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Free Actions of Semiderivations on Semiprime Semirings

Research Article

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Abstract: Motivated by the study of Laradji and Taheem in [1] in this paper we authors introduce the notion of dependent elements of semiderivations on semiprime semirings. Also in this paper we authors study and investigate dependent elements of semiderivations and proved that semiderivations and related mappings on semiprime semirings are free actions.

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1. Introduction and Preliminaries

Throughout this paper S will represent an associative semiring. In [2] Josovukman and Irena Kosi Ulbl worked on dependent elements of derivations on rings. Dependent elements were implicitly used by Kallman [3] to extend the notion of free action of automorphisms of abelian von Neumann algebras of Murray and von Neumann [4, 5]. They were later on introduced by Choda et al. [6]. Several other authors have studied dependent elements in operator algebras. A brief account of dependent elements in W^*-algebras has been also appeared in the book of Stratila [7]. The purpose of this paper is to investigate dependent elements of some mappings related to semiderivations on semiprime semirings. In this paper we characterize dependent elements of semiderivations on semiprime semirings and also we study the semiderivation f and related mappings on a semiprime semiring S are free actions.

Definition 1.1. *A semiring S is a nonempty set S Equipped with two binary operations $+$ and $\cdot$ such that*

(1). $(S, +)$ is a commutative monoid with identity element 0

(2). $(S, \cdot)$ is a monoid with identity element 1

(3). Multiplication left and right distributes over addition.

Definition 1.2. *A semiring S is said to be prime if $xsy = 0$ implies $x = 0$ or $y = 0$ for all $x, y \in S$*

Definition 1.3. *A semiring S is said to be semiprime if $xsx = 0$ implies $x = 0$ for all $x \in S$*

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Definition 1.4. *An additive mapping $d : S \to S$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in S$*

Definition 1.5. *An additive mapping $f : S \to S$ is called a semiderivation associated with a function $g : S \to S$ if for all $x, y \in S$*

(i). $f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$,

(ii). $f(g(x)) = g(f(x))$.

If $g = I$ i.e., an identity mapping of S then all semiderivations associated with g are merely ordinary derivations. If g is any endomorphism of S, then semiderivations are of the form $f(x) = x - g(x)$.

Definition 1.6. *Let S be a semiprime semiring. An element a in S is called a dependent element on a mapping $f : S \to S$ if $f(x)a = ax$ for all $x \in S$*

Definition 1.7. *A mapping $F : S \to S$ is called a free action in case zero is the only element dependent on F.*

2. Results

Theorem 2.1. *Let S be a semiprime semiring with a semiderivation f associated with a function $g : S \to S$. Then f is a free action.*

Proof. Let a in S be a dependent element of f. Hence we have to prove that $a = 0$. Now, for all $x \in S$ and $a \in S$, we have the relation

$$f(x)a = ax \tag{1}$$

Put xy for x in (1), we get

$$f(x)g(y)a + (xa - ax)y = 0, \text{ for all } x, y \in S. \tag{2}$$

Put yz for y in (2), we get

$$f(x)g(yz)a + (xa - ax)yz = 0, \text{ for all } x, y, z \in S. \tag{3}$$

Right multiply equation (2) by z and subtracting from (3), we get

$$f(x)g(y)(g(z)a - az) = 0, \text{ for all } x, y, z \in S. \tag{4}$$

Replace y by ay in (4), we get

$$axy(za - az) = 0, \text{ for all } x, y, z \in S. \tag{5}$$

Replace x by zx in (5), we get

$$azxy(za - az) = 0, \text{ for all } x, y, z \in S. \tag{6}$$

Left multiply equation (5) by z and subtracting from (6) we get, $(az - za)s(az - za) = 0$ for all $z \in S$. Since S is prime

$$(az - za) = 0, \quad \text{for all } z \in S. \tag{7}$$

Substituting (7) in (2) for all $z \in S$ we get $f(z)g(y)a = 0$. Since g is surjective

$$f(z)ya = 0 \tag{8}$$

Putting $y = ay$ in (8) we get $f(z)aya = 0$, for all $y, z \in S$. ie., $azya = 0$ by (1) for all $y, z \in S$. Hence $asa = 0$. By the semiprimeness of S we have $a = 0$. Thus f is a free action. □

Theorem 2.2. *Let S be a semiprime semiring and let $f : S \to S$ be a semiderivation associated with the function $g : S \to S$. Then the mapping $x \to xf(x)$ for all $x \in S$ is a free action.*

Proof. Consider the mapping $D : S \to S$ defined by $D(x) = xf(x)$ for all $x \in S$. Let a in S be a dependent element of D. Hence we have to prove that $a = 0$. Now, for all $x \in S$ and $a \in S$, we have the relation

$$D(x)a = ax. \tag{9}$$

Equivalently

$$xf(x)a = ax \quad \text{for all} \quad x \in S. \tag{10}$$

Linearizing (10) with respect to x

$$xf(y)a + yf(x)a = 0 \quad \text{for all} \quad x, y \in S. \tag{11}$$

Replacing x and y by a in (11) we get,

$$2af(a)a = 0. \tag{12}$$

Replacing y by xa in (11), $xf(xa)a + xaf(x)a = 0$ for all $x \in S$. That is

$$axa + xxf(a)a + xaf(x)a = 0 \;\text{ for all }\; x \in S. \tag{13}$$

Replacing x by a in (13), we get $a^3 + a2af(a)a = 0$. Hence $a^3 = 0$, by (12). ie., $a = 0$. Hence D is a free action. □

