International Journal of Mathematics And its Applications
Volume 4, Issue 2–A (2016), 103–109.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Indefinitely Oscillating Functions–Part II

Research Article

T.El Bouayachi[1]* and N.Yebari[2]

1 Département de Mathématiques, Université Abdelmalek Essaâdi, FST Tanger, Tanger, Morocco.

2 Département de Mathématiques, Université Abdelmalek Essaâdi, FS Tétouan, Tétouan, Morocco.

Abstract: This article details a class of indefinitely oscillating functions in $H^s[0,\infty)$. It is a class of functions of the Sobolev space $H^s[0,\infty)$ which have for all m integer one primitive of the order m in the same space.

Keywords: Class of indefinitely oscillating functions, Sobolev space, primitive of the order m.

1. Introduction

This is the second part of a work [13] on indefinitely oscillating functions. In [13] we have considered the case where the functions are in Sobolev spaces are cdefined on the whola real axis. In this paper, we will consider functions in Sobolev spaces which are only defined on a half axis. We will consider functions which are only defined on a half axis, $[0,\infty)$ for example. The difficulty comes from that we cannot use Fourier's transform. We overcome this difficulty either defining the space of functions indefinitely oscillating on $[0,\infty)$ as the space of restrictions to $[0,\infty)$ of functions indefinitely oscillating on $\mathbb{R}$, or defining an indefinitely oscillating function f on $[0,\infty)$ using the scalar product $\langle f,\varphi(a,b)\rangle$, where $\varphi\in C_0^\infty([-1,1])$ and $\varphi(a,b)(x)=\frac{1}{a}\varphi\left(\frac{x-b}{a}\right)$ where $b\geq a+1$. We will show that the two definitions are equivalent.

The motivation for studying indefinitely oscillating functions is given by chirps. We observe that a chirp is an asymptotic signal which is of the form $s(t)=A(t)e^{i\lambda\Phi(t)}$, where A and Φ are two smooth functions and $\lambda\gg 1$ (actually $\Phi'(t)\to\infty$ when $t\to t_0$).

2. Indefinitely Oscillating Functions on the Half Real Axis

Here we cannot use Fourier's transform. We have two ways to study Indefinitely oscillating functions within this context:

- Restriction to a half real axis of functions indefinitely oscillating on the whole $\mathbb{R}$.
- Direct definition.

Before studying the case of functions indefinitely oscillating on $[0,\infty)$ relatively to the space $H^s[0,\infty)$, we will recall the results concerning functions indefinitely oscillating on $[0,\infty)$ relatively to the space $L^\infty([0,\infty))$.

* E-mail: figo407@gmail.com

2.1. The $L^{\infty}([0,\infty))$-case

Definition 2.1. *Let f be a function defined on the half real axe $[0,\infty)$. We say that f is indefinitely oscillating in the $L^{\infty}([0,\infty))$-sense if $f \in L^{\infty}([0,\infty))$ and if for every integer m there exists $f_m \in L^{\infty}([0,\infty))$ such that $f = \left(\frac{df_m}{dx}\right)^m$, in the distributional sense on $[0,\infty)$.*

Theorem 2.2. *Let f a function indefinitely oscillating in $L^{\infty}([0,\infty))$-sense. Then f is the restriction to $[0,\infty)$ of a function g indefinitely oscillating on $\mathbb{R}$.*

Proof. We start defining the generalized moments μ_k of f by

$$\mu_k = \lim_{\epsilon \to 0} \int_0^{\infty} e^{-\epsilon x} x^k f(x)\,dx.$$

Let us first prove the existence of this limit. We write

$$\int_0^{\infty} e^{-\epsilon x} x^k \frac{df_1}{dx}(x)\,dx = \left[e^{-\epsilon x} x^k f_1(x)\right]_0^{\infty} - \int_0^{\infty} \frac{d}{dx}\left(e^{-\epsilon x} x^k\right) f_1(x)\,dx.$$

