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Invariant Submanifold of $\tilde{\psi}$ $(5, -3)$ Structure Manifold

Research Article

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Abstract: In this paper, we have studied various properties of a $\tilde{\psi}$ $(5, -3)$ structure manifold and its invariant submanifold. Under two different assumptions, the nature of induced structure ψ, has also been discussed.

Keywords: Invariant submanifold, Nijenhuis tensor, projection operators and complementary distributions.

1. Introduction

Let V^m be a C^∞ m-dimensional Riemannian manifold imbedded in a C^∞ n-dimensional Riemannian manifold M^n, where $m < n$. The imbedding being denoted by R.Nivas & S.Yadav [7] and K.Yano [6]

$$f \ : \ V^m \longrightarrow M^n.$$

Let B be the mapping induced by f i.e. $B = df$.

$$df \ : \ T(V) \longrightarrow T(M).$$

Let $T(V, M)$ be the set of all vectors tangent to the submanifold $f(V)$. It is well known that according to A.Bejanc [1] and B.Prasad [2]

$$B \ : \ T(V) \longrightarrow T(V, M).$$

Is an isomorphism. The set of all vectors normal to $f(V)$ forms a vector bundle over $f(V)$, which we shall denote by $N(V, M)$. We call $N(V, M)$ the normal bundle of V^m. The vector bundle induced by f from $N(V, M)$ is denoted by $N(V)$. We denote by $C \ : \ N(V) \longrightarrow N(V, M)$ the natural isomorphism and by $\eta_s^r(V)$ the space of all C^∞ tensor fields of type $(r,\ s)$ associated with $N(V)$. Thus $\zeta_0^0(V) = \eta_0^0(V)$ is the space of all C^∞ functions defined on V^m while an element of $\eta_0^1(V)$ is a C^∞ vector field normal to V^m and an element of $\zeta_0^1(V)$ is a C^∞ vector field tangential to V^m.

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Let $\bar{X}$ and $\bar{Y}$ be vector fields defined along $f(V)$ and $\tilde{X}$, $\tilde{Y}$ be the local extensions of $\bar{X}$ and $\bar{Y}$ respectively. Then $\left[\tilde{X},\, \tilde{Y}\right]$ is a vector field tangential to M^n and its restriction $\left[\tilde{X},\, \tilde{Y}\right] \big/ f(V)$ to $f(V)$ is determined independently of the choice of these local extension $\tilde{X}$ and $\tilde{Y}$. Thus $\left[\bar{X},\, \bar{Y}\right]$ is defined as

$$\left[\bar{X},\, \bar{Y}\right] \;=\; \left[\tilde{X},\, \tilde{Y}\right] \big/ f(V) \tag{1}$$

Since B is an isomorphism

$$[BX,\; BY] \;=\; B\; [X, Y] \;\; \text{for all} \;\; X, Y \;\in\; \zeta_0^1\,(V)\,. \tag{2}$$

Let $\bar{G}$ be the Riemannain metric tensor of M^n, we define g and g^* on V^m and $N(V)$ respectively as

$$g\;(X_1,\; X_2) \;=\; \tilde{G}\;(BX_1,\; BX_2)\; f, \;\; \text{and} \tag{3}$$

$$g^*\;(N_1,\; N_2) \;=\; \tilde{G}\;(CN_1,\; CN_2) \tag{4}$$

For all $X_1,\; X_2 \;\in\; \zeta_0^1\,(V)$ and $N_1,\, N_2 \in\; \eta_0^1\,(V)$. It can be verified that g and g^* are the induced metrics on V^m and $N(V)$ respectively. Let $\tilde{\nabla}$ be the Riemannian connection determined by $\tilde{G}$ in M^n, then $\tilde{\nabla}$ induces a connection ∇ in $f(V)$ defined by

$$\nabla_{\bar{X}} \bar{Y} = \tilde{\nabla}_{\tilde{X}} \tilde{Y} \big/ f(V) \tag{5}$$

where $\bar{X}$ and $\bar{Y}$ are arbitrary C^∞ vector fields defined along $f(V)$ and tangential to $f(V)$. Let us suppose that M^n is a C^∞ $\tilde{\psi}$ $(5,-3)$ structure manifold with structure tensor $\tilde{\psi}$ of type (1, 1) satisfying

$$\tilde{\psi}^5 \;-\; \tilde{\psi}^3 \;=\; 0 \tag{6}$$

Let $\tilde{L}$ and $\tilde{M}$ be the complementary distributions corresponding to the projection operators

$$\tilde{l} \;=\; \tilde{\psi}^4, \qquad\qquad \tilde{m} \;=\; I \;-\; \tilde{\psi}^4 \tag{7}$$

where I denotes the identity operator. From (6) and (7), we have

$$(a)\;\; \tilde{l} + \tilde{m} = I \qquad (b)\;\; \tilde{l}^2 = \tilde{l} \qquad (c)\;\; \tilde{m}^2 = \tilde{m} \qquad (d)\;\; \tilde{l}\,\tilde{m} = \tilde{m}\tilde{l} = 0. \tag{8}$$

