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A New Family of Additive Weibull-Generated Distributions

Research Article

Amal S. Hassan[1] and Saeed E. Hemeda[1]*

1 Institute of Statistical Studies & Research, Cairo University, Egypt.

Abstract: The additive Weibull distribution is an extended version of the most widely used Weibull distribution which is a quite flexible model for analyzing lifetime data. In this paper, a new family of distributions called additive Weibull generated (AW-G) is defined and studied.The new family includes the Weibull-G family which was previously worked out by Bourguignon et al. [7]. Four sub-models are introduced and studied in some details. Explicit expressions for the ordinary moments, quantile, order statistics and reliability are obtained. The method of maximum likelihood is used to estimate the model parameters of the family. The additive Weibull-G family may serve as a viable alternative to other distributions for modeling data arising in various fields such as the physical and biological sciences, survival analysis and engineering. The importance of the additive Weibull-G family is concluded by means of two real data sets.

Keywords: Additive Weibull, Reliability, Moments, Maximum likelihood estimation.

1. Introduction

Classical distributions have been widely utilized over the past decades for modeling real data in several areas. Furthermore, in many applied areas such as lifetime analysis, finance, and insurance, there is a strong need for extended forms of classical distributions. Therefore, some attempts have been made to define newly generated families of probability distributions that extend well-known distributions and provide great flexibility in modeling data in practice. Some of the well-known generators are as follows: the beta-G by Eugene et al. [12], gamma-G by Zografos and Balakrishanan [17], Kumaraswamy-G by Cordeiro and de Castro [9], generalized beta-G by Alexander et al. [2], transformed-transformer (T-X) by Alzaatreh et al. [4] among others.

A more general family called the Weibull G-family was introduced by Bourguignon et al. [7]. They considered a baseline cumulative distribution function (cdf) $G(x;\xi)$, probability density function (pdf) $g(x;\xi)$ with parameter vector ξ and the Weibull distribution as generator. They defined the cdf and pdf of the Weibull-G family as follows

$$F(x;a,b,\xi)=\int_0^{\frac{G(x;\xi)}{\overline{G}(x;\xi)}} abt^{b-1}e^{-at^b}dt=1-e^{-a\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^b},\ x\geq 0,\ a,\ b>0, \tag{1}$$

$$f(x;a,b,\xi)=abg(x;\xi)\frac{[G(x;\xi)]^{b-1}}{[1-G(x;\xi)]^{b+1}}e^{-a\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^b};x\geq 0,\ \ a,b>0. \tag{2}$$

* E-mail: saidhemeda25jan@gmail.com

Where, $\overline{G}(x;\xi) = 1 - G(x;\xi)$. In this article, we introduce the additive Weibull generated family, by using the additive Weibull distribution as a generator instead of the Weibull generator in the Weibull-G family and study some of its mathematical properties. We hope that the extended family yields a better fit in more practical situations.This paper is organized as follows: The new family of distributions is defined in Section 2. In the following section, some special models of the family are presented. Section 4 provides some mathematical properties of the new family. In Section 5, the estimation of the model parameters is performed by the method of maximum likelihood. In Section 6, two illustrative applications based on real data are investigated. A conclusion is provided in Section 7.

2. The Additive Weibull-G Family

In this section, the new family of distributions called AW-G family is introduced. To obtain the new family, first we give a brief review about the additive Weibull distribution, then the pdf of AW-G is derived. Additionally, the reliability and hazard rate functions are obtained and studied analytically. Further, the asymptotic of pdf, cdf and hazard function is discussed. The additive Weibull distribution was proposed by Xie and Lai [16] by combining the failure rates of two Weibull distributions. Almalki and Yuan [3] introduced a four-parameter additive Weibull distribution as a competitive model. Further, Lemonte et al. [13] studied some statistical properties of the AW distribution.Recently, Elbatal and Aryal [11] introduced the Transmuted Additive Weibull Distribution. The cdf and pdf of AW distribution with shape parameters b, d and scale parameters a, c are given by

$$F(x;a,b,c,d) = 1 - e^{-c\,x^{d} - a\,x^{b}} \quad ; x \geq 0,\ a,\ b,\ c,\ d > 0, \tag{3}$$

$$f(x;a,b,c,d) = (c\,d\,x^{d-1} + a\,b\,x^{b-1})e^{-c\,x^{d} - a\,x^{b}} \quad ; x \geq 0,\ a,\ b,\ c,\ d > 0. \tag{4}$$

We obtain the cdf of new family by replacing the Weibull generator defined in (1) by the additive Weibull generator defined in (4) as the following

$$F(x;\Phi) = \int_{0}^{\frac{G(x;\xi)}{\bar{G}(x;\xi)}} (c\,d\,x^{d-1} + a\,b\,x^{b-1})e^{-c\,x^{d} - a\,x^{b}}\,dx = 1 - \exp\left\{-c\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{d} - a\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{b}\right\}, \tag{5}$$

where, $\Phi = (a,b,c,d,\xi)$ and $\overline{G}(x;\xi) = 1 - G(x;\xi)$. The corresponding AW–G pdf takes the following form

$$f(x;\Phi) = \left\{cdg(x;\xi)\frac{[G(x;\xi)]^{d-1}}{[\bar{G}(x;\xi)]^{d+1}} + abg(x;\xi)\frac{[G(x;\xi)]^{b-1}}{[\bar{G}(x;\xi)]^{b+1}}\right\}.\exp\left\{-c\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{d} - a\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{b}\right\};\ x \geq 0,\ a,b,c,d > 0. \tag{6}$$

Or, it can be written in the following form

$$f(x;\Phi) = \frac{g(x;\xi)}{[\bar{G}(x;\xi)]^{2}}\left\{cd\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{d-1} + ab\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{b-1}\right\}.\exp\left\{-c\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{d} - a\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{b}\right\};\ x \geq 0,\ a,b,c,d > 0. \tag{7}$$

One can notice that: If $b = d$, an AW-G family reduces to W-G family with scale parameter $a + c$ as previously derived by Bourguignon [7]. Also, we obtain the same result when $a = 0$ or $c = 0$.

Furthermore, the reliability function of AW-G family is

$$R(x;\Phi) = \exp\left\{-c\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{d} - a\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{b}\right\}. \tag{8}$$

The hazard rate function of AW-G family is

$$h(x;\Phi)=\frac{g(x;\xi)}{\left[\bar{G}(x;\xi)\right]^{2}}\left\{c\,d\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{d-1}+a\,b\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{b-1}\right\}. \tag{9}$$

Additionally, the shapes of the hazard rate function can be described analytically as follows:The critical points of the AW-G family are the roots of the equation $\frac{d}{dx}h(x;\Phi)$, where

$$h'(x;\Phi)=c\,d\frac{d}{dx}\left[g(x;\xi)\frac{[G(x;\xi)]^{d-1}}{\left[\bar{G}(x;\xi)\right]^{d+1}}\right]+a\,b\frac{d}{dx}\left[g(x;\xi)\frac{[G(x;\xi)]^{b-1}}{\left[\bar{G}(x;\xi)\right]^{b+1}}\right],$$

$$\frac{d}{dx}\left[g(x;\xi)\frac{[G(x;\xi)]^{d-1}}{\left[\bar{G}(x;\xi)\right]^{d+1}}\right]=[g(x;\xi)]^{2}\,[G(x;\xi)]^{d-2}\left[\bar{G}(x;\xi)\right]^{-d-2}[2G(x;\xi)+d-1]+g'(x;\xi)\,[G(x;\xi)]^{d-1}\left[\bar{G}(x;\xi)\right]^{-d-1}.$$

