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Harmonic Maps on S-Manifolds

Research Article

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Abstract: We study in this paper harmonic maps and harmonic morphisms on S-manifolds. We also give some results and applications on the spectral theory of a harmonic map for which the target manifold is a S-space form.

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1. Introduction

Harmonic maps on Riemannian manifolds have been studied for many years, starting with the paper of J. Eells and J.H. Sampson [3]. Due to their analytic and geometric properties harmonic maps have become an important and attractive field of research. The study of harmonic maps on Riemannian manifolds endowed with some structures has its origin in a paper of Lichnerowicz's [14], in which he proved that a holomorphic map between Kähler manifolds is not only a harmonic map but also attains the minimum of the energy in its homotopy class. More general, Rawnsley [15] studied structure preserving harmonic maps between f-manifolds by using twistorial methods. Harmonic maps on $\mathcal{C}$-manifolds were studied by Gherghe and Kenmotsu in [9]. On contact geometry, similar results as Lichnerowicz's, were obtained by Ianuş, Gherghe, Pastore, Chinea [2, 8].

The purpose of this paper is to obtain some results concerning harmonic maps and harmonic morphisms on S-manifolds. After we recall some well known facts about harmonic maps, harmonic morphisms and S-manifolds (section 2), we prove that any structure preserving map from S-manifold to a Kähler manifold is harmonic and that there are no non constant harmonic holomorphic maps from a Kähler manifold to S-manifold (section 3). In the same section we give some conditions for a map from S-manifold to a Kähler manifold to be a harmonic morphism. In the last section we obtain some results on spectral theory of harmonic maps for which the target manifold is an S-space form.

2. Preliminaries

As a generalization of both almost complex (in even dimension) and almost contact (in odd dimension) structures, Yano introduced in [19] the notion of f-structure on a smooth manifold of dimension $2n + s$, i.e. a tensor field of type (1,1) and rank $2n$ satisfying $f^3 + f = 0$. The existence of such a structure is equivalent to a reduction of the structural group of the

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tangent bundle to $U(n)\times O(s)$. Let N be a $(2n+s)$-dimensional manifold with an f-structure of rank $2n$. If there exist s global vector fields $\xi_1,\xi_2,\dots,\xi_s$ on N such that:

$$f\xi_\alpha=0,\qquad \eta_\alpha\circ f=0,\qquad f^2=-I+\sum\xi_\alpha\otimes\eta_\alpha, \tag{1}$$

where η_α are the dual 1-forms of ξ_α, we say that the f-structure has complemented frames. For such a manifold there exists a Riemannian metric g such that

$$g(X,Y)=g(fX,fY)+\sum\eta_\alpha(X)\eta_\alpha(Y)$$

for any vector fields X and Y on N. See [1]. An f-structure f is normal, if it has complemented frames and

$$[f,f]+2\sum\xi_\alpha\otimes d\eta_\alpha=0,$$

where $[f,f]$ is Nijenhuis torsion of f. Let Ω be the fundamental 2-form defined by $\Omega(X,Y)=g(X,fY)$, X, $Y\in T(N)$. A normal f-structure for which the fundamental form Ω is closed, $\eta_1\wedge\cdots\wedge\eta_s\wedge(d\eta_\alpha)^n\neq 0$ for any α, and $d\eta_1=\cdots=d\eta_s=\Omega$ is called to be an S-structure. A smooth manifold endowed with an S-structure will be called an S-manifold. These manifolds were introduced by Blair in [1]. We have to remark that if we take $s=1$, S-manifolds are natural generalizations of Sasakian manifolds. In the case $s\geq 2$ some interesting examples are given in [1, 11]. If N is an S-manifold, then the following formulas are true (see [1]):

$$\overline{\nabla}_X\xi_\alpha=-fX,\qquad X\in T(N),\qquad \alpha=1,\dots,s, \tag{2}$$

$$(\overline{\nabla}_Xf)Y=\sum\{g(fX,fY)\xi_\alpha+\eta_\alpha(Y)f^2X\},\quad X,Y\in T(N), \tag{3}$$

where $\overline{\nabla}$ is the Riemannian connection of g. Let L be the distribution determined by the projection tensor $-f^2$ and let M be the complementary distribution which is determined by f^2+I and spanned by $\xi_1,\dots,\xi_s$. It is clear that if $X\in L$ then $\eta_\alpha(X)=0$ for any α, and if $X\in M$, then $fX=0$. A plane section π on N is called an invariant f-section if it is determined by a vector $X\in L(x)$, $x\in N$, such that $\{X,fX\}$ is an orthonormal pair spanning the section. The sectional curvature of π is called the f-sectional curvature. If N is an S-manifold of constant f-sectional curvature k, then its curvature tensor has the form

