex.

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