International Journal of Mathematics And its Applications
Volume 4, Issue 2–B (2016), 35–40.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Nearness and Convergence Properties of Mild Solutions of Impulsive Nonlocal Cauchy Problem

Research Article

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Abstract: In the present paper, we study nearness and convergence properties of mild solutions of nonlocal impulsive cauchy problem by using generalised form of Grownwall type inequality.

MSC: 34A12, 47GXX, 47H10, 47B38.

Keywords: Cauchy Problem, impulsive, nonlocal condition.

1. Introduction

Impulsive differential equations are excellent source of many real world processes which subject to certain perturbations and the duration of those perturbations is negligible in comparison with the duration of the whole process. These type of equations occurs in many areas such as physics, economics, pharmaco-kinematics, medicine, biology, space-craft, industrial robotics etc. They have been an object of investigation for many researchers in the recent years. For more details and information one can refer the monographs [1, 2] and papers [3–7] and the references cited therein.

Also, differential equations with nonlocal condition are more suitable for describing many natural phenomena and processes which cannot be modelled with usual classical initial condition. That is the reason several researcher devoted their study to the problems with nonlocal condition. For more information, one can see the papers [4, 6, 8–12] and the references given therein.

Let X be a Banach space with the norm $\|\cdot\|$. Let $C = \mathcal{C}([-r, 0], X), 0 < r < \infty$, be the Banach space of all continuous functions $\psi : [-r, 0] \to X$ endowed with supremum norm $\|\psi\|_C = \sup\{\|\psi(t)\| : -r \leq t \leq 0\}$ and B denote the set $\{x : [-r, T] \to X | x(t)$ is piecewise continuous at $t \neq \tau_k$,left continuous at $t = \tau_k$, and the right limit $x(\tau_k + 0)$ exists for $k = 1, 2, ..., m\}$. Clearly, B is a Banach space with the supremum norm $\|x\|_B = \sup\{\|x(t)\| : t \in [-r, T] \setminus \{\tau_1, \tau_2, ..., \tau_m\}\}$. For any $x \in B$ and $t \in [0, T] \setminus \{\tau_1, \tau_2, ..., \tau_m\}$, we denote x_t the element of C given by $x_t(\theta) = x(t + \theta), \quad$ for $\quad \theta \in [-r, 0]$ and ϕ is a given element of C.

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In the present paper, we consider impulsive Cauchy problem with nonlocal condition of the type

$$x'(t) = Ax(t) + f(t, x_t), t \in [0, T], t \neq \tau_k, k = 1, 2, ..., m \tag{1}$$

$$x(t) + (g(x_{t_1}, ..., x_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0, \tag{2}$$

$$\Delta x(\tau_k) = I_k x(\tau_k), \quad k = 1, 2, ..., m, \tag{3}$$

where, A is the infinitesimal generator of strongly continuous semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$ and $I_k(k = 1, 2, ..., m)$ are the linear operators acting in a Banach space X. The function $f : [0, T] \times C \to X$ is continuous and the function ϕ is given element of C. $g : C^p \to C$, where $0 < t_1 < t_2 < ... < t_p \leq T$, $p \in$ N is a given function. For example, $\big(g(x_{t_1}, ..., x_{t_p})\big)(\theta) = \sum_{k=1}^{p} c_k x(t_k + \theta), x \in PC([-r, T], X)$, $c_k's$ are constants. $\Delta x(\tau_k) = x(\tau_k + \theta) - x(\tau_k - \theta)$ and impulsive moments τ_k are such that $0 < \tau_1 < \tau_2 < ... < \tau_k < T$, $k \in$ N.As mentioned in theory of impulsive differential equations, we assume that at points of discontinuity τ_i of solution $t \to x(t), x(\tau_i) \equiv x(\tau_i - 0)$. In general, the derivative $x'(\tau_i)$ does not exist but according to equation (1), there exists a limit $x'(\tau_i \pm 0)$ and by convention, we assume $x'(\tau_i) \equiv x'(\tau_i - 0)$.
In [4], Shaochun Ji and Shu Wen investigated existence of mild solutions of the problem:

$$u'(t) = Au(t) + f(t, u(t)), t \in [0, b], t \neq t_i, i = 1, 2, ..., p$$

$$\Delta u(t_i) = u(t_i^+) - u(t_i)^- = I_i(u(t_i)), i = 1, 2, ..., p$$

$$u(0) = g(u) + u_0$$

using Hausdroff measure of noncompactness and Darbo Sadovskii fixed point theorem. Also,in [10], R. S. Jain and M. B. Dhakne studied existence, uniqueness and continuous dependance on initial data of mild solutions of the problem:

$$x'(t) = Ax(t) + f(t, x_t), t \in [0, T]$$

$$x(t) + (g(x_{t_1}, ..., x_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0$$

using modified version of Banach contraction theorem. Our aim is to further extend the study for nearness and convergence properties of mild solutions of above models or their generalised forms studied in [4],[7],[10] for the case of impulsive effect.

The paper is organised as follows: In Section 2, we give preliminaries and hypothesises. In section 3, we discuss main results and their proofs.

2. Preliminaries and Hypotheses

Definition 2.1. *A function $x \in B$ satisfying the equations:*

$$\begin{aligned} x(t) &= T(t)\phi(0) - T(t)(g(x_{t_1}, ..., x_{t_p}))(0) \\ &+ \int_0^t T(t-s) f(s, x_s) ds + \sum_{0<\tau_k<t} T(t-\tau_k) I_k x(\tau_k), \quad t \in (0, T], \\ x(t) &+ (g(x_{t_1}, ..., x_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0, \end{aligned}$$

is said to be the mild solution of the initial value problem (1)-(3)

Lemma 2.2 ([2]). *Let a nonnegative piecewise continuous function $u(t)$ satisfy for $t \geq t_0$, the inequality*

$$u(t) \leq C + \int_{t_0}^{t} v(s)u(s)ds + \sum_{t_0<\tau_i<t} \beta_i u(\tau_i)$$

where $C \geq 0$, $\beta_i \geq 0$, $v(t) > 0$, τ_i are the first kind discontinuity points of the function $u(t)$. Then the following estimate holds for the function $u(t)$,

$$u(t) \leq C \prod_{t_0<\tau_i<t} (1+\beta_i) exp(\int_{t_o}^{t} v(s)ds).$$

Let us introduce the following hypothesis which are assumed thereafter for our convenience.

(H_1) Let $f : [0,T] \times C \to X$ such that for every $w \in B$ and $t \in [0,T]$, $f(.,w_t) \in B$ and there exists a constant $L > 0$ such that

$$\|f(t,\psi) - f(t,\phi)\| \leq L(\|\psi - \phi\|_C), \qquad \phi, \psi \in C.$$

(H_2) Let $g : C^p \to C$ such that there exists a constant $G \geq 0$ satisfying

$$\|(g(x_{t_1}, x_{t_2}, ..., x_{t_p}))(t) - (g(y_{t_1}, y_{t_2}, ..., y_{t_p}))(t)\| \leq G\|x - y\|_B, \quad t \in [-r, 0].$$

(H_3) Let $I_k : X \to X$ are functions such that there exists L_k satisfying

$$\|I_k(\psi) - I_k(\phi)\| \leq L_k\|\psi - \phi\| \quad \psi, \phi \in X, \quad k = 1, 2, ..., m.$$

Also in this paper we assume that, there exist a positive constant $K_0 > 1$ such that $\|T(t)\| \leq K_0$, for every $t \in [0,T]$.

3. Nearness and Convergence of Solutions

Consider the nonlocal cauchy problem (1)-(3), along with the nonlocal initial value problem

$$y'(t) = Ay(t) + \bar{f}(t, y_t), \quad t \in [0,T], t \neq \tau_k, k = 1, 2, \cdots m \tag{4}$$

$$y(t) + (\bar{g}(y_{t_1}, ..., y_{t_p}))(t) = \bar{\phi}(t), \quad -r \leq t \leq 0, \tag{5}$$

$$\Delta y(\tau_k) = \bar{I}_k y(\tau_k), \quad k = 1, 2, ..., m, \tag{6}$$

where, $\bar{f} : [0,T] \times C \to X$, $\bar{g} : C^p \to C$ and $\bar{\phi} \in C$.