$$\begin{aligned}\overline{R}(X,Y,Z,W)=&\sum_{\alpha,\beta}\{g(fX,fW)\eta_\alpha(Y)\eta_\beta(Z)-g(fX,fZ)\eta_\alpha(Y)\eta_\beta(W)+g(fY,fZ)\eta_\alpha(X)\eta_\beta(W)-g(fY,fW)\eta_\alpha(X)\eta_\beta(Z)\}\\&+\frac{1}{4}(k+3s)\{g(fX,fW)g(fY,fZ)-g(fX,fZ)g(fY,fW)\}\\&+\frac{1}{4}(k-s)\{F(X,W)F(Y,Z)-F(X,Z)F(Y,W)-2F(X,Y)F(Z,W)\},\end{aligned} \tag{4}$$

X, Y, Z, W $\in T(N)$. Such a manifold $N(k)$ will be called an S-space form. The Euclidean space E^{2n+s} and the hyperbolic space H^{2n+s} are examples of S-space forms.

Now we recall some well known facts about harmonic maps on Riemannian manifolds. Let $F:(M,g)\longrightarrow(N,h)$ be a smooth map between two Riemannian manifolds of dimensions m and n respectively. The energy density of F is a smooth function $e(F):M\longrightarrow[0,\infty)$ given by

$$e(F)_p=\frac{1}{2}Tr_g(F^*h)(p)=\frac{1}{2}\sum_{i=1}^m h(F_{*p}u_i,F_{*p}u_i),$$

for any $p \in M$ and any orthonormal basis $\{u_1, \dots, u_m\}$ of T_pM. If M is a compact Riemannian manifold, the energy $E(F)$ of F is the integral of its energy density:

$$E(F) = \int_M e(F)\upsilon_g,$$

where υ_g is the volume measure associated with the metric g on M. A map $F \in C^\infty(M, N)$ is said to be harmonic if it is a critical point of the energy functional E on the set of all maps between (M, g) and (N, h). Now, let (M, g) be a compact Riemannian manifold. If we look at the Euler-Lagrange equation for the corresponding variational problem, a map $F : M \longrightarrow N$ is a harmonic if and only if $\tau(F) \equiv 0$, where $\tau(F)$ is tension field which is defined by

$$\tau(F) = Tr_g \tilde{\nabla} dF,$$

where $\tilde{\nabla}$ is the connection induced by the Levi-Civita connection on M and the F-pull back connection of the Levi Civita connection on N. We take now a smooth variation $F_{s,t}$ with two parameters $s, t \in (-\epsilon, \epsilon)$ such that $F_{0,0} = F$. The corresponding variation vector fields are denoted by V and W. The second variation formula of E is:

$$H_F(V, W) = \frac{\partial^2}{\partial s \partial t}(E(F_{s,t}))\big|_{(s,t)=(0,0)} = \int_M h(J_F(V), W)\upsilon_g,$$

where J_F is a second order self-adjoint elliptic operator acting on the space of variation vector fields along F (which can be identified with $\Gamma(F^{-1}(TN))$) and is defined by

$$J_F(V) = -\sum_{i=1}^{m}(\widetilde{\nabla}_{u_i}\widetilde{\nabla}_{u_i} - \widetilde{\nabla}_{\nabla_{u_i}u_i})V - \sum_{i=1}^{m} R^N(V, dF(u_i))dF(u_i), \tag{5}$$

for any $V \in \Gamma(F^{-1}(TN))$ and any local orthonormal frame $\{u_1, \dots, u_m\}$ on M. Here R^N is the curvature tensor of (N, h) and $\widetilde{\bigtriangledown}$ is the pull-back connection by ϕ of the Levi-Civita connection of N (see [6] and [16] for more details on harmonic maps).

3. Harmonic Maps and Harmonic Morphisms on S- manifolds

A smooth map $F : M \to N$ from an S-manifold $M(f, \xi_\alpha, \eta_\alpha, g)$ to a Kähler manifold $N(J, h)$ is called to be a (f, J)-holomorphic if its differential interwines the structures, that is $dF \circ f = J \circ dF$. In the following theorem we prove the harmonicity of such maps.