The first part in the right hand side is equal to 0. Then we have

$$\begin{aligned}\int_0^{\infty} e^{-\epsilon x} x^k f(x)\,dx &= (-1)^k \int_0^{\infty} \left(\frac{d}{dx}\right)^k \left(e^{-\epsilon x} x^k\right) f_k(x)\,dx \\ &= (-1)^k \begin{pmatrix} k!\int_0^{\infty} e^{-\epsilon x} f_k(x)\,dx + \epsilon^{k-2}\dfrac{k^2(k-2)^2}{2}\int_0^{\infty} e^{-\epsilon x} x^{k-2} f_k(x)\,dx \\ -k^2\epsilon^{k-1}\int_0^{\infty} e^{-\epsilon x} f_k(x)\,x^{k-1}dx + \epsilon^k \int_0^{\infty} e^{-\epsilon x} x^k f_k(x)\,dx \end{pmatrix}.\end{aligned}$$

We then argue by recurrence on the integer. Assume that $\lim_{\epsilon \to 0} \int_0^{\infty} e^{-\epsilon x} x^q f(x)\,dx$ exists for $q = 0, 1, \ldots, k-1$. We remark that if f is an indefinitely oscillating function on $[0,\infty)$ its primitives functions $f_1, \ldots, f_k, \ldots$ are indefinitely oscillating function too. Because of the hypothesis, the quantity $\int_0^{\infty} e^{-\epsilon x} x^q f_n(x)\,dx$ has a limit when ϵ goes to 0, for every integer n and for every integer q with $0 \le q \le k-1$. However we precisely have

$$\begin{aligned}\int_0^{\infty} e^{-\epsilon x} x^k f(x)\,dx = (-1)^k\, k! \int_0^{\infty} e^{-\epsilon x} f_k(x)\,dx + +\epsilon^{k-2}\frac{k^2(k-2)^2}{2}\int_0^{\infty} e^{-\epsilon x} x^{k-2} f_k(x)\,dx \\ - k^2\epsilon^{k-1}\int_0^{\infty} e^{-\epsilon x} f_k(x)\,x^{k-1}dx + \epsilon^k \int_0^{\infty} e^{-\epsilon x} x^k f_k(x)\,dx.\end{aligned}$$

In order to prove the desired result, it is sufficient to show the existence of

$$\lim_{\epsilon \to 0} \epsilon^k \int_0^{\infty} e^{-\epsilon x} x^k f_k(x)\,dx.$$

Integrating again and using the hypothesis, we get $\epsilon^{k+1}\int_0^{\infty} e^{-\epsilon x} x^k f_{k+1}(x)\,dx$. Another last integration gives $\epsilon^{k+2}\int_0^{\infty} e^{-\epsilon x} x^k f_{k+2}(x)\,dx$, which can be estimated from above by $\|f_{k+2}\|_{\infty}\,\epsilon \int_0^{\infty} x^k e^{-x}dx$, which goes to 0 when ϵ goes to 0.

The demonstration will now be complete if the property is satisfied at the zero rank. We have

$$\int_0^{\infty} e^{-\epsilon x} f(x)\,dx = \left[f_1(x)\,e^{-\epsilon x}\right]_0^{\infty} + \epsilon\int_0^{\infty} f_1(x)\,e^{-\epsilon x}dx = -f_1(0) + \epsilon\int_0^{\infty} f_1(x)\,e^{-\epsilon x}dx.$$

An integration gives

$$\begin{aligned}\epsilon\int_0^{\infty} f_1(x)\,e^{-\epsilon x} &= \epsilon\left[f_2(x)\,e^{-\epsilon x}\right]_0^{\infty} + \epsilon^2\int_0^{\infty} f_2(x)\,e^{-\epsilon x}dx \\ &= -\epsilon f_2(0) + \epsilon^2\int_0^{\infty} f_2(x)\,e^{-\epsilon x}dx.\end{aligned}$$

We have $0 \leq \epsilon^2 \left|\int_0^\infty f_2(x) e^{-\epsilon x} dx\right| \leq \epsilon \|f_2\|_\infty \int_0^\infty e^{-x} dx$ and this last quantity goes to 0 when ϵ goes to 0. We deduce $\mu_k = \lim_{\epsilon\to 0} \int_0^\infty e^{-\epsilon x} x^k f(x)\, dx = (-1)^{k+1} k! f_{k+1}(0)$ or again

$$\lim_{\epsilon\to 0} \int_0^\infty e^{-\epsilon x} f_k(x)\, dx = -f_{k+1}(0).$$