Similarly

$$\frac{d}{dx}\left[g(x;\xi)\frac{[G(x;\xi)]^{b-1}}{\left[\bar{G}(x;\xi)\right]^{b+1}}\right]=[g(x;\xi)]^{2}\,[G(x;\xi)]^{b-2}\left[\bar{G}(x;\xi)\right]^{-b-2}[2G(x;\xi)+b-1]+g'(x;\xi)\,[G(x;\xi)]^{b-1}\left[\bar{G}(x;\xi)\right]^{-b-1}.$$

There is more than one root to the previous equation. Suppose that $b=d=1$, $x=x_0$ is a root of the equation:

$$h'(x;\Phi)=\frac{2(a+c)\,[g(x;\xi)]^{2}}{\left[\bar{G}(x;\xi)\right]^{3}}+\frac{(a+c)g'(x;\xi)}{\left[\bar{G}(x;\xi)\right]^{2}}.$$

Then x_0 is the critical point which refers to a local maximum if $h''(x;\Phi)>0\,(<0)\;\forall\,x<x_\circ$ and a local minimum if $h''(x;\Phi)>0\,(<0)\;\forall\,x>x_\circ$. It gives an inflection point if either $h''(x;\Phi)>0\;\forall\,x\neq x_\circ$or $h''(x;\Phi)<0\;\forall\,x\neq x_\circ$, where, $h''(x;\Phi)=\frac{d^2\,h(x;\Phi)}{d\,x^2}$.

Furthermore, the asymptotic of cdf (5), pdf (7) and hazard function (9) respectively, are given as $G(x;\xi)\to\infty$ then $\frac{G(x;\xi)}{\bar{G}(x;\xi)}\to -1$ and there are two cases:

(i). If $d,b>1$ are even numbers, then $F(x;\Phi)\to 1-\exp(c+a)$,

$$f(x;\Phi)\to\frac{-(cd+ab)[1-\exp(-c-a))]g(x;\Phi)}{[\bar{G}(x;\Phi)]^{2}},\quad h(x;\Phi)\to\frac{-(cd+ab)g(x;\Phi)}{[\bar{G}(x;\Phi)]^{2}}.$$

(ii). If $d,b>1$ are odd numbers, then $F(x;\Phi)\to 1-\exp(-c-a)$,

$$f(x;\Phi)\to\frac{(cd+ab)[1-\exp(c+a))]g(x;\Phi)}{[\bar{G}(x;\Phi)]^{2}},\quad h(x;\Phi)\to\frac{(cd+ab)g(x;\Phi)}{[\bar{G}(x;\Phi)]^{2}}.$$

3. Some Special Models for AW-G Family

In this section, some new special distributions, namely, AW-uniform, AW-Weibull, AW-Burr XII, and AW-log logistic are introduced.

3.1. AW-Uniform

Considering the baseline distribution is uniform on the interval $(0,\theta)$,the cdf of the AW-uniform (AWU) distribution is given by

$$F(x;a,b,c,d,\theta)=1-\exp\left\{-c\left(\frac{x}{\theta-x}\right)^{d}-a\left(\frac{x}{\theta-x}\right)^{b}\right\};0<x<\theta<\infty,\;a,\;b,\;c,\;d>0.$$

The corresponding pdf is given by

$$f(x;a,b,c,d,\theta)=\frac{\theta}{(\theta-x)^2}\left[c\,d\left(\frac{x}{\theta-x}\right)^{d-1}+a\,b\left(\frac{x}{\theta-x}\right)^{b-1}\right]$$
$$\exp\left\{-c\left(\frac{x}{\theta-x}\right)^{d}-a\left(\frac{x}{\theta-x}\right)^{b}\right\};\ 0<x<\theta<\infty,\ a,\ b,\ c\,,d>0.$$

The survival and hazard rate functions are given respectively as follows

$$R(x;a,b,c,d,\theta)=\exp\left\{-c\left(\frac{x}{\theta-x}\right)^{d}-a\left(\frac{x}{\theta-x}\right)^{b}\right\},$$
$$h(x;a,b,c,d,\theta)=\frac{\theta}{(\theta-x)^2}\left[c\,d\left(\frac{x}{\theta-x}\right)^{d-1}+a\,b\left(\frac{x}{\theta-x}\right)^{b-1}\right].$$

3.2. AW -Weibull

Suppose that the baseline distribution is Weibull with the following pdf and cdf

$$g(x;\lambda,\theta)=\lambda\,\theta\,x^{\theta-1}\,e^{-\lambda x^{\theta}}\ ;x,\lambda,\ \theta>0,\quad G(x;\lambda,\theta)=1-e^{-\lambda x^{\theta}}\ ;x,\lambda,\ \theta>0.$$

The cdf of AW-Weibull (AWW) distribution is obtained by substituting the previous pdf and cdf in (5) as follows

$$F(x;a,b,c,d,\lambda,\theta)=1-\exp\left\{-c\left(e^{\lambda x^{\theta}}-1\right)^{d}-a\left(e^{\lambda x^{\theta}}-1\right)^{b}\right\}\ ;x>0.$$

The corresponding pdf is

$$f(x;a,b,c,d,\lambda,\theta)=\lambda\theta x^{\theta-1}e^{\lambda\,x^{\theta}}\left[c\,d\left(e^{\lambda\,x^{\theta}}-1\right)^{d-1}+a\,b\left(e^{\lambda\,x^{\theta}}-1\right)^{b-1}\right]$$
$$.\exp\left[-c\left(e^{\lambda x^{\theta}}-1\right)^{d}-a\left(e^{\lambda x^{\theta}}-1\right)^{b}\right];x>0,a,b,c,d,\theta>0.$$

The survival and hazard rate functions take, respectively, the following forms

$$R(x;a,b,c,d,\lambda,\theta)=\exp\left[-c\left(e^{\lambda x^{\theta}}-1\right)^{d}-a\left(e^{\lambda x^{\theta}}-1\right)^{b}\right],$$
$$h(x;a,b,c,d,\lambda,\theta)=\lambda\theta x^{\theta-1}e^{\lambda x^{\theta}}\left[c\,d\left(e^{\lambda x^{\theta}}-1\right)^{d-1}+a\,b\left(e^{\lambda x^{\theta}}-1\right)^{b-1}\right].$$

3.3. AW-Burr XII

Considering the baseline distribution is Burr XII (see Burr [8]) with the following pdf and cdf

$$g(x;\alpha,\theta)=\alpha\,\theta\,x^{\theta-1}\,(1+x^{\alpha})^{-(\theta+1)}\,;x\geq 0,\ \alpha,\ \theta>0,$$
$$G(x;\alpha,\theta)=1-\ (1+x^{\alpha})^{-\theta}\,;x\geq 0,\ \alpha,\ \theta>0.$$

The cdf of AW-Burr XII (AWBXII) distribution is obtained by substituting the pdf and cdf of Burr-XII in (5) as follows

$$F(x;a,b,c,d,\alpha,\theta)=1-\exp\left\{-c\left[(1+x^{\alpha})^{\theta}-1\right]^{d}-a\left[(1+x^{\alpha})^{\theta}-1\right]^{b}\right\}\ ;x>0.$$