Theorem 3.1. *Any (f, J)-holomorphic map from an S-manifold $M(f, \xi_\alpha, \eta_\alpha, g)$ to a Kähler manifold $N(J, h)$ is a harmonic map.*

Proof. The tension field of any (f, J)-holomorphic can be computed by using the following formula see [8].

$$J(\tau(F)) = F_*(divf) - tr_g\beta, \tag{6}$$

where $\beta(X, Y) = (\widetilde{\nabla}_X J)F_*Y$, $\widetilde{\nabla}$ being the connection induced in the pull-back bundle F^*TN. Let $\{e_1, ..., e_m, fe_1, ..., fe_m, \xi_1, \dots, \xi_s\}$ be a local orthonormal f-adapted basis on TM, then we have

$$\begin{aligned}
divf &= \sum_{i=1}^{2m+s}(\nabla_{e_i} f)e_i = \sum_{i=1}^{m}(\nabla_{e_i} f)e_i + \sum_{i=1}^{m}(\nabla_{fe_i} f)fe_i + \sum_{\alpha=1}^{s}(\nabla_{\xi_\alpha} f)\xi_\alpha \\
&= \sum_{i=1}^{m}\sum_{\alpha}\{g(fe_i, fe_i)\xi_\alpha + \eta_\alpha(e_i)f^2e_i\} + \{g(f^2e_i, f^2e_i)\xi_\alpha + \eta_\alpha(fe_i)f^2fe_i\} + \{g(f\xi_\alpha, f\xi_\alpha)\xi_\alpha + \eta_\alpha(\xi_\alpha)f^2\xi_\alpha\} \\
&= \sum_{i=1}^{m}\sum_{\alpha} 2g(e_i, e_i)\xi_\alpha = \sum_{\alpha} 2m\xi_\alpha.
\end{aligned}$$

But, as F is (f, J)-holomorphic, it is easy to see that $F_*(\xi_\alpha) = 0$ for any α and we get $F_*(divf) = 0$. Finally, as N is a Kähler manifold we have $\nabla J = 0$, and thus the second term of the formula 6 vanishes. Therefore $\tau(F) = 0$ and thus F is harmonic. □

Example 3.2. *It is well known that the canonical example of a Sasakian manifold, the odd dimensional sphere S^{2n+1}, is a circle bundle over the complex projective space PC^n by the Hopf-fibration. Let $\acute{\pi} : S^{2n+1} \longrightarrow PC^n$ denote the Hopf-fibration; then using the diagonal map $\triangle$ we define a principal toroidal bundle over PC^n by the following diagram:*

$$\begin{array}{ccc} H^{2n+s} & \overset{\widehat{\triangle}}{\rightarrow} & S^{2n+1} \times \cdots \times S^{2n+1} \\ \downarrow \pi & & \downarrow \acute{\pi} \times \cdots \times \acute{\pi} \\ PC^n & \overset{\triangle}{\rightarrow} & PC^n \times \cdots \times PC^n \end{array}$$

that is, $H^{2n+s} = \{(p_1, \ldots, p_s) \in S^{2n+1} \times \cdots \times S^{2n+1} | \acute{\pi}(p_1) = \cdots = \acute{\pi}(p_s)\}$. Using the Theorem 3.1 in [1] we can prove that H^{2n+s} is an S-manifold. Now, it is not very difficult to see that the map π is a (f, J)-holomorphic map between the S-manifold H^{2n+s} and the Kähler manifold CP^n. Therefore by the Theorem 3.1 we obtain that π is harmonic. Also for $s = 1$, we reobtain that the Hopf fibration $\acute{\pi} : S^{2n+1} \rightarrow PC^n$ is a harmonic map between a Sasakian and a Kähler manifold.

Now a natural question is to see if the same thing is true for maps defined from a Kähler manifold to an S-manifold. A smooth map $F : N \rightarrow M$ from a Kähler manifold $N(J, h)$ to an S-manifold $M(f, \xi_\alpha, \eta_\alpha, g)$ is called to be a (J, f)-holomorphic if $dF \circ J = f \circ dF$. For such a map we can prove the following theorem.