Let D_l and D_m be the subspaces inherited by complementary projection operators l and m respectively. We define

$$D_l \;=\; \{X \;\in\; T_p\,(V) \;:\; lX \;=\; X,\; mX \;=\; 0\}$$

$$D_m \;=\; \{X \;\in\; T_p\,(V) \;:\; mX \;=\; X,\; lX \;=\; 0\}\,.$$

Thus $T_p\,(V) \;=\; D_l \;+\; D_m$. Also

$$Ker\; l \;=\; \{X \;:\; lX \;=0\} \;=\; D_m$$

$$Ker\; m \;=\; \{X \;:\; mX \;=0\} \;=\; D_l$$

at each point p of $f(V)$.

2. Invariant Submanifold of $\tilde{\psi}\,(5,-3)$ Structure Manifold

H.B.Pandey & A.Kumar [5], we call V^m to be invariant submanifold of M^n if the tangent space $T^p\,(f\,(V))$ of $f\,(V)$ is invariant by the linear mapping $\tilde{\psi}$ at each point p of $f\,(V)$. Thus

$$\tilde{\psi}BX = B\psi X, \quad \text{for all } \; X \;\in\; \zeta_0^1\,(V)\,, \tag{9}$$

and ψ being a (1, 1) tensor field in V^m.

Theorem 2.1. *Let $\tilde{N}$ and N be the Nijenhuis tensors determined by $\tilde{\psi}$ and ψ in M^n and V^m respectively, then*

$$\tilde{N}\,(BX,\; BY) = BN\,(X,\; Y)\,, \quad \textit{for all } \; X,\; Y \;\in\; \zeta_0^1\,(V)\,. \tag{10}$$

Proof. We have, by using (2) and (9)

$$\begin{aligned} \tilde{N}\,(BX,\; BY) &= \left[\tilde{\psi}BX, \tilde{\psi}BY\right] + \tilde{\psi}^2\,[BX, BY] - \tilde{\psi}\left[\tilde{\psi}BX, BY\right] - \tilde{\psi}\left[BX, \tilde{\psi}BY\right] \\ &= [B\psi X, B\psi Y] + \tilde{\psi}^2 B\,[X,Y] - \tilde{\psi}\,[B\psi X, BY] - \tilde{\psi}\,[BX, B\psi Y] \\ &= B\,[\psi X, \psi Y] + B\psi^2\,[X,Y] - \tilde{\psi}B\,[\psi X, Y] - \tilde{\psi}B\,[X, \psi Y] \\ &= B\left\{[\psi X, \psi Y] + \psi^2\,[X,Y] - \psi\,[\psi X, Y] - \psi\,[X, \psi Y]\right\} \\ &= BX + B\psi^3 X \end{aligned} \tag{11}$$

□

3. Distribution $\tilde{M}$ Never Being Tangential to $f\,(V)$

Theorem 3.1. *If the distribution $\tilde{M}$ is never tangential to $f\,(V)$, then*

$$\tilde{m}\,(BX) = 0 \;\; \textit{for all } \; X \in \; \zeta_0^1 \;\; (V) \tag{12}$$

and the induced structure ψ on V^m satisfies

$$\psi^4 = \;\; I \tag{13}$$

Proof. If possible $\tilde{m}\,(BX) \neq 0$. From (9) We get,

$$\tilde{\psi}^4 BX = B\psi^4 X; \tag{14}$$

from (7) and (14)

$$\begin{aligned} \tilde{m}\,(BX) &= \left(I - \tilde{\psi}^4\right) BX \\ &= BX - B\psi^4 X \end{aligned}$$

$$\tilde{m}\,(BX) = B\left(X - \psi^4 X\right) \tag{15}$$

This relation shows that $\tilde{m}\,(BX)$ is tangential to $f\,(V)$ which contradicts the hypothesis. Thus $\tilde{m}\,(BX)$= 0. Using this result in (15) and remembering that B is an isomorphism, We get