The corresponding pdf is

$$f(x;a,b,c,d,\alpha,\theta)=\alpha\,\theta\,x^{\alpha-1}\,(1+x^{\alpha})^{\theta-1}\left[c\,d\left[(1+x^{\alpha})^{\theta}-1\right]^{d-1}+a\,b\left[(1+x^{\alpha})^{\theta}-1\right]^{b-1}\right]$$
$$.\exp\left\{-c\left[(1+x^{\alpha})^{\theta}-1\right]^{d}-a\left[(1+x^{\alpha})^{\theta}-1\right]^{b}\right\};x>0,\ a,\ b,\ c,\alpha,\theta>0.$$

The survival and hazard rate functions are obtained, respectively, as follows

$$R(x;a,b,c,d,\alpha,\theta)=\exp\left\{-c\left[(1+x^{\alpha})^{\theta}-1\right]^{d}-a\left[(1+x^{\alpha})^{\theta}-1\right]^{b}\right\},$$
$$h(x;a,b,c,d,\alpha,\theta)=\alpha\,\theta\,x^{\alpha-1}\,(1+x^{\alpha})^{\theta-1}\left[c\,d\left[(1+x^{\alpha})^{\theta}-1\right]^{d-1}+a\,b\left[(1+x^{\alpha})^{\theta}-1\right]^{b-1}\right].$$

3.4. AW log-logistic

Assuming that the baseline distribution is log-logistic (see (Bennett [6]) with the following pdf and cdf

$$g(x;\lambda,\alpha)=\alpha\,\lambda^{-\alpha}\,x^{\alpha-1}\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-2}\;;x\geq 0,\ \lambda,\ \alpha>0,$$
$$G(x;\lambda,\alpha)=1-\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-1}.$$

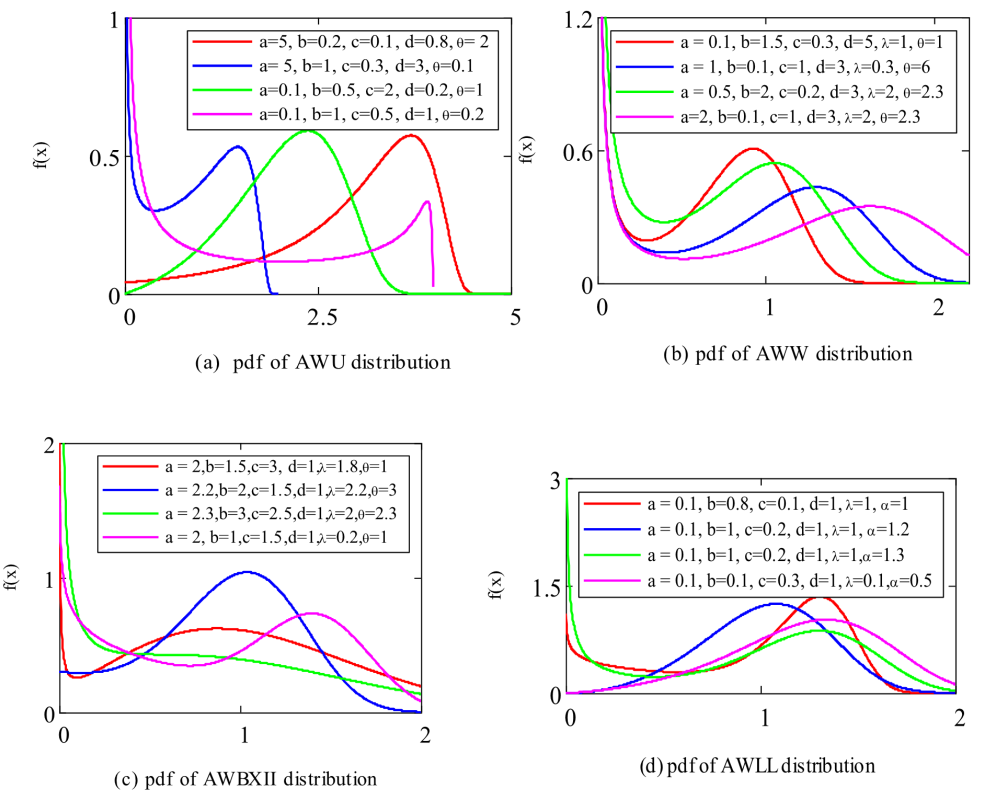
As previously mentioned, the cdf and pdf of AW-log-logistic (AWLL) distribution are obtained by substituting the previous pdf and cdf in (5) and (7) as follows

$$F(x;a,b,c,d,\lambda,\alpha)=1-\exp\left\{-c\left(\frac{x}{\lambda}\right)^{\alpha\,d}-a\left(\frac{x}{\lambda}\right)^{\alpha\,b}\right\}.$$
$$f(x;a,b,c,d,\lambda,\alpha)=\frac{\alpha}{x}\left[c\,d\left(\frac{x}{\lambda}\right)^{\alpha\,d}+a\,b\left(\frac{x}{\lambda}\right)^{\alpha\,b}\right]\exp\left\{-c\left(\frac{x}{\lambda}\right)^{\alpha\,d}-a\left(\frac{x}{\lambda}\right)^{\alpha\,b}\right\}\;;x\geq 0,a,b,c,d,\lambda,\alpha>0.$$

The survival and hazard rate functions take, respectively, the following forms

$$R(x;a,b,c,d,\lambda,\alpha)=\exp\left\{-c\left(\frac{x}{\lambda}\right)^{\alpha\,d}-a\left(\frac{x}{\lambda}\right)^{\alpha\,b}\right\},$$
$$h(x;a,b,c,d,\lambda,\alpha)=\frac{\alpha}{x}\left[c\,d\left(\frac{x}{\lambda}\right)^{\alpha\,d}+a\,b\left(\frac{x}{\lambda}\right)^{\alpha\,b}\right].$$

Plots of pdf and hazard rate function for some parameter values for the selected distributions are represented through Figures 1 and 2.



