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On Generalized H-Birecurrent Finsler Space

Research Article

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Abstract: In this paper, we introduced a Finsler space for which the h-curvature tensor H^i_{jkh} (curvature tensor of Berwald) satisfies the condition

$$\mathcal{B}_m\mathcal{B}_n\ H^i_{jkh} = a_{mn}\ H^i_{jkh} + b_{mn}(\delta^i_j g_{kh} - \delta^i_k g_{jh}) - 2\ y^r \mu_n \mathcal{B}_r(\delta^i_j C_{khm} - \delta^i_k C_{jhm}),\ H^i_{jkh} \neq 0$$

C_{jhm} is (h) hv-torsion tensor, where $\mathcal{B}_m\mathcal{B}_n$ is Berwald's covariant differential operator of the second order with respect to x^n and x^m, successively, $\mathcal{B}_r$ is Berwald's covariant differential operator of the first order with respect to x^r, a_{mn} and b_{mn} are non-zero covariant tensors field of second order and μ_n is non-zero covariant vector field. We called this space a generalized H-birecurrent space. The aim of this paper is to develop some properties of a generalized H-birecurrent space by obtaining Berwald's covariant derivative of the second order for the (h)v-torsion tensor H^i_{jk} and the deviation tensor H^i_j. The non-vanishing of Ricci tensor H_{kh}, the curvature vector H_k and the curvature scalar H are investigated. Different results regarding the covariant tensors field a_{mn} and b_{mn} have been established. Some conditions have been pointed out which reduce a generalized H-birecurrent space F_n $(n > 2)$ into Landsberg space. We obtained an identity for a generalized H-birecurrent space. The conditions which reduce a generalized H-birecurrent space F_n $(n > 2)$ in to a space of curvature scalar are given.

Keywords: Finsler space, Generalized H-birecurrent Finsler space, Ricci tensor, Landsberg space, Finsler space of scalar curvature.

1. Introduction

A Finsler space of recurrent curvature was introduced and studied by P.N. Pandey [5, 6], P.N. Pandey and V.J. Dwivedi [7], R. Verma [9], S. Dikshit [10], F.Y.A. Qasem [1], N.S.H. Hussien [4], N.L. Youssef and A. Soleiman [3] and others. P.N. Pandey, S.S. Saxena and A. Goswami [6] introduced and studied a generalized H-recurrent Finsler space. Let F_n be an n-dimensional Finsler spaces equipped with the metric function F satisfies the requisite conditions [2].

Let the components of the corresponding metric tensor and Berwald's connection coefficients be denoted by g_{ij} and G^i_{jk} [1] respectively. These are positively homogeneous of degree zero in the directional arguments. Due to their homogeneity in the directional arguments, we have [2]

$$a)\ \ C_{ijk}y^i = C_{kij}y^i = C_{jki}y^i = 0 \ \text{ and }\ b)\ \ G^i_{jkh}y^j = G^i_{hjk}y^j = G^i_{khj}y^j = 0, \tag{1}$$

where $C_{ijk} = \dot{\partial}_k g_{ij}$ and $G^i_{jkh} = \dot{\partial}_h\ G^i_{jk}$ are the components of tensors, they are symmetric in the their lower indices and $\dot{\partial}_h = \frac{\partial}{\partial y^h}$.

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[1] *The indices i, j, k,... assume positive integral values from 1 to n.*

[2] *Unless stated otherwise. Henceforth all geometric object are assumed to be functions of line-element.*

The relations between the metric function F, the components of the metric tensor and the vector y_i are given by

$$a)\ \ y_i = g_{ij}y^j,\ \ b)\ \ g_{ij} = \dot{\partial}_i y_j = \dot{\partial}_j y_i,\ \text{ and }\ c)\ \ y_i y^i = F^2. \tag{2}$$