Theorem 2.3. *Let S be a 2-torsion free semiprime semiring and let $f : S \to S$ be a semiderivation associated with the function $g : S \to S$. Then the mapping $x \to xf(x) + f(x)x$ for all $x \in S$ is a free action.*

Proof. Consider the mapping $D : S \to S$ defined by $D(x) = xf(x) + f(x)x$ for all $x \in S$. Let a in S be a dependent element of D. Hence we have to prove that $a = 0$. Now, for all $x \in S$ and $a \in S$, we have the relation

$$D(x)a = ax. \tag{14}$$

Equivalently

$$xf(x)a + f(x)xa = ax \;\text{ for all }\; x \in S. \tag{15}$$

Linearizing (15) with respect to x

$$xf(y)a + yf(x)a + f(x)ya + f(y)xa = 0 \;\text{ for all }\; x, y \in S. \tag{16}$$

Replacing x by a in (16) we get $2(af(a)a + f(a)a) = 0$. Since S is 2-torsion free we get

$$af(a)a + f(a)a = 0. \tag{17}$$

Replacing x by a in (15) we get, $af(a)a + f(a)aa = a^2$. Hence $a^2 = 0$, by (17). ie., $a = 0$. Hence D is a free action. □

Theorem 2.4. *Let S be a semiprime semiring and let f_1 and f_2 be semiderivations on S associated with functions $g_1 : S \to S$ and $g_2 : S \to S$. Then the mapping $x \to f_1(x) + f_2(x)$ for all $x \in S$ is a free action.*

Proof. Consider the mapping $D : S \to S$ defined by $D(x) = f_1(x) + f_2(x)$ for all $x \in S$. Let a in S be a dependent element of D. Hence we have to prove that $a = 0$. Now, for all $x \in S$ and $a \in S$, we have the relation

$$D(x)a = ax. \tag{18}$$

Equivalently

$$f_1(x)a + f_2(x)a = ax \text{ for all } x \in S. \tag{19}$$

Replacing x by xy in (19), $f_1(x)g_1(y)a + xf_1(y)a + f_2(x)g_2(y)a + xf_2(y)a = axy$ for all $x, y \in S$. That is

$$D(x)ya + xD(y)a = axy \text{ for all } x, y \in S. \tag{20}$$

Replacing x by xa in (18),

$$D(xa)a = axa \text{ for all } x \in S. \tag{21}$$

Replacing y by a in (20),

$$D(x)a^2 + xD(a)a = axa \text{ for all } x \in S. \tag{22}$$

$axa + xa^2 = axa$, by (18). That is

$$xa^2 = 0, \text{ for all } x \in S. \tag{23}$$

Replacing x by a in (23), $a^3 = 0$ which implies $a = 0$. □

Theorem 2.5. *Let S be a semiprime semiring and let f_1 and f_2 be semiderivations on S associated with functions $g_1 : S \to S$ and $g_2 : S \to S$. Then the following conditions hold:*

(1). the mapping $x \to f_1^2(x) + f_2(x)$ for all $x \in S$ is a free action.

(2). the mapping $x \to f_1(x) + f_2^2(x)$ for all $x \in S$ is a free action.

Proof.

(1). Consider the mapping $D : S \to S$ defined by $D(x) = f_1^2(x) + f_2(x)$ for all $x \in S$. Let a in S be a dependent element of D. Hence we have to prove that $a = 0$. Now, for all $x \in S$ and $a \in S$, we have the relation

$$D(x)a = ax. \tag{24}$$

Equivalently

$$f_1^2(x)a + f_2(x)a = ax \text{ for all } x \in S. \tag{25}$$

Replacing x by xy in (25), $f_1^2(x)g_1^2(y)a + f_2(x)g_2(y)a + xf_1^2(y)a + xf_2(y) + 2f_1(x)f_1(g_1(y))a = axy$ for all $x, y \in S$. That is

$$D(x)ya + xD(y)a + 2f_1(x)f_1(y)a = D(xy)a, \text{ for all } x, y \in S. \tag{26}$$

Replacing x by xa in (24),

$$D(xa)a = axa, \quad \text{for all } x \in S. \tag{27}$$

Replacing y by a in (26),

$$D(x)a^2 + xD(a)a + 2f_1(x)f_1(a)a = D(xa)a, \text{ for all } x \in S. \tag{28}$$

$axa + xa^2 + 2f_1(x)f_1(a)a = axa$, by (24) for all $x \in S$.

$$xa^2 + 2f_1(x)f_1(a)a = 0, \text{ by (24) for all } x \in S. \tag{29}$$

Replacing x by yx in (28), $D(yx)a^2 + yxD(a)a + 2f_1(yx)f_1(a)a = D(yxa)a$, for all $x, y \in S$. $yxa^2 + 2f_1(yx)f_1(a)a = 0$ for all $x, y \in S$. $2f_1(y)g_1(x)f_1(a)a = 0$ for all $x, y \in S$,by (29). Replacing $g_1(x)$ by $f_1(x)$

$$2f_1(y)f_1(x)f_1(a)a = 0 \quad \text{for all } x, y \in S. \tag{30}$$

Left multiplying (29) by $f_1(y)$ and using (30), we get, $f_1(y)xa^2 = 0$for all $x, y \in S$. Replacing x by $f_1(a)$ and y by a, we get,

$$(f_1(a))^2a^2 = 0 \tag{31}$$

Right multiplying (29) by a we get, $xa^3 + 2f_1(x)f_1(a)a^2 = 0$ for all $x, y \in S$. Replacing x by a and using (31) we get, $a^4 = 0$, which implies $a = 0$.

(2). Proof is similar as (1)

□

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