(a) pdf of AWU distribution

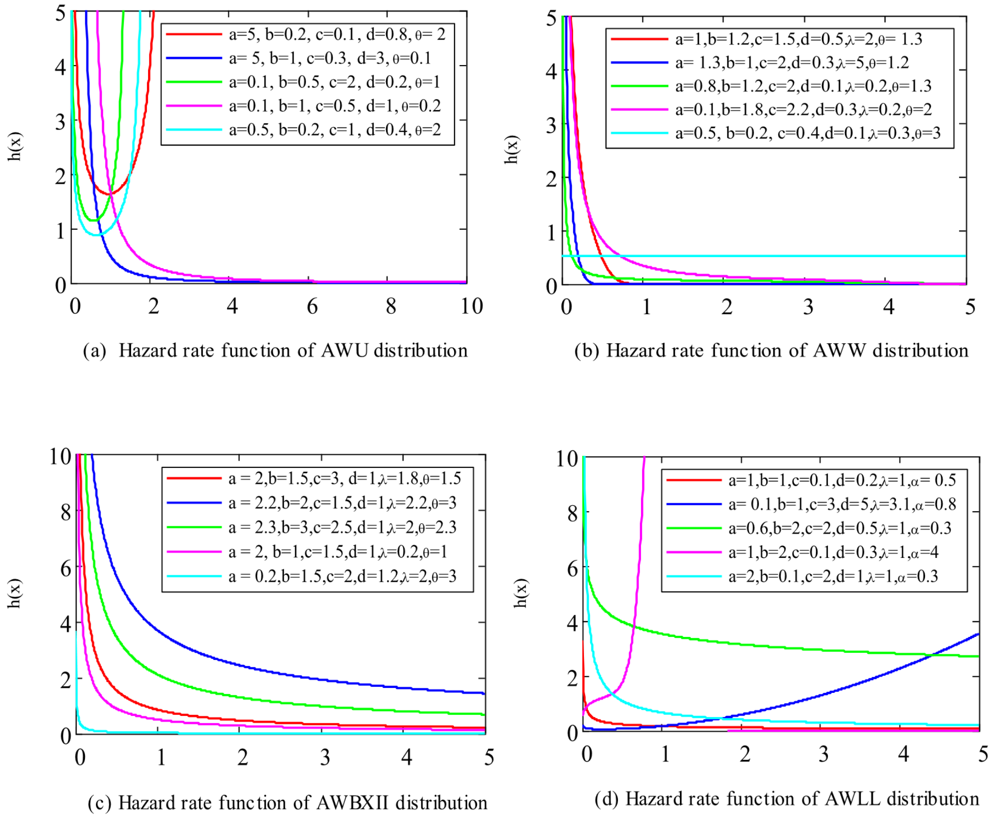
(b) pdf of AWW distribution

(c) pdf of AWBXII distribution

(d) pdf of AWLL distribution

Figure 1. **Plots of the pdfs for some values of parameters.**

From Figure 1, it appears that the shape of the distribution depend heavily on parameter values. In fact, the shape could be left skewed, symmetric and right skewed, which will depend on the values of the parameter. Thus this distribution could be suitable to model many kind of data.



(a) Hazard rate function of AWU distribution

(b) Hazard rate function of AWW distribution

(c) Hazard rate function of AWBXII distribution

(d) Hazard rate function of AWLL distribution

Figure 2. **Plots of hazard rate functions for some values of parameters.**

4. Some Mathematical Properties

In this section, some general results of the AW-G family are derived.

4.1. Mixture representation

Expansion forms for the cdf, pdf and reliability are obtained. Since the power series for the following exponential function can be written as

$$\exp\left\{-c\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{d}-a\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{b}\right\}=\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}\left[c\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{d}+a\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{b}\right]^{i}$$

$$=\sum_{i=0}^{\infty}\frac{(-1)^{i}c^{i}}{i!}\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{i\,d}\left[1+\frac{a}{c}\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{b-d}\right]^{i}. \tag{10}$$

$$\left[1+\frac{a}{c}\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{b-d}\right]^{i}=\sum_{i,j=0}^{\infty}\frac{i!}{j!\,(i-j)!}\left[\frac{a}{c}\right]^{j}\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{j\,(b-d)}. \tag{11}$$

Substituting (11) into (10), the power series for the exponential function will be

$$\exp\left\{-c\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{d}-a\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{b}\right\}=\sum_{i,j=0}^{\infty}\frac{(-1)^{i}\,a^{j}\,c^{i-j}}{j!(i-j)!}\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{i\,d+j(b-d)}. \tag{12}$$

Substituting (12) into cdf (5) then, the expansion for the cdf of AW-G family can be written as

$$F(x;\Phi)=1-\sum_{i,j=0}^{\infty}\frac{(-1)^{i}\,a^{j}\,c^{i-j}}{j!(i-j)!}\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{i\,d+j(b-d)}. \tag{13}$$

The expansion of the reliability function is

$$R(x;\Phi)=\sum_{i,j=0}^{\infty}\frac{(-1)^i\,a^j\,c^{i-j}}{j!(i-j)!}\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{i\,d+j(b-d)}. \tag{14}$$

To obtain the expansion of pdf of AW-G family, substituting (12) into (6) as follows

$$f(x;\Phi)=\sum_{i,j=0}^{\infty}\frac{(-1)^i}{j!\,(i-j)!}\left[\frac{a^j\,c^{i-j+1}\,d\,g(x;\xi)\,\left[G(x;\xi)\right]^{i\,d+j(b-d)+d-1}}{\left[\bar{G}(x;\xi)\right]^{i\,d+j(b-d)+d+1}}+\frac{a^{j+1}\,b\,c^{i-j}\,g(x;\xi)\,\left[G(x;\xi)\right]^{i\,d+j(b-d)+b-1}}{\left[\bar{G}(x;\xi)\right]^{i\,d+j(b-d)+b+1}}\right]$$

$x>0,\ a,\ b,\ c,d>0$. Therefore, $f(x;\Phi)=f_1(x;\Phi)+f_2(x;\Phi)$, where,

$$f_1(x;\Phi)=\sum_{i,j=0}^{\infty}\frac{(-1)^i}{j!\,(i-j)!}\frac{a^j\,c^{i-j+1}\,d\,g(x;\xi)\,\left[G(x;\xi)\right]^{i\,d+j(b-d)+d-1}}{\left[\bar{G}(x;\xi)\right]^{i\,d+j(b-d)+d+1}},$$

and

$$f_2(x;\Phi)=\sum_{i,j=0}^{\infty}\frac{(-1)^i}{j!\,(i-j)!}\frac{a^{j+1}\,b\,c^{i-j}\,g(x;\xi)\,\left[G(x;\xi)\right]^{i\,d+j(b-d)+b-1}}{\left[\bar{G}(x;\xi)\right]^{i\,d+j(b-d)+b+1}}.$$

Then,

$$\left[\bar{G}(x;\xi)\right]^{-(i\,d+j(b-d)+d+1)}=\sum_{q=0}^{\infty}\frac{(i\,d+j(b-d)+d+q)!}{q!\,(i\,d+j(b-d)+d)!}\left[G(x;\xi)\right]^q\ \ (i\,d+j(b-d)+d+1)>0.$$

Similarly

$$\left[\bar{G}(x;\xi)\right]^{-(i\,d+j(b-d)+b+1)}=\sum_{p=0}^{\infty}\frac{(i\,d+j(b-d)+b+p)!}{p!\,(i\,d+j(b-d)+b)!}\left[G(x;\xi)\right]^p\ \ (i\,d+j(b-d)+b+1)>0.$$

Therefore

$$f_1(x;\Phi)=\sum_{i,j,q=0}^{\infty}\frac{(-1)^i}{j!\,q!\,(i-j)!}\frac{(i\,d+j(b-d)+d+q)!}{(i\,d+j(b-d)+d)!}\,a^j\ c^{i-j+1}\,d\,g(x;\xi)\left[G(x;\xi)\right]^{i\,d+j(b-d)+d-1+q},$$

$$f_2(x;\Phi)=\sum_{i,j,p=0}^{\infty}\frac{(-1)^i}{j!\,p!\,(i-j)!}\frac{(i\,d+j(b-d)+b+p)!}{p!\,(i\,d+j(b-d)+b)!}\,a^{j+1}\ b\ c^{i-j}\,g(x;\xi)\ \left[G(x;\xi)\right]^{i\,d+j(b-d)+b-1+p}.$$