Theorem 3.3. *Let N(J,h) be a Kähler manifold, $M(f, \xi_\alpha, \eta_\alpha, g)$ be an S-manifold and $F : N \longrightarrow M$ be a (J, f)-holomorphic map. Then F is a harmonic map if and only if F is a constant map.*

Proof. For such a map we have a similar formula as (6),

$$f(\tau(F)) = F_*(divJ) - tr_h\beta,$$

where $\beta(X, Y) = (\widetilde{\nabla}_X f)(F_*Y)$. Suppose that M is a Kähler manifold of real dimension $2n$. Then we have:

$$divJ = \sum_{i=1}^{2n} (\nabla_{e_i} J)e_i = 0,$$

where $\{e_i\}_{i=1\ldots 2n}$ is an orthonormal local basis on TN. Now, using the formula (3) we obtain

$$Tr_h\beta = \sum_{i=1}^{2n} (\widetilde{\nabla}_{e_i} f)(F_*e_i) = \sum_{i=1}^{2n} \sum_{\alpha} \left\{g(F_*e_i, F_*e_i)\xi_\alpha + \eta_\alpha^2(F_*e_i))\xi_\alpha - \eta_\alpha(F_*e_i)F_*e_i\right\}.$$

As F is a (J, f)-holomorphic map, we have $\eta_\alpha(F_*e_i) = -\eta_\alpha(F_*J^2e_i)$=$-\eta_\alpha(fF_*Je_i)$=0 and thus

$$f(\tau(F)) = -\sum_{i=1}^{2n} \sum_{\alpha} g(F_*e_i, F_*e_i)\xi_\alpha. \tag{7}$$

If we apply again f in the last formula we get $f^2(\tau(F)) = 0$ and thus $f^3(\tau(F)) = 0$. But f satisfies the condition $f^3 + f = 0$ and thus we get $f(\tau(F)) = 0$. Finally from formula 7 we obtain that F is harmonic if and only if F is a constant map. □

Harmonic morphism are maps which pull back germs of real valued harmonic functions on the target manifold to germs of harmonic functions on the domain, that is, a smooth map $F : (M, g) \to (N, h)$ is a harmonic morphism if for any harmonic function $f_1 : U \to \mathbb{R}$, defined on an open subset U of N such that $\pi^{-1}(U)$ is non-empty, the composition $f_1 \circ F : \pi^{-1}(U) \to \mathbb{R}$ is a harmonic function. The following characterization of harmonic morphisms is due to Fuglede and Ishihara: A smooth map F is a harmonic morphism if and only if F is a horizontally conformal harmonic map (see [4] and [5]).

Definition 3.4. *A smooth map $F : (M^m, g) \to (N^n, h)$ is horizontally conformal if for each point $x \in M$ at which $dF_x \neq 0$, the restriction $dF_x \mid_{T^H M} \to T_{F(x)}N$ is conformal and surjective, where $T^H M$ is orthogonal compliment of $T^v_x M = ker(dF_x)$. Thus in this case there is a non negative function λ on M satisfying $F^*h = \lambda^2 g$ on $T^H M$. The function λ is called dilation of F. λ^2 is a smooth function and is equal to $\frac{|dF|^2}{n}$, $n = dimN$.*

Now we look for harmonic morphisms defined on S-manifolds.

Theorem 3.5. *Let $F : M \longrightarrow N$ be a horizontally conformal (f, J)-holomorphic map from an S-manifold $M(f, \xi, \eta, g)$ into an almost Hermitian manifold $N(J, h)$. Then F is a harmonic morphism if and only if N is a semi-Kähler manifold.*

Proof. It can be proved similarly as in ([8]) that for a horizontally conformal (f, J)-holomorphic map F from an S-manifold $M(f, \xi_\alpha, \eta_\alpha, g)$ to an almost Hermitian manifold $N(J, h)$, any two of the following conditions imply the third: (i) $divJ = 0$ (ii) $dF(divf) = 0$ (iii) F is harmonic and so is harmonic morphism. Now let $\{e_1, \ldots, e_m, fe_1, \ldots, fe_m, \xi_1, \ldots, \xi_s\}$ be a local f-adapted frame on TM. We have seen that

$$divf = \sum_{i=1}^{2m+s} (\nabla_{e_i} f)e_i = \sum_\alpha 2m\xi_\alpha.$$

As F is a (f, J)-holomorphic map we have $dF(\xi_\alpha) = 0$. Because F is a horizontally conformal (f, J)-holomorphic map, it follows that F is a harmonic morphism if and only if $divJ = 0$, i.e. N is semi-Kähler. □