Let us come back to the proof of the theorem. Applying Borel's theorem, there exists a function h belonging to the Schwartz class now taken on $(-\infty, 0]$ such that

$$-\mu_k = \lim_{\epsilon\to 0} \int_{-\infty}^0 x^k h(x)\, dx.$$

We show that $h + f\chi_{([0,\infty))}$ is the function g that we are looking for. It is sufficient to prove that g is indefinitely oscillating on the whole $\mathbb{R}$. First $g \in L^\infty(\mathbb{R})$. By definition $g_1(x) = \int_{-\infty}^x g(t)\, dt$. If $x < 0$ then $g_1(x) = \int_{-\infty}^x h(t)\, dt$ and $g_1 \in L^\infty((-\infty, 0])$. If $x = 0$ then $g_1(0) = \int_{-\infty}^0 h(t)\, dt = -\mu_0 = f_1(0)$. If $x \geq 0$ then $g_1(x) = \int_{-\infty}^0 h(t)\, dt + \int_0^x f(t)\, dt = f_1(0) + f_1(x) - f_1(0)$, which implies $\|g_1\|_\infty \leq C$. In a similar way, $g_2(x) = \int_{-\infty}^x g_1(t)\, dt$. For $x \geq 0$ one has

$$\begin{aligned} g_2(x) &= \int_{-\infty}^x g_1(t)\, dt + \int_0^x f_1(t)\, dt = \int_{-\infty}^0 g_1(t)\, dt + f_2(x) - f_2(0) \\ &= -\int_{-\infty}^0 th(t)\, dt + f_2(x) - f_2(0). \end{aligned}$$

But $\int_{-\infty}^0 th(t)\, dt = -\mu_1 = -f_2(0)$, through the construction of h. We observe that $-\mu_k = \int_{-\infty}^0 x^k h(x)\, dx = \int_{-\infty}^0 h_k(x)\, dx = h_{k+1}(0)$. Arguing by recurrence, that is assuming that $\|g_k\|_\infty \leq C_k \leq \infty$ for $k < n$ and showing that g_n satisfies the same inequality, we have

$$g_n(0) = \int_{-\infty}^0 g_{n-1}(t)\, dt = f_n(0) = \frac{(-1)^{n-1}}{(n-1)!} \int_{-\infty}^0 x^{n-1} h(x)\, dx.$$

For $x < 0$ one has $g_n(x) = h_n(x)$, which belongs to the Schwartz class. For $x \geq 0$ one has

$$\begin{aligned} g_n(x) &= \int_{-\infty}^x g_{n-1}(t)\, dt = \int_{-\infty}^0 h_{n-1}(t)\, dt + \int_0^x f_{n-1}(t)\, dt \\ &= f_n(0) + f_n(x) - f_n(0). \end{aligned}$$

But f_n is bounded by hypothesis. So it is for $g_n(x)$. □

2.1.1. The $L^2([0,\infty))$-case

Definition 2.3. *A function $f \in L^2([0,\infty))$ is indefinitely oscillating in the $L^2([0,\infty))$-sense if, for every n, there exists $f_n \in L^2([0,\infty))$ such that $f = \frac{d^n f_n}{dx^n}$, in the distributional sense.*

Let f be a function indefinitely oscillating in the L^2-sense. Using similar notations, we show that $f_n \in H^n([0,\infty))$. If $0 < x < y$ we have $f_1(y) - f_1(x) = \int_x^y f(t)\, dt$. Hence $|f_1(y) - f_1(x)| \leq \sqrt{y-x}\, \|f\|_2$. Hence f_1 is uniformly continuous on $[0,\infty)$ and can be extended at 0. It is the same for every $f_n, n \geq 1$ .Let us now show that f_n goes to 0 at infinity for every $n \geq 1$.

Lemma 2.4. *Let u be a uniformly continuous function on $[0,\infty)$. If $u \in L^2([0,\infty))$ then $\lim_{x\to\infty} u(x) = 0$.*

Proof. The proof of this lemma is well-known. We will repeat it for the reader's convenience. To prove this lemma, we argue by contradiction. Hence, there exists a sequence $(x_k)_k$ which goes to ∞ such that $|u(x_k)| \geq \delta > 0$. As u is uniformly continuous, there exists $\epsilon > 0$ such that $|u(x_k + t)| \geq \frac{\delta}{2}$ if $|t| \leq \epsilon$. Up to a subsequence, we can assume that the intervals $[x_k - \epsilon, x_k + \epsilon]$ are disjoint two by two. Then $\|u\|_2 \geq \sum_k \int_{x_k-\epsilon}^{x_k+\epsilon} |u(s)|^2\, ds = \infty$. □