Therefore

$$f(x;\Phi)=\sum_{i,j,q=0}^{\infty}\omega_{i\,j\,q}\,g(x;\xi)\left[G(x;\xi)\right]^{i\,d+j(b-d)+d+q-1}+\sum_{i,j,p=0}^{\infty}\omega_{i\,j\,p}\,g(x;\xi)\ \left[G(x;\xi)\right]^{i\,d+j(b-d)+b+p-1},$$

$$f(x;\Phi)=\sum_{i,j,q=0}^{\infty}\omega_{i\,j\,q}\ h_{i\,j\,q}(x;\xi)+\sum_{i,j,p=0}^{\infty}\omega_{i\,j\,p}\ h_{i\,j\,p}(x;\xi). \tag{15}$$

where,

$$h_{i\,j\,q}(x;\xi)=g(x;\xi)\left[G(x;\xi)\right]^{i\,d+j(b-d)+d+q-1},\qquad h_{i\,j\,p}(x;\xi)=g(x;\xi)\ \left[G(x;\xi)\right]^{i\,d+j(b-d)+b+p-1}.$$

$$\omega_{i\,j\,q}=\frac{(-1)^i\,(i\,d+j(b-d)+d+q)!}{j!\,q!\,(i-j)!\,(i\,d+j(b-d)+d)!}\,a^j\,d\,c^{i-j+1},\qquad \omega_{i\,j\,p}=\frac{(-1)^i\,\Gamma(i\,d+b(j+d)+b+p)}{j!\,p!\,(i-j)!\,\Gamma(i\,d+b(j+d)+b)}\,a^{j+1}\,b\ c^{i-j}.$$

4.2. Quantile function

Quantile functions are used in widespread in general statistics and often to obtain percentiles. The quantile function of AW-G family, say $Q(u)=F^{-1}(u)$, is straightforward to be computed by inverting (5) as follows $u=1-e^{-c\left[\frac{x_G}{1-x_G}\right]^d-a\left[\frac{x_G}{1-x_G}\right]^b}$. After some simplifications, the previous equation is reduced to

$$\ln(1-u)+\left[\frac{x_G}{1-x_G}\right]^d\left[c+a\left(\frac{x_G}{1-x_G}\right)^{b-d}\right]=0, \tag{16}$$

where, $x_G=Q(u)$. By solving the nonlinear equation (16), numerically, the AW-G family random variable X can be generated, where u has the uniform distribution on the unit interval.

4.3. Moments

Moments are necessary and important in study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis). In this subsection, the rth moment for AW-G family about the origin will be derived. The r^{th} moment of random variable X can be obtained from pdf (15) as follows

$$\mu'_r = \int_0^\infty x^r f(x;\Phi)dx = \sum_{i,j,q=0}^{\infty} \omega_{i\,j\,q} \int_0^\infty x^r\, h_{i\,j\,q}(x;\xi)\,dx + \sum_{i,j,p=0}^{\infty} \omega_{i\,j\,p} \int_0^\infty x^r\, h_{i\,j\,p}(x;\xi)\,dx.$$

Therefore,

$$\mu'_r = \omega_{i\,j\,q}\, I_{i,j,q,r} + \omega_{i\,j\,p}\, I_{i,j,p,r} \quad , r = 1, 2, ..., \tag{17}$$

where, $I_{i,j,q,r} = \sum\limits_{i\,,\,j\,,\,q=0}^{\infty} \int\limits_0^\infty x^r\, h_{i\,j\,q}(x;\xi)\,dx$ and $I_{i,j,p,r} = \sum\limits_{i\,,\,j\,,\,p=0}^{\infty} \int\limits_0^\infty x^r\, h_{i\,j\,p}(x;\xi)\,dx$.
In particular, the mean and variance of AW-G family are obtained as follows: $E(X) = \omega_{i\,j\,q}\, I_{i,j,q,1} + \omega_{i\,j\,p}\, I_{i,j,p,1}$, where, $I_{i,j,q,1} = \sum\limits_{i\,,\,j\,,\,q=0}^{\infty} \int\limits_0^\infty x\, \omega_{i\,j\,q}\; h_{i\,j\,q}(x;\xi)\,dx$ and $I_{i,j,p,1} = \sum\limits_{i\,,\,j\,,\,p=0}^{\infty} \int\limits_0^\infty x\, \omega_{i\,j\,p}\; h_{i\,j\,p}(x;\xi)\,dx$. The variance is $Var(X) = \omega_{i\,j\,q}\, I_{i,j,q,2} + \omega_{i\,j\,p}\, I_{i,j,p,2} - [\omega_{i\,j\,q}\, I_{i,j,q,1} + \omega_{i\,j\,p}\, I_{i,j,p,1}]^2$. where, $I_{i,j,q,2} = \sum\limits_{i\,,\,j\,,\,q=0}^{\infty} \int\limits_0^\infty x^2\, \omega_{i\,j\,q}\; h_{i\,j\,q}(x;\xi)\,dx$, and $I_{i,j,p,2} = \sum\limits_{i\,,\,j\,,\,p=0}^{\infty} \int\limits_0^\infty x^2\, \omega_{i\,j\,p}\; h_{i\,j\,p}(x;\xi)\,dx$. Additionally, measures of skewness and kurtosis of family can be obtained, based on (17), according to the following relations

$$\gamma_1 = \frac{\mu'_3 - 3\mu'_2\mu'_1 + 2{\mu'}_1^3}{\left(\mu'_2 - {\mu'}_1^2\right)^{3/2}}, \qquad \gamma_2 = \frac{\mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2{\mu'}_1^2 - 3{\mu'}_1^4}{\left(\mu'_2 - {\mu'}_1^2\right)^2}.$$

Furthermore, the moment generating function of AW-G family is as follows $M_X(t) = \sum\limits_{r=0}^{\infty} \frac{t^r\,\mu'_r}{r!}$ where, μ'_r is the r^{th} moment about origin, then the moment generating function of AW-G family is obtained by using (17) as follows

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\omega_{i\,j\,q}\, I_{i,j,q,r} + \omega_{i\,j\,p}\, I_{i,j,p,r}\right].$$

4.4. Distribution of Order Statistics

Let X_1, X_2,..., X_n be a simple random sample from AW-G family with cdf (5) and pdf (7) and $X_{1:n}$, $X_{2:n}$,..., $X_{n:n}$ denote the order statistics obtained from this sample. The pdf of $X_{r:n}$ is obtained through the following

$$f_{r:n}(x;\Phi) = \frac{1}{B(r, n-r+1)} \left[F(x;\Phi)\right]^{r-1} \left[1 - F(x;\Phi)\right]^{n-r} f(x;\Phi).$$

$$f_{r:n}(x;\Phi) = \frac{1}{B(r, n-r+1)} \sum_{s=0}^{n-r} (-1)^s \begin{pmatrix} n-r \\ s \end{pmatrix} \left[F(x;\Phi)\right]^{r+s-1} f(x;\Phi).$$

Using the cdf (5) and pdf (7), the pdf of r^{th} order statistic from AW-G family takes the following form

$$f_{r:n}(x;\Phi) = \frac{1}{B(r, n-r+1)} \sum_{s=0}^{n-r} (-1)^s \begin{pmatrix} n-r \\ s \end{pmatrix} \left[1 - \exp\left\{-c\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^d - a\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^c\right\}\right]^{r+s-1}$$
$$.\left\{\frac{c\,dg(x;\xi)}{\left[\bar{G}(x;\xi)\right]^2}\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{d-1} + \frac{a\,b\,g(x;\xi)}{\left[\bar{G}(x;\xi)\right]^2}\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{b-1}\right\} \exp\left\{-c\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^d - a\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^b\right\}. \tag{18}$$