4. Spectral Geometry on S-manifolds

Let $F : (M, g) \longrightarrow (N, h)$ be a harmonic map defined on a compact manifold M. The corresponding Jacobi operator is an elliptic self-adjoint operator which has discrete spectrum of eigenvalues with finite multiplicities, denoted by

$$Spec(J) = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \ldots \uparrow \infty\}.$$

Then the trace $Z(t) = \sum_{j=1}^{\infty} exp(-t\lambda_j)$ of the heat kernel for the operator J has the asymptotic expansion

$$Z(t) \sim (4\pi t)^{-m/2}\{a_0(J) + a_1(J)t + a_2(J)t^2 + \ldots\} \quad as \quad t \to \infty. \tag{8}$$

Using the results of Gilkey (see [10]) Urakawa obtained the expression for the first three coefficients (see [17]):

Theorem 4.1. *For a harmonic map $F : (M^m, g) \longrightarrow (N^n, h)$, the first three coefficient of the expansions are given by*

$$a_0(J_F) = nVol(M, g), \tag{9}$$

$$a_1(J_F) = \frac{n}{6}\int_M \tau_g \upsilon_g + \int_M Tr(R_F)\upsilon_g, \tag{10}$$

$$a_2(J_F) = \frac{n}{360}\int_M (5\tau_g^2 - 2\|\rho_g\|^2 + 2\|R_g\|^2)d\upsilon_g + \frac{1}{360}\int_M [-30\|R^{\tilde{\nabla}}\|^2 + 60\tau_g Tr(R_F) + 180 Tr(R_F^2)]d\upsilon_g, \tag{11}$$

where $R^{\tilde{\nabla}}$ is the curvature tensor of the connection $\tilde{\nabla}$ on the induced bundle, which is defined by $R^{\tilde{\nabla}} = F^* R_h$ (R_h is the Riemann curvature tensor of (N,h)), R_g, ρ_g, τ_g are the curvature tensor, Ricci tensor, scalar curvature on M respectively, and R_F is the endomorphism of the induced bundle defined by $R_F(V) = Tr_g F^* R(V, -)$.

The spectral geometry for the Jacobi operators of harmonic maps into a Sasakian or cosymplectic space form was studied by Kang and Kim (see [13]). We now consider the spectral geometry for the case when the target manifold is an S-space form. We recall that if N is an S-manifold whose invariant f-sectional curvature is a constant k, then its curvature tensor has the form (4), (see [12]). Let N(k) be a (2n+s)-dimensional S-manifold with constant f-sectional curvature k. Let $\psi : (M^m, g) \longrightarrow N(k)$ be a harmonic map from a compact Riemannian manifold into an S-manifold. Considering the following notations $\lambda = \frac{k+3s}{4}$, $\mu = \frac{k-s}{4}$ we get

$$\begin{aligned} Tr(R_\psi) &= \sum_{i=1}^{m}\sum_{a=1}^{2n+s} h(R(v_a, \psi_* e_i)\psi_* e_i, v_a) \\ &= 2[s + \lambda(2n-1) + 3\mu]e(\psi) + [2n + s - 2 + \lambda(2 - s - 2n) - 3\mu]\|\psi^*\eta_\alpha\|^2, \qquad (12) \\ Tr(R_\psi^2) &= \sum_{i,j=1}^{m}\sum_{a=1}^{2n+s} h(R(v_a, \psi_* e_i)\psi_* e_i, R(v_a, \psi_* e_j)\psi_* e_j) \\ &= [6\lambda\mu + (2n-2)\lambda^2 + s]4e(\psi)^2 + (\lambda^2 + 9\mu^2)\|\psi^* h|^2 - 6\lambda\mu\|\psi^* f\|^2 \\ &\quad + [-18\mu^2 - 2\lambda^2 + 2s]\sum_{i,j=1}^{m}\sum_{\alpha=1}^{s} h(\psi_* e_i, \psi_* e_j)\eta_\alpha(\psi_* e_i)\eta_\alpha(\psi_* e_j) \\ &\quad + [6\lambda\mu + 9\mu^2 + (2n-1)\lambda^2 + (2n-s) - 6\mu - 2(2n-1)\lambda]\|\psi^*\eta_\alpha\|^4 \\ &\quad + [-4s + 12\mu + 4(2n-1)\lambda - 24\lambda\mu - 4(2n-2)\lambda^2]e(\psi)\|\psi^*\eta_\alpha\|^2 \qquad (13) \\ \|R^{\tilde\nabla}\|^2 &= \sum_{i,j=1}^{m}\sum_{a=1}^{2n+1} h(R(\psi_* e_i, \psi_* e_j)v_a, R(\psi_* e_i, \psi_* e_j)v_a) \\ &= 8(\lambda^2 + \mu^2)e(\psi)^2 + [8(s-2)\lambda^2 - 8\mu^2 + 8s]e(\psi)\|\psi^*\eta_\alpha\|^2 - (2\lambda^2 + 2\mu^2)\|\psi^* h\|^2 + [-4(s-2)\lambda^2 + 4\mu^2 - 4s]. \\ &\quad \cdot\sum_{i,j=1}^{m}\sum_{\alpha=1}^{s} h(\psi_* e_i, \psi_* e_j)\eta_\alpha(\psi_* e_i)\eta_\alpha(\psi_* e_j) + [8(n+1)\mu^2 + 12\lambda\mu]\|\psi^* f\|, \qquad (14) \end{aligned}$$