Remark 2.5. *If $f \in L^2([0,\infty))$ is indefinitely oscillating, its primitives given by the definition $f_1, f_2, \dots, f_n, \dots$ are indefinitely oscillating in the $L^\infty([0,\infty))$-sense. The f generalized moment μ_k exists for every integer $k \geq 1$.*

Lemma 2.6. *If $f \in L^2([0,\infty))$ is indefinitely oscillating in the $L^2([0,\infty))$-sense, then $\lim_{\epsilon\to 0}\int_0^{+\infty} e^{-\epsilon x} f(x)\,dx$ exists and is equal to $-f_1(0)$.*

Proof. Integrating by parts, we get

$$\int_0^{+\infty} e^{-\epsilon x} f(x)\,dx = \left[e^{-\epsilon x} f_1(x)\right]_0^{+\infty} + \epsilon \int_0^{+\infty} e^{-\epsilon x} f_1(x)\,dx.$$

As $f_1(+\infty) = 0$, the first term of the right hand side is equal to $-f_1(0)$ and the second term can be bounded using Cauchy-Schwarz by $\frac{\|f_1\|_2}{\sqrt{\epsilon}}$. □

Theorem 2.7. *If $f \in L^2([0,\infty))$ is indefinitely oscillating, there exists a function $g \in L^2(\mathbb{R})$, indefinitely oscillating, whose restriction to $[0,\infty)$ is f.*

Proof. In order to use Borel's theorem, it is necessary to start defining the generalized moments of f. Let $f \in L^2([0,\infty))$ be an indefinitely oscillating function. For every n, there exists $f_n \in L^2([0,\infty))$ such that $f = \frac{d^n f_n}{dx^n}$. We have already proved that for every $n \geq 1$ f_n is uniformly continuous on $[0,\infty)$ and $\lim_{x\to\infty} f_n(x) = 0$.
Let us prove that for every $f \in L^2([0,\infty))$ which is an indefinitely oscillating function, then for every integer k $\lim_{\epsilon\to 0}\int_0^\infty e^{-\epsilon x} x^k f(x)\,dx = (-1)^{k+1} k! f_{k+1}(0)$ or again $\lim_{\epsilon\to 0}\int_0^\infty e^{-\epsilon x} f_k(x)\,dx = -f_{k+1}(0)$.
We apply Borel's theorem. There exists a function h in the Schwartz class taken on $(-\infty, 0]$, such that $h(x) + f(x)_{\chi[0,\infty)}$ belongs to $L^2(\mathbb{R})$ and is indefinitely oscillating on the whole real axis. Using a similar argument than that of the L^∞-case, we prove that $h(x) + f(x)_{\chi[0,\infty)}$ is the desired function g. □

2.1.2. The $H^s([0,\infty))$-case

Definition 2.8. *A function $f \in H^s([0,\infty))$ is indefinitely oscillating in the H^s-sense if, for every integer n, there exists a function (or a distribution) $f_n \in H^s([0,\infty))$ such that $f = \frac{d^n f_n}{dx^n}$, in the distributional sense.*

Theorem 2.9. *Every function (resp. distribution) $f \in H^s([0,\infty))$ which is indefinitely oscillating in the H^s-sense is the restriction to $[0,\infty)$ of a function (resp. distribution) $g \in H^s(\mathbb{R})$ which is indefinitely oscillating in the H^s-sense.*

Proof. One starts assuming that $s \geq 0$. The functions $f_1, f_2, \dots, f_n, \dots$ are uniformly continuous and can be extended by 0. We apply Borel's theorem and build a function h in the Schwartz class taken in $(-\infty, 0]$ and such that

$$\begin{aligned} f_1(0) &= \textstyle\int_{-\infty}^0 h(x)\,dx, \\ f_2(0) &= \textstyle\int_{-\infty}^0 h_1(x)\,dx \qquad \text{where } h_1(x) = \int_{-\infty}^x h(t)\,dt, \\ f_m(0) &= \textstyle\int_{-\infty}^0 h_{m-1}(x)\,dx \quad \text{where } h_{m-1}(x) = \int_{-\infty}^x h_{m-2}(t)\,dt, \end{aligned}$$