Since

$$\left[1 - \exp\left\{-c\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^d - a\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^b\right\}\right]^{r+s-1} = \sum_{w=0}^{\infty} (-1)^w \begin{pmatrix} r+s-1 \\ w \end{pmatrix} \exp\left\{-w\,c\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^d - w\,a\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^b\right\}.$$

Substituting in (18), therefore

$$f_{r:n}(x;\Phi)=\frac{1}{B(r,n-r+1)}\sum_{w=0}^{\infty}\sum_{s=0}^{n-r}(-1)^{s+w}\begin{pmatrix}n-r\\ s\end{pmatrix}\begin{pmatrix}r+s-1\\ w\end{pmatrix}.\left\{\frac{c\,dg(x;\xi)}{\left[\bar{G}(x;\xi)\right]^{2}}\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{d-1}+\frac{a\,b\,g(x;\xi)}{\left[\bar{G}(x;\xi)\right]^{2}}\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{b-1}\right\}$$
$$.\exp\left\{-c(w+1)\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{d}-a(w+1)\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{b}\right\}. \tag{19}$$

Using the exponential expansion form for the following term,

$$\exp\left[(w+1)\left\{-\,c\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{d}-\,a\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{b}\right\}\right]=\sum_{i,j=0}^{\infty}\frac{(-1)^{i}\,(w+1)^{i}\,a^{j}\,c^{i-j}}{i!\,j!\,(i-j)!}\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{i\,d+j(b-d)}.$$

Substituting in (19)

$$f_{r:n}(x;\Phi)=\frac{1}{B(r,n-r+1)}\sum_{s=0}^{n-r}\sum_{i,j,w=0}^{\infty}\frac{(-1)^{i+s+w}}{i!\,j!\,(i-j)!}\begin{pmatrix}r+s-1\\ w\end{pmatrix}\begin{pmatrix}n-r\\ s\end{pmatrix}(w+1)^{i}\,a^{j}\,c^{i-j}\,\frac{g(x;\xi)}{\left[\bar{G}(x;\xi)\right]^{2}}$$
$$.\left\{c\,d\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{(i+1)d+j(b-d)-1}+a\,b\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{i\,d+j(b-d)+b-1}\right\}.$$

Therefore,

$$f_{r:n}(x;\Phi)=\eta_{r}\,\frac{g(x;\xi)}{\left[\bar{G}(x;\xi)\right]^{2}}\left\{c\,d\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{(i+1)d+j(b-d)-1}+a\,b\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{i\,d+j(b-d)+b-1}\right\}. \tag{20}$$

where,

$$\eta_{r}=\frac{1}{B(r,n-r+1)}\sum_{s=0}^{n-r}\sum_{i,j,w=0}^{\infty}\frac{(-1)^{i+s+w}}{i!\,j!\,(i-j)!}\begin{pmatrix}r+s-1\\ w\end{pmatrix}\begin{pmatrix}n-r\\ s\end{pmatrix}(w+1)^{i}\,a^{j}\,c^{i-j}.$$

Let $\overset{*}{G}(x;\xi)=G(x;\xi)/\bar{G}(x;\xi)$, therefore, the pdf of r^{th} order statistic $X_{r:n}$ from AW-G family will be as follows

$$f_{r:n}(x;\Phi)=\eta_{r}\,\frac{g(x;\xi)}{\left[\bar{G}(x;\xi)\right]^{2}}\left\{c\,d\left[\overset{*}{G}(x;\xi)=\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{d(i+1)+j(b-d)-1}+a\,b\left[\overset{*}{G}(x;\xi)=\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{i\,d+j(b-d)+b-1}\right\}. \tag{21}$$

In particular, the pdf of the smallest order statistic $X_{1:n}$ is obtained from (21), by substituting $r=1$

$$f_{1:n}(x;\Phi)=\eta_{1}\,\frac{g(x;\xi)}{\left[\bar{G}(x;\xi)\right]^{2}}\left\{c\,d\left[\overset{*}{G}(x;\xi)\right]^{d(i+1)+j(b-d)-1}+a\,b\left[\overset{*}{G}(x;\xi)\right]^{i\,d+j(b-d)+b-1}\right\},$$

where,

$$\eta_{1}=n\sum_{i,j,w=0}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^{i+w+s}}{i!\,j!\,(i-j)!}\begin{pmatrix}s\\ w\end{pmatrix}\begin{pmatrix}n-1\\ s\end{pmatrix}(w+1)^{i}\,a^{j}\,c^{i-j}.$$

Also, the pdf of the largest order statistic $X_{n:n}$ is obtained from (21), by substituting $r=n$

$$f_{n:n}(x;\Phi)=\eta_{n}\left\{c+a\,b\left[\overset{*}{G}(x;\xi)\right]^{b-1}\right\}\exp\left\{-c\left[\overset{*}{G}(x;\xi)\right]^{d}-a\left[\overset{*}{G}(x;\xi)\right]^{b}\right\}^{w+1},$$

where,

$$\eta_{n}=n\sum_{i,j,w=0}^{\infty}\frac{(-1)^{i+w+s}}{i!\,j!\,(i-j)!}\begin{pmatrix}n+s-1\\ w\end{pmatrix}(w+1)^{i}\,a^{j}\,c^{i-j}.$$

5. Estimation of Model Parameters

In this section, the maximum likelihood estimators of the model parameters $\Phi = (a,\ b,\ c,\ d, \xi)$ of AW-G family from complete samples are derived. Let $X_1\ ,\quad X_2\ ,\ \ldots, X_n$ be a simple random sample from AW-G family with observed values $x_1\ ,\quad x_2\ ,\ \ldots x_n$. The log likelihood function of (6) is obtained as follows

$$ln\, L(\Phi) = \sum_{i=1}^{n} ln\left[c\, d\, g(x_i;\xi)\frac{[G(x_i;\xi)]^{d-1}}{[\bar{G}(x_i;\xi)]^{d+1}} + a\, b\, g(x;\xi)\frac{[G(x_i;\xi)]^{b-1}}{[\bar{G}(x_i;\xi)]^{b+1}}\right] - c\sum_{i=1}^{n}\left[\frac{G(x_i;\xi)}{\bar{G}(x_i;\xi)}\right]^{d} - a\sum_{i=1}^{n}\left[\frac{G(x_i;\xi)}{\bar{G}(x_i;\xi)}\right]^{b}.$$

For simplicity, let

$$Z_i = c\, d\, g(x_i;\xi)\frac{[G(x_i;\xi)]^{d-1}}{[\bar{G}(x_i;\xi)]^{d+1}} + a\, b\, g(x_i;\xi)\frac{[G(x_i;\xi)]^{b-1}}{[\bar{G}(x_i;\xi)]^{b+1}},$$

and $ln\, L(\Phi)$ to be ℓ, then

$$\ell = \sum_{i=1}^{n} ln\, Z_i - c\sum_{i=1}^{n}\left[\frac{G(x_i;\xi)}{\bar{G}(x_i;\xi)}\right]^{d} - a\sum_{i=1}^{n}\left[\frac{G(x_i;\xi)}{\bar{G}(x_i;\xi)}\right]^{b}.$$