where $\|\psi^*\eta_\alpha\|^2 = \sum_{i=1}^{m}\sum_{\alpha}\eta_\alpha(\psi_* e_i)\eta_\alpha(\psi_* e_i)$, $\|\psi^* f\|^2 = \sum_{i=1}^{m} h(\psi_* e_i, f\psi_* e_j)$, $\|\psi^* h\|^2 = \sum_{i=1}^{m} h(\psi_* e_i, \psi_* e_j)$, and $\{e_i : i = 1 \ldots m\}$ and $\{v_a : a = 1 \ldots 2n+s\}$ are local orthonormal basis on M and N respectively. Thus substituting (10) $\sim$ (12) into (7) $\sim$ (9), we get

Theorem 4.2. *Let $\psi : (M, g) \longrightarrow N(k)$ be a harmonic map from a m-dimensional compact Riemanniann manifold (M, g) to a $(2n+s)$-dimensional S-space form $N(k)$. Then the coefficients $a_0(J_\psi)$, $a_1(J_\psi)$ and $a_2(J_\psi)$ of the asymptotic expansion for the Jacobi operator J_ψ are respectively given by*

$$\begin{aligned} a_0(J_\psi) &= (2n+s)Vol(M, g), \\ a_1(J_\psi) &= \frac{2n+s}{6}\int_M \tau_g \upsilon_g + 2(s + \lambda(2n-1) + 3\mu)E(\psi) + (2n + s - 2 + \lambda(2 - s - 2n) - 3\mu)\int_M \|\psi^*\eta_\alpha\|^2\upsilon_g, \\ a_2(J_\psi) &= \frac{2n+s}{360}\int_M (5\tau_g^2 - 2\|\rho_g\|^2 + \|R_g\|^2)d\upsilon_g + \frac{2}{3}\int_M (\lambda^2 + 7\mu^2)\|\psi^* h\|^2\upsilon_g \\ &\quad - \frac{1}{12}\int_M ((-4s + 20)\lambda^2 - 112\mu^2 + 16s)\sum_{i,j=1}^{m} h(\psi_* e_i, \psi_* e_j)\eta_\alpha(\psi_* e_i)\eta_\alpha(\psi_* e_j)\upsilon_g \\ &\quad + \frac{2}{3}\int_M [(6n-7)\lambda^2 - \mu^2 + 18\lambda\mu + 3s]e(\psi)^2\upsilon_g - \frac{2}{3}\int_M [(n+1)\mu^2 + 6\lambda\mu]\|\psi^* F\|^2\upsilon_g \\ &\quad + \frac{1}{360}\int_M [6\lambda\mu + 9\mu^2 + (2n-1)\lambda^2 + (2n-s) - 6\mu - 2(2n-1)\lambda]\|\psi^*\eta_\alpha\|^4\upsilon_g \\ &\quad + \frac{1}{3}\int_M (s + \lambda(2n-1) + 3\mu)\tau_g e(\psi)\upsilon_g + \frac{1}{6}\int_M [2n + s - 2 + \lambda(2 - s - 2n) - 3\mu]\|\psi^*\eta_\alpha\|^2\tau_g\upsilon_g \\ &\quad - \frac{2}{3}\int_M [(s - 2 + 6(n-1))\lambda^2 - \mu^2 + 4s - 9\mu - 3(2n-1)\lambda + 18\lambda\mu]\|\psi^*\eta_\alpha\|^2 e(\psi)\upsilon_g. \end{aligned}$$

A first application of the above theorem is the following.