for every $m \geq 1$. Then we define $g(x) = h(x) + f(x)\,\chi_{[0,\infty)}$. This approach is possible only in the case where $0 \leq s \leq \frac{1}{2}$. The reason is that if $f \in H^s([0,\infty))$, it is true that f is the restriction to $[0,\infty)$ of a function in $H^s(\mathbb{R})$, but it is not necessarily obtained by multiplying it by the indicator function $\chi_{[0,\infty)}$. Then we reach the end of the proof observing that if $-\frac{1}{2} < s < \frac{1}{2}$ and $f \in H^s(\mathbb{R})$, then $f\chi_I \in H^s(\mathbb{R})$ for every interval I, with either a finite or infinite length.
Let $sign(x) = \frac{x}{|x|}$ and $f \in H^s(\mathbb{R})$ with $0 \leq s \leq \frac{1}{2}$. Observe that the Fourier transform of $sign(x) f(x)$ is Fourier $(sign(.)\,f) = \frac{1}{i\pi}\widehat{f} * V_P\left(\frac{1}{\xi}\right)$. As one remarks, Borel's theorem can be applied only in the case $s \geq 0$ and $0 \leq s < \frac{1}{2}$.
Let us now remark that f is indefinitely oscillating in the H^s-sense (either on the half axis or on the whole axis) if and only if $D^m f$ is indefinitely oscillating in the H^{s-m}-sense for every m relative integer. We can generalize this observing that f is

indefinitely oscillating in the H^s-sense (either on $\mathbb{R}$ or on some subset of $\mathbb{R}$) if and only if $D^\alpha f$ is indefinitely oscillating in the $H^{s-\alpha}$-sense for every real α. Here $D^\alpha f$ means the fractional derivative of order α of f, $0<\alpha<1$, which is defined through $D^\alpha f(x) = c_\alpha \int_x^\infty \frac{f(y)}{(x-y)^{1-\alpha}} dy$. This integral exists as f is defined on $[T,\infty)$. Let f a function indefinitely oscillating. If $f \in H^s$ then $D^\alpha f \in H^{s-\alpha}$. For every m, there exists $f_m \in H^{s+m}$ such that $f = \frac{d^m f_m}{dx^m}$. This implies $D^\alpha f_m \in H^{s+m-\alpha}$. Hence $D^\alpha f$ is indefinitely oscillating in the $H^{s-\alpha}$-sense. Conversely, if $D^\alpha f$ is indefinitely oscillating in the $H^{s-\alpha}$-sense, we define $g = D^\alpha f$. Then $g_m \in H^{s-\alpha+m}$. As $0 < 1-\alpha < 1$ we have $D^{1-\alpha} g_1 \in H^s$ and $D^{1-\alpha} g_1 = f$. One deduces $D^{1-\alpha} g_m \in H^{s+m-1}$ and $D^{1-\alpha} g_m = f_{m-1}$.
Let us now move to the case where $-1<\alpha<0$. Take a indefinitely oscillating function f. For every m, there exists $f_m \in H^{s+m}$ such that $f = \frac{d^m f_m}{dx^m}$. $f_1 \in H^{s+1}$. One deduces that $D^{1+\alpha} f_1 \in H^{s-\alpha}$. (The operator $D^{1+\alpha}$ is a derivative operator since $0<1+\alpha<1$). As f_1 is indefinitely oscillating in the H^{s+1}-sense, $D^{1+\alpha} f_1$ is indefinitely oscillating in the $H^{s-\alpha}$-sense. □

2.1.3. Generalization to an Arbitrary Banach Space

We take a Banach space E satisfying the following conditions:

(1). $E \subset D'([0,\infty))$ and the embedding is continuous.

(2). For every sequence $(f_j)_j$ such that $f_j \in E$, $\|f_j\|_E \leq C$ and $(f_j)_j$ converges to f in the distributional sense, then $f \in E$.

(3). Let $f \in E$ and $\tau \geq 0$. Then $T^\tau f$, defined by $T^\tau f(x) = f(x+\tau)$, belongs to E with $\|T^\tau f\|_E \leq \|f\|_E$.

Definition 2.10. *A function (or distribution) $f \in E$ is indefinitely oscillating relatively in the E-sense, if and only if, for every integer $n \geq 1$, there exists $f_n \in E$ such that $f = \left(\frac{df_n}{dx}\right)^n$, in the distributional sense.*

Definition 2.11. *A function $\psi \in C_0^\infty(0,1)$ is admissible if $\int_0^\infty u^{-q}\psi(u)\,du = \gamma_q \neq 0$ for every integer $q \geq 1$ (this is compatible with the fact that ψ has a null integral).*

Theorem 2.12. *With the previous notations, the two following properties of the function $f \in E$ are equivalent:*

(1). f is indefinitely oscillating relatively to E

(2). The quantity $W_{(a,b)} = \int_0^\infty f(x)\frac{1}{a}\psi\left(\frac{x-b}{a}\right)dx$ with $a,b>0$, satisfies $\left\|W_{(a,.)}\right\|_E \leq C_N a^{-N}$, for every $a \geq 1$.