Differentiating ℓ with respect to each parameter and setting the result equals to zero, the maximum likelihood estimators will be obtained. The partial derivatives of ℓ with respect to each parameter are given by

$$\frac{\partial \ell}{\partial a} = \sum_{i=1}^{n}\frac{Z'_{i\,a}}{Z_i} - \sum_{i=0}^{n}\left[\frac{G(x_i;\xi)}{\bar{G}(x_i;\xi)}\right]^{b},\ \frac{\partial \ell}{\partial b} = \sum_{i=1}^{n}\frac{Z'_{i\,b}}{Z_i} - a\sum_{i=1}^{n}\left[\frac{G(x_i;\xi)}{\bar{G}(x_i;\xi)}\right]^{b} ln\left[\frac{G(x_i;\xi)}{\bar{G}(x_i;\xi)}\right],$$

$$\frac{\partial \ell}{\partial c} = \sum_{i=1}^{n}\frac{Z'_{i\,c}}{Z_i} - \sum_{i=1}^{n}\left[\frac{G(x_i;\xi)}{\bar{G}(x_i;\xi)}\right]^{d},\ \frac{\partial \ell}{\partial d} = \sum_{i=1}^{n}\frac{Z'_{i\,d}}{Z_i} - c\sum_{i=1}^{n}\left[\frac{G(x_i;\xi)}{\bar{G}(x_i;\xi)}\right]^{d} ln\left[\frac{G(x_i;\xi)}{\bar{G}(x_i;\xi)}\right],$$

$$\frac{\partial \ell}{\partial \xi} = \sum_{i=1}^{n}\frac{Z'_{i\,\xi}}{Z_i} - \sum_{i=1}^{n} g(x_i;\xi)\left[c\, d\frac{[G(x_i;\xi)]^{d-1}}{[\bar{G}(x_i;\xi)]^{d+1}} + a\, b\frac{[G(x_i;\xi)]^{b-1}}{[\bar{G}(x_i;\xi)]^{b+1}}\right],\quad \text{where}$$

$$Z'_{i\,a} = \frac{\partial Z_i}{\partial a} = b\, g(x_i;\xi)\frac{[G(x_i;\xi)]^{b-1}}{[\bar{G}(x_i;\xi)]^{b+1}},$$

$$Z'_{ib} = a\, g(x_i;\xi)\ \left[1 + b\, ln\,[G(x_i;\xi)] - b\, ln\left[\bar{G}(x_i;\xi)\right]\right]\ \frac{[G(x_i;\xi)]^{b-1}}{[\bar{G}(x_i;\xi)]^{b+1}},$$

$$Z'_{ic} = \frac{\partial Z_i}{\partial c} = d\, g(x_i;\xi)\frac{[G(x_i;\xi)]^{d-1}}{[\bar{G}(x_i;\xi)]^{d+1}}, Z'_{i\,d} = c\, g(x_i;\xi)\ \left[1 + d\, ln\,[G(x_i;\xi)] - d\, ln\left[\bar{G}(x_i;\xi)\right]\right]\ \frac{[G(x_i;\xi)]^{d-1}}{[\bar{G}(x_i;\xi)]^{d+1}},\ \text{and}$$

$$Z'_{i\,\xi} = \left[c\, d\frac{[G(x_i;\xi)]^{d-1}}{[\bar{G}(x_i;\xi)]^{d+1}} + a\, b\frac{[G(x_i;\xi)]^{b-1}}{[\bar{G}(x_i;\xi)]^{b+1}}\right]\left[g'(x_i;\xi) + g(x_i;\xi)\left(\ln\,[G(x_i;\xi)] - \ln\left[\bar{G}(x_i;\xi)\right]\right)\right].$$

The maximum likelihood estimates (MLEs) of the model parameters are determined by solving the non-linear equations

$$\frac{\partial \ell}{\partial a} = 0,\ \frac{\partial \ell}{\partial b} = 0,\ \frac{\partial \ell}{\partial c} = 0,\ \frac{\partial \ell}{\partial \xi} = 0.$$

These equations cannot be solved analytically but some software's can be used to solve them numerically.

6. Application of AW-G Family to Real Data

Two real data sets are considered to demonstrate the flexibility of new AW-G family as compared with some of their sub-models. For both data sets, MLEs of the model parameters are obtained. The model selection is accomplished using the Kolmogorov-Smirnov *(K-S)* statistic and its p-value, minus of log-likelihood function $(\mathbf{-2\ell})$**,** Akaike information criterion *(AIC)*, Bayesian information criterion *(BIC)*, the Hannan-Quinn information criterion *(HQIC)* and the corrected Akaike information criterion *(CAIC)*. Generally, the smaller values of these statistics, the better fit to real data (see Akaike [1] and Schwarz [15]). Additionally, the histogram plots and the estimated pdf of the models for each data set are showed.

Furthermore, the plots of empirical cdf and estimated cdf of models for both data sets are displayed.We used the software program: MathCAD 14, Parametric Technology Corporation (2007). The first real data set represents the ages of 155 patients of breast tumors taken from (June-November 2014), whose entered in (Breast Tumors Early Detection Unit, Benha Hospital University, Egypt). The data are listed in Table 1.

46	32	50	46	44	42	69	31	25	29	40	42	24	17	35
48	49	50	60	26	36	56	65	48	66	44	45	30	28	40
40	50	41	39	36	63	40	42	45	31	48	36	18	24	35
30	40	48	50	60	52	47	50	49	38	30	52	52	12	48
50	45	50	50	50	53	55	38	40	42	42	32	40	50	58
48	32	45	42	36	30	28	38	54	90	80	60	45	40	50
50	40	50	50	50	60	39	34	28	18	60	50	20	40	50
38	38	42	50	40	36	38	38	50	50	31	59	40	42	38
40	38	50	50	50	40	65	38	40	38	58	35	60	90	48
58	45	35	38	32	35	38	34	43	40	35	54	60	33	35
36	43	40	45	56										

Table 1. **The first real data set.**

For the above real data set, we fit the AWW and AWLL as compared with the three other competitive distributions. These distributions are the transmuted exponentiated modified Weibull (TEMW) (see Ashour and Eltehiwy [5]), the exponentiated modified Weibull (EMW) (see Elbatal [10]), AW and Weibull (W).

Model	MLE of parameters	-2ℓ	AIC	CAIC	HQIC	BIC	K-S	p-value
AWW	$\hat{a}=0.650$	623.67	668.23	682.19	716.05	738.45	0.00213	0.562
	$\hat{b}=0.002$							
	$\hat{c}=0.076$							
	$\hat{d}=0.031$							
	$\hat{\lambda}=28.007$							
	$\hat{\theta}=8.301$							
AWLL	$\hat{a}=0.014$	665.03	680.14	701.35	725.26	750.11	0.0313	0.5041
	$\hat{b}=0.032$							
	$\hat{c}=1.061$							
	$\hat{d}=0.505$							
	$\hat{\lambda}=2.080$							
	$\hat{\alpha}=14.007$							
EMW	$\hat{a}=0.382$	700.41	734.10	765.34	792.21	803.03	0.0724	0.2797
	$\hat{b}=0.024$							
	$\hat{c}=2.133$							
	$\hat{\alpha}=25.301$							
TEMW	$\hat{a}=4.161$	901.24	923.28	941.51	956.06	978.31	0.0971	0.1983
	$\hat{b}=5.89$							
	$\hat{c}=3.687$							
	$\hat{d}=1.121$							
	$\hat{\lambda}=9.810$							
AW	$\hat{a}=0.901$	1007.08	1037.71	1051.93	1067.46	1080.11	0.1823	0.1317
	$\hat{b}=1.007$							
	$\hat{c}=0.043$							
	$\hat{d}=1.677$							
W	$\hat{a}=1.412$	1031.61	1050.72	1072.23	1097.50	1101.09	0.2032	0.0966
	$\hat{b}=0.008$							