$$\psi^4 = I, \tag{16}$$

□

Theorem 3.2. *Let $\tilde{M}$ be never tangential to $f(V)$, then*

$$\underset{\tilde{m}}{\tilde{N}}(BX, BY) = 0 \tag{17}$$

Proof. We have

$$\underset{\tilde{m}}{\tilde{N}}(BX, BY) = [\tilde{m}BX, \tilde{m}BY] + \tilde{m}^2 [BX, BY] - \tilde{m}[\tilde{m}BX, BY] - \tilde{m}[BX, \tilde{m}BY] \tag{18}$$

Using (2), (8) (c) and (12), we get (17). □

Theorem 3.3. *Let $\tilde{M}$ be never tangential to $f(V)$, then*

$$\underset{\tilde{l}}{\tilde{N}}(BX, BY) = 0 \tag{19}$$

Proof. We have

$$\underset{\tilde{l}}{\tilde{N}}(BX, BY) = \left[\tilde{l}BX, \tilde{l}BY\right] + \tilde{l}^2 [BX, BY] - \tilde{l}\left[\tilde{l}BX, BY\right] - \tilde{l}\left[BX, \tilde{l}BY\right] \tag{20}$$

Using (2), (8) (a), (b) and (12) in (20); we get (19). □

Theorem 3.4. *Let $\tilde{M}$ be never tangential to $f(V)$. Define*

$$\tilde{H}\left(\tilde{X}, \tilde{Y}\right) = \tilde{N}\left(\tilde{X}, \tilde{Y}\right) - \tilde{N}\left(\tilde{m}\tilde{X}, \tilde{Y}\right) - \tilde{N}\left(\tilde{X}, \tilde{m}\tilde{Y}\right) + \tilde{N}\left(\tilde{m}\tilde{X}, \tilde{m}\tilde{Y}\right). \tag{21}$$

For all $\tilde{X}, \tilde{Y} \in \zeta_0^1(M)$, then

$$\tilde{H}(BX, BY) = BN(X, Y). \tag{22}$$

Proof. Using $\tilde{X} = BX$, $\tilde{Y} = BY$ and (10), (12) in (21) We get (22). □

4. Distribution $\tilde{M}$ Always Being Tangential to $f(V)$

Theorem 4.1. *Let $\tilde{M}$ be always tangential to $f(V)$, then*

$$(a)\ \ \tilde{m}(BX) = Bm\,X \qquad (b)\ \ \tilde{l}(BX) = Bl\,X. \tag{23}$$

Proof. from (15), we get (23) (a). Also

$$l = \psi^4 \tag{24}$$

$$lX = \psi^4 X$$

$$BlX = B\psi^4\,X. \tag{25}$$

Using (9) in (25)

$$BlX = \tilde{\psi}^4 \quad BX \ = \tilde{l}(BX), \tag{26}$$

which is (23) (b). □

Theorem 4.2. *Let* $\tilde{M}$*be always tangential to* $f(V)$*, then* l *and* m *satisfy*

$$(a)\;\; l+m=I \quad (b)\;\; lm=ml=0 \quad (c)\;\; l^2=l \quad (d)\;\; m^2=m. \tag{27}$$

Proof. Using (8) and (23) We get the results. □

Theorem 4.3. *If* $\tilde{M}$ *is always tangential to* $f(V)$*, then*

$$\psi^5-\psi^3=0. \tag{28}$$

Proof. From (9)

$$\tilde{\psi}^5\; BX \;=\; B\,\psi^5\; X \tag{29}$$

Using (6) in (29)

$$\tilde{\psi}^3 BX = B\psi^5 X$$
$$B\psi^3 X = B\psi^5 X \;\text{ or}$$
$$\psi^5-\psi^3=0.$$

Which is (28) □

Theorem 4.4. *If* $\tilde{M}$ *Is always tangential to* $f(V)$ *then as in* (21)

$$\tilde{H}(BX,BY)=BH(X,Y). \tag{30}$$

Proof. From (21) we get

$$\tilde{H}(BX,BY)=\tilde{N}(BX,BY)-\tilde{N}(\tilde{m}BX,BY)-\tilde{N}(BX,\tilde{m}BY)+\tilde{N}(\tilde{m}BX,\tilde{m}BY) \tag{31}$$

Using (23) (a) and (10) in (31) we get (30). □

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