Corollary 4.3. *Let ψ, $\widetilde{\psi}$ be two harmonic maps from a compact Riemannian manifold M into a S-space form $N(k)$. If $Spec(J_\psi) = Spec(J_{\widetilde{\psi}})$ and the structure vector fields $\xi_\alpha : \alpha = 1, \dots, s$ are normal to $\psi(M)$ and $\widetilde{\psi}(M)$, then $E(\psi) = E(\widetilde{\psi})$.*

Proof. Since the vector field ξ_α for each α is normal to $\psi(M)$ and $\widetilde{\psi}(M)$, then

$$\|\psi^*\eta_\alpha\|^2 = \sum_{i=1}^{m}\sum_{\alpha}\eta_\alpha(\psi_* e_i)\eta_\alpha(\psi_* e_i) = \sum_{i=1}^{m}\sum_{\alpha} g(\psi_* e_i, \xi_\alpha)g(\psi_* e_i, \xi_\alpha) = 0$$

and similar for $\|\widetilde{\psi}^*\eta\|$. On the other hand, as $Spec(J_\psi) = Spec(J_{\widetilde{\psi}})$ we have $a_1(\psi) = a_1(\widetilde{\psi})$ and using the expression of the first coefficient in Theorem 4.2, we get $E(\psi) = E(\widetilde{\psi})$. □

Let M be an m-dimensional submanifold immersed in an S-space form N. M is said to be an invariant submanifold if $\xi_\alpha \in T(M)$ for any α and $fX \in T(M)$ for any $X \in T(M)$. It is known that an invariant immersion $\psi : (M, g) \to (N, h)$ of a Riemannian manifold (M, g) into an S-manifold is minimal (see [12]). On the other hand any isometric immersion is harmonic if and only if is minimal. Thus any invariant immersion of a compact Riemannian manifold into an S-space form is an example of a harmonic map for which the target manifold is an S-space form. Another application of the Theorem 4.2 is the following.

Proposition 4.4. *Let ψ, $\widetilde{\psi}$ be invariant immersions of compact Riemannian manifolds (M, g) and $(\widetilde{M}, \widetilde{g})$ into an S-space form N. Assume that $Spec(J_\psi) = Spec(J_{\widetilde{\psi}})$. If ψ is a totally geodesic immersion, then so is $\widetilde{\psi}$.*

Proof. As $Spec(J_\psi) = Spec(J_{\widetilde{\psi}})$, using the formulas for the coefficients we have: $\frac{n}{6}\int_M \tau_g dv_g + \int_M Tr(R_\psi)dv_g = \frac{n}{6}\int_{\widetilde{M}} \tau_{\widetilde{g}} dv_{\widetilde{g}} + \int_{\widetilde{M}} Tr(R_{\widetilde{\psi}})dv_{\widetilde{g}}$. Now, using the Corollary 4.3 we have $\int_M \tau_g dv_g = \int_{\widetilde{M}} \tau_{\widetilde{g}} dv_{\widetilde{g}}$, and we get

$$\int_M Tr(R_\psi)dv_g = \int_{\widetilde{M}} Tr(R_{\widetilde{\psi}})dv_{\widetilde{g}}, \tag{15}$$

where

$$Tr(R_\psi) = \sum_{i=1}^{m}\sum_{a=1}^{2n+s} h(R_h(v_a, \psi_* e_i)\psi_* e_i, v_a); \quad Tr(R_{\widetilde{\psi}}) = \sum_{i=1}^{\widetilde{m}}\sum_{a=1}^{2n+s} h(R_h(v_a, \widetilde{\psi}_* \widetilde{e}_i)\widetilde{\psi}_* \widetilde{e}_i, v_a).$$

Now, the proposition follows by using the Gauss equation. □

Remark 4.5.

1. *For the case s=1, we get the results when the target manifold is Sasakian, obtained by Tae Ho Kang and Hyun Suk Kim (see [13]).*

2. *We know that an f-invariant submanifold M imbedded in an S-manifold N such that the vectors $\xi_\alpha(\alpha = l, \dots, s)$ are never tangent to $i(M)$ is a Kähler minimal submanifold in N (see [12]). For the case when $s = 0$, the manifold N is Kähler and we reobtain the results of Urakawa (see [17]) when the target manifold is a complex space form.*

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