Table 2. **MLEs, -2ℓ, AIC, HQIC, CAIC, BIC, K-S and p-value for the first data set.**

The fitted densities for the data set are displayed in Figure 3 (together with the data histogram), respectively

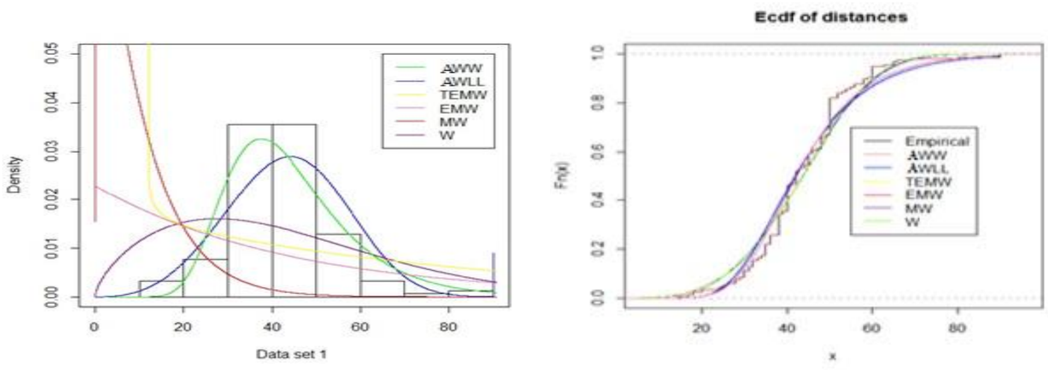


Figure 3. **Plots of the estimated pdfs and cdfsfor selected models for first data set.**

The second real data set represents an uncensored data set from Nichols and Padgett [14] on breaking stress of carbon fibres (in Gba). The data are listed in Table 3.

3.70	2.74	2.73	2.50	3.60	3.11	3.27	2.87	1.47	4.42
2.41	3.19	3.22	1.69	3.28	3.09	1.87	3.15	4.90	3.75
2.43	2.95	2.97	3.39	2.96	2.53	2.67	2.93	3.22	3.39
2.81	4.20	3.33	2.55	3.31	3.31	2.85	3.56	3.15	2.55
2.59	2.38	2.77	1.92	1.42	3.68	2.97	1.36	0.98	2.67
4.91	3.68	1.84	1.59	3.19	1.57	0.81	5.56	1.73	1.59
2.00	2.48	0.85	1.61	2.79	4.70	2.03	1.61	2.21	1.89
2.88	2.82	2.05	3.65						

Table 3. **The second real data set.**

As a second application to the suggested family, the AWBXII and AWU distributions are compared with TEMW and EMW distributions. Table 4 records the MLE of the model parameters and the above mentioned measures.

Model	MLE of parameters	-2ℓ	AIC	CAIC	BIC	K-S	p-value
AWBXII	$\hat{a}=0.276$	574.21	1018.171	1024.435	1027.404	0.0871	0.0674
	$\hat{b}=0.0021$						
	$\hat{c}=2.1528$						
	$\hat{d}=1.4330$						
	$\widehat{\alpha}=0.3068$						
	$\widehat{\theta}=17.006$						
AWU	$\hat{a}=0.013$	578.71	1021.245	1026.068	1032.400	0.1900	0.0257
	$\hat{b}=1.0236$						
	$\hat{c}=0.6205$						
	$\hat{d}=9.04$						
	$\widehat{\theta}=1.8002$						
TEMW	$\hat{a}=0.243$	810.26	1452.150	1487.272	1534.740	0.2209	0.0016
	$\hat{b}=0.0016$						
	$\hat{c}=8.0067$						
	$\hat{d}=2.1201$						
	$\widehat{\lambda}=13.012$						
EMW	$\hat{a}=5.598$	959.80	1927.801	1935.702	1939.813	0.0673	0.0921
	$\hat{b}=3.6871$						
	$\hat{c}=0.0207$						
	$\widehat{\alpha}=0.3030$						

Table 4. **MLEs, -2ℓ, AIC, HQIC, CAIC, BIC, K-S and p-value for the second data set.**

The fitted densities for the second data sets are displayed in Figure 4(together with the data histogram), respectively.

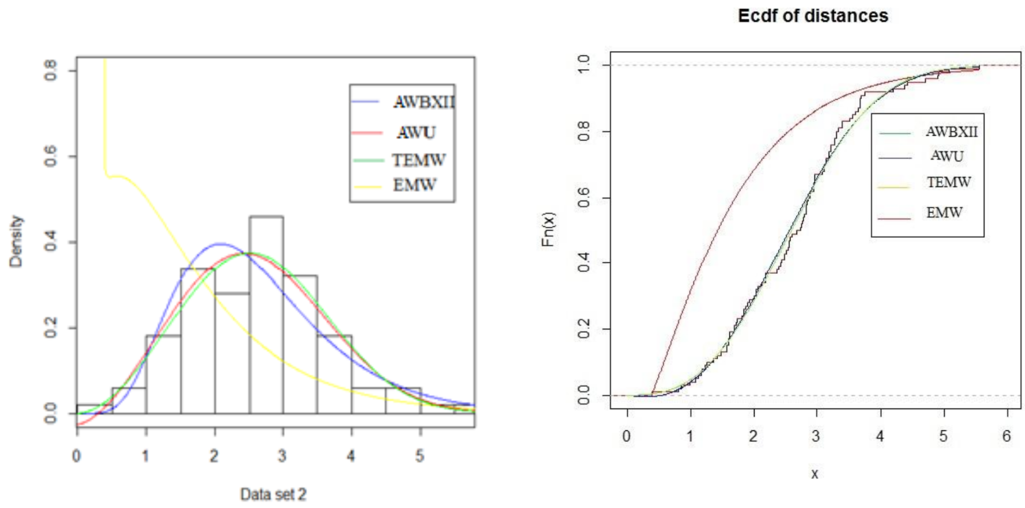


Figure 4. **Plots of the estimated pdfs and cdfsof the selected models for the second data set.**

Based on the values in Tables 3 and 4, the new AWW, AWLL, AWBXII and AWU models provide adequate fits as compared to other models in both applications with small values for *AIC, CAIC*, *BIC* and *K-S*. Based on the first data set, the new AWW model is much better than the TEMW, EMW, AW and W models. Based on the second data set, the new AWBXII is better than the AWU, TEMW and EMW models, while the AWU and TEMW models approximately give the same fit to the second data set. Figures 3 and 4 also support the results in Tables 3 and 4.

7. Conclusions

We introduce and study a new generated family of distributions, called the additive Weibull-G. The AW- G family generalizes the Weibull-G family (see [7]) and includes several new distributions. Properties of the AW-G family include: an expansion for the density function and expressions for the quantile function, ordinary moments, moment generating function, reliability and order statistics. Four new distributions, namely, AWU, AWW, AWBXII and AWLL are defined and discussed in some details. The maximum likelihood method is employed to estimate the model parameters. Two real data sets are used to demonstrate the flexibility of some distributions belonging to the introduced family. These special models give better fits than other models. We hope the findings of the paper will be quite useful for the practitioners in various fields of probability, statistics and applied sciences.

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