The next theorem deals with nearness of solutions of initial value problem (1)-(3) and initial value problem (4)-(6).

Theorem 3.1. *Suppose that the functions f, g in (1)-(3)satisfy hypotheses (H_1)-(H_3) and there exist nonnegative constants $\epsilon_1, \epsilon_2, \delta, \delta_k$ such that*

$$\|f(t,\phi) - \bar{f}(t,\phi)\| \leq \epsilon_1, \tag{7}$$

$$\|(g(x_{t_1}, ..., x_{t_p}))(t) - (\bar{g}(x_{t_1}, ..., x_{t_p}))(t)\| \leq \epsilon_2, \tag{8}$$

$$\|\phi(t) - \bar{\phi}(t)\| \leq \delta. \tag{9}$$

$$\|I_k(\phi) - \bar{I}_k(\phi)\| \leq \delta_k. \tag{10}$$

Let $x(t)$ and $y(t)$ be respectively solutions of initial value problem (1)-(3) and (4)-(6) on [-r, T]. Then the following inequality holds:

$$\|x-y\|_B \leq \frac{K_0 \prod\limits_{0<\tau_k<t}(1+K_0L_k)exp\big(K_0LT\big)}{1-K_0G \prod\limits_{0<\tau_k<t}(1+K_0L_k)exp\big(K_0LT\big)}[\delta+\epsilon_2+\epsilon_1T+\delta_k]$$

Proof. Using the facts that $x(t)$ and $y(t)$ be respectively solutions of initial value problem (1)-(2) and (4)-(6) and hypotheses (H_1)-(H_3), we obtain, for $t \in [r, 0]$,

$$\begin{aligned}
\|x(t)-y(t)\| &= \|\phi(t)-(g(x_{t_1},...,x_{t_p}))(t)-\bar{\phi}(t)+(\bar{g}(y_{t_1},...,y_{t_p}))(t)\| \\
&\leq \|\phi(t)-\bar{\phi}(t)\|+\|(g(x_{t_1},...,x_{t_p}))(t)-(\bar{g}(y_{t_1},...,y_{t_p}))(t)\| \\
&\leq \|\phi(t)-\bar{\phi}(t)\|+\|(g(x_{t_1},...,x_{t_p}))(t)-(g(y_{t_1},...,y_{t_p}))(t)\| \\
&+\|(g(y_{t_1},...,y_{t_p}))(t)-(\bar{g}(y_{t_1},...,y_{t_p}))(t)\| \\
&\leq \delta+G\|x-y\|_B+\epsilon_2.
\end{aligned} \tag{11}$$

For $t \in (0, T]$,

$$\begin{aligned}
\|x(t)-y(t)\| &\leq \|T(t)\|[\|\phi(0)-\bar{\phi}(0)\| \\
&+\|(g(x_{t_1},...,x_{t_p}))(0)-(\bar{g}(y_{t_1},...,y_{t_p}))(0)\|] \\
&+\int_0^t \|T(t-s)\|\Big\|f(s,x_s)-\bar{f}(s,y_s)\Big\|ds \\
&+\sum_{0<\tau_k<t}\|T(t-\tau_k)\|\|I_kx(\tau_k)-\bar{I}_ky(\tau_k)\| \\
&\leq K_0[\delta+G\|x-y\|_B+\epsilon_2]+K_0L\int_0^t \Big[\|x_s-y_s\|\Big]ds+K_0\epsilon_1T \\
&+K_0\sum_{0<\tau_k<t}\|x(\tau_k)-y(\tau_k)\|+\delta_kK_0 \\
&\leq K_0[\delta+G\|x-y\|_B+\epsilon_2+\epsilon_1T+\delta_k] \\
&+K_0L\int_0^t \Big[\|x_s-y_s\|\Big]ds+\sum_{0<\tau_k<t}K_0L_k\|x(\tau_k)-y(\tau_k)\|,
\end{aligned} \tag{12}$$