Proof. 1. implies 2. is evident.

Lemma 2.13. *Let $\omega \in L^1(0,\infty)$ and $f \in E$. Then $\int_0^\infty (T^\tau f)\,\omega(\tau)\,d\tau \in E$.*

Using this result, we write

$$\begin{aligned} W_{(a,b)} &= \int_0^\infty f(x)\frac{1}{a}\psi\left(\frac{x-b}{a}\right)dx = \int_0^\infty \left(\frac{d}{dx}\right)^n f_n(x)\frac{1}{a}\psi\left(\frac{x-b}{a}\right)dx \\ &= (-1)^n a^{-n}\int_0^\infty f_n(x)\frac{1}{a}\psi^{(n)}\left(\frac{x-b}{a}\right)dx = (-1)^n a^{-n}\int_0^\infty \left(T^{(x)}f_n\right)(b)\frac{1}{a}\psi^{(n)}\left(\frac{x}{a}\right)dx \\ &= (-1)^n a^{-n} W_{(n)_{(a,b)}}. \end{aligned}$$

Owing to Lemma 13, $W_{(n)_{(a,b)}} \in E$ with a norm uniformly bounded by a constant. The estimate on $W_{(a,b)}$ follows.
2. implies 1. is more subtle. We use the identity $\int_0^\infty a^{q-1}\psi\left(\frac{t-x}{a}\right)\frac{da}{a} = \gamma_q (t-x)_+^{q-1}$. We conclude that we have $(q-1)!\gamma_q f_q(x) = \int_0^\infty a^{q-1} W_{(a,x)} da$, where f_q is the q-th primitive of f. If $q \geq 1$ $\int_0^1 a^{q-1} W_{(a,x)} da \in E$, as $\left\|W_{(a,.)}\right\|_E \leq C\|f\|_E$ (see Lemma 13). Now $\int_1^\infty a^{q-1} W_{(a,x)} da \in E$ since (2) let to satisfy that the integral is the Bochner integral. □

3. Some Interesting Examples of Indefinitely Oscillating Functions

Some links between functions indefinitely oscillating in the L^{∞}- and H^s-sense can be more interesting. It is evident to see that if f is indefinitely oscillating in the H^s-sense and if $s > \frac{1}{2}$, then f is indefinitely oscillating in the L^{∞}-sense (because of the Sobolev embeddings). One has also the following property.

Lemma 3.1. *Let $f \in H^s(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$ for $s < \frac{1}{2}$ be an indefinitely oscillating in the $H^s(\mathbb{R})$-sense. Then f is a function indefinitely oscillating in the $L^{\infty}(\mathbb{R})$-sense.*

Proof. By hypothesis, one has $\|\Delta_j f\|_2 \le C_N 2^{jN}$, for every N and for some $j \le -1$. As $\widehat{\Delta_j f}(\xi)$ is taken in the dyadic corona $\alpha 2^j \le |\xi| \le \beta 2^j$ with $0 < \alpha < \beta$, it comes $\left\|\widehat{\Delta_j f}\right\|_{L^1} = \int_{\alpha 2^j}^{\beta 2^j} \left|\widehat{\Delta_j f}(\xi)\right| d\xi$. Applying Cauchy-Schwarz' inequality we have $\left\|\widehat{\Delta_j f}\right\|_{L^1} \le C_N 2^{jN} 2^{\frac{j}{2}}$ for every N and for some $j \le -1$. But $\Delta_j f(x) = \int \widehat{\Delta_j f}(\xi) e^{i\xi x} d\xi$. Hence we have $\|\Delta_j f\|_{\infty} \le \left\|\widehat{\Delta_j f}\right\|_{L^1} \le C_N 2^{jN} 2^{\frac{j}{2}}$. □