Define the function $z : [-r, T] \to \mathbb{R}$ by $z(t) = \sup\{\|x_(s)-y(s)\| : -r \leq s \leq t\}, t \in [0, T]$. Let $t^* \in [-r, t]$ be such that $z(t) = \|x(t^*)-y(t^*)\|$. If $t^* \in [0, t]$, then from the above inequality, we have,

$$\begin{aligned}
z(t) &= \|x(t^*)-y(t^*)\| \\
&\leq K_0[\delta+G\|x-y\|_B+\epsilon_2+\epsilon_1T+\delta_k] \\
&+\int_0^t K_0L\Big[z(s)\Big]ds+\sum_{0<\tau_k<t}K_0L_kz(\tau_k)
\end{aligned} \tag{13}$$

If $t^* \in [-r, 0]$, then $z(t) \leq \delta+G\|x-y\|_B+\epsilon_2+\delta_k$ and since $K_0 \geq 1$, the inequality (13) holds good for $t \in [-r, T]$. Now applying the lemma 2.2 to the inequality (2.2) , we get,

$$\begin{aligned}
z(t) \leq\ & K_0[\delta+G\|x-y\|_B+\epsilon_2+\epsilon_1T+\delta_k]\times \\
&\prod_{0<\tau_k<t}(1+K_0L_k)exp(\int_0^t K_0Lds)
\end{aligned}$$

Consequently,

$$\|x-y\|_B \leq K_0[\delta + G\|x-y\|_B + \epsilon_2 + \epsilon_1 T + \delta_k]\times \prod_{0<\tau_k<t} ((1+K_0L_k)exp\big(K_0LT\big)$$

which implies,

$$\|x-y\|_B \leq \frac{K_0 \prod\limits_{0<\tau_k<t} (1+K_0L_k)exp\big(K_0LT\big)}{1-K_0G \prod\limits_{0<\tau_k<t} ((1+K_0L_k)exp\big(K_0LT\big)}[\delta + \epsilon_2 + \epsilon_1 T + \delta_k]$$

This completes the proof. □

Remark 3.2. *The result given in the above theorem, relates the solutions of initial value problem (1)-(3) and (4)-(6) in the sense that if f and $\bar{f}$, $\phi(t)$ is close to $\bar{\phi}(t)$ and g is close to $\bar{g}$, then not only the solutions of initial value problem (1)-(3) and (4)-(6) are close to each other, but also depend continuously on the functions involved therein.*

Consider the initial value problem (1)-(3) with the initial value problem

$$y'(t) = Ay(t) + f_n\Big(t, y_t\Big), \quad t \in [0,T], \tag{14}$$

$$y(t) + (g_n(y_{t_1}, ..., y_{t_p}))(t) = \phi_n(t), \quad -r \leq t \leq 0, n = 1,2,... \tag{15}$$

$$\Delta y(\tau_k) = I_{kn} y(\tau_k), \quad k = 1,2,...,m, \tag{16}$$

$f_n : [0,T]\times C \to X$, $g_n : C^p \to C$ and $\phi_n(t)$ is a sequence in X.

An immediate consequence of the above theorem , we have the following corollary:

Corollary 3.3. *Suppose that the functions f, g, in (1)-(3) satisfy the hypotheses (H_1)-(H_3) and there exist nonnegative constants $\epsilon_n, \epsilon_n^*, \delta_n, \delta_{kn}$ such that*

$$\|f(t,\phi) - f_n(t,\phi)\| \leq \epsilon_n, \tag{17}$$

$$\|(g(x_{t_1}, ..., x_{t_p}))(t) - (g_n(x_{t_1}, ..., x_{t_p}))(t)\| \leq {\epsilon_n}^*, \tag{18}$$

$$\|\phi(t) - \phi_n(t)\| \leq \delta_n. \tag{19}$$

$$\|I_k\phi(\tau_k) - I_{kn}\phi(\tau_k)\| \leq \delta_{kn}. \tag{20}$$

with $\epsilon_n \to 0, {\epsilon_n}^* \to 0, \delta_n \to 0, \delta_{kn} \to 0$ as $n \to \infty$. If $x(t)$ and $y_n(t), n = 1,2,...$ be respectively solutions of initial value problems (1)-(3) and (14)-(16) on [-r,T]. Then as $n \to \infty$, $y_n(t) \to x(t)$ on [-r, T].

Remark 3.4. *The result obtained in this corollary provides sufficient conditions that ensures solutions of initial value problem (14)-(16) will converge to solutions of initial value problem (1)-(3).*

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