Remark 3.2. *If $f \in H^s(\mathbb{R})$ is indefinitely oscillating in the $L^{\infty}(\mathbb{R})$-sense, f is not necessarily indefinitely oscillating in the $H^s(\mathbb{R})$-sense. We build such function in the following way. Let $\widehat{\varphi}$ be a function with compact support and indefinitely differentiable (it is in the Schwartz class) such that $\widehat{\varphi}(\xi) \ge 0$. Write $f(x) = \sum_0^{+\infty} 2^{-k^2} \varphi\left(\frac{x}{2^{k^2}}\right) e^{i2^{-k}x}$. Using Fourier's transform, it comes $\widehat{f}(\xi) = \sum_0^{+\infty} \widehat{\varphi}\left(2^{k^3}\left(\xi - 2^{-k}\right)\right) 2^{k^3-k^2}$. It clear that one has $\int_{2^{-j} \le |\xi| \le 2 2^{-j}} \left|\widehat{f}(\xi)\right| d\xi = O\left(2^{-Nj}\right)$. It can be observed by the proof of the preceding Lemma that f is indefinitely oscillating in the L^{∞}-sense, but $\lim_{j\to+\infty} \int_{2^{-j} \le |\xi| \le 2 2^{-j}} \left|\widehat{f}(\xi)\right|^2 d\xi = +\infty$. Hence f is not indefinitely oscillating in the L^2-sense.*

Remark 3.3. *We know that an indefinitely oscillating function in the H^s-sense can be written under the form $f = f_0 + f_1$, where f_0 is the principal component frequency which defines the oscillating characteristic of the function f, and f_1 is a function of H^s which is indefinitely oscillating since the support of $\widehat{f_1}$ does not contain 0. Finally f_0 is a function indefinitely oscillating in the L^2-sense. An interesting example of indefinitely oscillating function in the L^2-sense is the Grassmann wavelet defined by*

$$f(x) = \begin{cases} e^{-(\log x)^2} & \text{if } x > 0, \\ e^{-(\log|x| + i\pi)^2} & \text{if } x < 0. \end{cases}$$

We can easily show that f is in the Schwartz class. Hence $\widehat{f}$ is in the Schwartz class too. Paley-Wiener's theorem implies that $\widehat{f}(\xi) = 0$ for $\xi < 0$ and $\frac{\widehat{f}(\xi)}{\xi^N}$ is in the Schwartz class, for every N. One deduces that 0 is a zero of infinity order for $\widehat{f}$. Hence f is indefinitely oscillating in the L^2-sense. We can observe that the same function is also indefinitely oscillating in the L^{∞}-sense.

References

[1] S.Alinhac and P.Gérard, *Opérateurs pseudo-diffé rentiels et théorème de Nash-Moser*, Savoirs Actuels, (1991).

[2] T.El Bouayachi, *Chirp et analyse temps-fréquence avec recherche de la fréquence instantanée*, Thèse de doctorat de l'Université Paris Dauphine, (1994).

[3] G.Bourdaud, *Localisation et multiplicateurs des espaces de Sobolev homogènes*, Manuscripta mathematica, (1988).

[4] S.Jaffard, *Habilitation à diriger des recherches.*

[5] S.Jaffard, *Oscillations Spaces: Properties and applications to fractal and multifractal functions*, J. Math. Phys., 36(8)(1998), 4129-4141.

[6] S.Jaffard, *On lacunary wavelets Series*, The Annals of Applied Probability, 10(1)(2000), 313-329.

[7] S.Jaffard, *PhD Thesis*, Université Paris Dauphine, Mars (1992).

[8] S.Jaffard, *Oscillation spaces: Properties and applications to fractal and multifractal functions*, Journal of Mathematical Physics, 39(8)(1998), 4129-4141.

[9] S.Jaffard and Y.Meyer, *Wavelet methods for pointwise regularity and local oscillating of functions*, American Mathematical Soc., (1996).

[10] Y.Meyer and H.Xu, *Wavelet analysis and chirp*, Applied and Computational Harmonic Analysis, 4(4)(1997), 366-379.

[11] Y.Meyer, *Ondelettes et opérateurs*, Vol. I, Hermann, Paris, (1991).

[12] Y.Meyer, *Analyse par ondelettes et analyse 2-microlocale des chirps généralisés*, Cahiers de mathématiques de la décision, CEREMADE, 27-11-92. N 9246.

[13] T.El Bouayachi and N.Yebari, *Indefinitely oscillating functions-Part I*, International Journal of Mathematics And its Applications, 4(1–D)(2016), 203-207.