The unit vector l^i in the direction of the directional argument is given by

$$a)\ \ l^i := \frac{y^i}{F} \tag{3}$$

and the associate vector of l_i is defined by

$$b)\ \ l_i := g_{ij}l^j = \dot{\partial}_i F = \frac{y_i}{F}.$$

Berwald covariant derivative of an arbitrary tensor field T^i_j with respect to x^h is given by

$$\mathcal{B}_h T^i_j := \partial_h T^i_j - \left(\dot{\partial}_r T^i_j\right) G^r_h + T^r_j G^i_{rh} - T^i_r G^r_{jh}. \tag{4}$$

Berwald covariant derivative of the vectors y^i and y_i with respect to x^k vanish identically i.e.

$$a)\ \ \mathcal{B}_k y^i = 0\ \text{ and }\ b)\ \ \mathcal{B}_k\ y_i = 0. \tag{5}$$

Berwald's covariant derivative of the metric tensor g_{ij} does not vanish in general ($\mathcal{B}_k g_{ij} \neq 0$) and is given by

$$\mathcal{B}_k g_{ij} = -2C_{ijk|r}y^r = -2y^h \mathcal{B}_h C_{ijk}, \tag{6}$$

where $|r$ is h-covariant derivative with respect to x^r (Cartan's second kind covariant differentiation). The processes of Berwald's covariant differentiation with respect to x^h and the partial differentiation with respect to directional argument y^k commute according to

$$\left(\dot{\partial}_k \mathcal{B}_h - \mathcal{B}_h \dot{\partial}_k\right) T^i_j = T^r_j G^i_{khr} - T^i_r G^r_{khr} \tag{7}$$

for an arbitrary tensor field T^i_j. The second covariant derivative of an arbitrary tensor field T^i_j with respect to x^h and x^k in the sense of Berwald may written as

$$\mathcal{B}_k \mathcal{B}_h T^i_j = \partial_k \mathcal{B}_h T^i_j - \left(\partial_s \mathcal{B}_h T^i_j\right) G^s_k + \left(\mathcal{B}_h T^r_j\right) G^i_{rk} - (\mathcal{B}_h T^i_r) G^r_{ik} - (\mathcal{B}_r T^i_j) G^r_{hk}. \tag{8}$$

The commutation formula for Berwwald's curvature differentiation as follows:

$$\mathcal{B}_h \mathcal{B}_k T^i_j - \mathcal{B}_k \mathcal{B}_h T^i_j = T^r_j H^i_{hkr} - T^i_r H^r_{hkj} - (\dot{\partial}_r T^i_j) H^r_{hk} \tag{9}$$

where H^i_{jkh} [3] defined by

$$H^i_{jkh} = 2\left\{\partial_{[j}\ G^i_{k]h} + G^i_{rh[j} G^r_{k]} + G^i_{r[j}\ G^r_{k]h}\right\} \tag{10}$$

are components of Berwald curvature tensor and

$$a)\ \ H^i_{jk} = H^i_{jkh}y^h\ \text{ and }\ b)\ \ H^i_{jkh} = \dot{\partial}_h\ H^i_{jk}. \tag{11}$$

[3] *In Rund's book, H^i_{jkh} defined here, is denoted by H^i_{hkj}. This difference must be noted. The square brackets denote the skew-symmetric part of the tensor with respect to the indices enclosed therein.*

It is clear from the definition that Berwald curvature tensor H^i_{jkh} is skew-symmetric in its first two lower indices and positively homogeneous of degree zero in the directional arguments y^i. Berwald's deviation tensor H^i_j is defined as

$$H^i_j = H^i_{jk} y^k. \tag{12}$$

The contraction of the indices i and j in H^i_{jkh}, H^i_{jk} and H^i_j gives

$$a)\ \ H_{kh} = H^i_{ikh},\ \ b)\ \ H_k = H^i_{ik},\ \ c)\ \ H = \tfrac{1}{n-1} H^i_i \ \text{ and }\ d)\ \ H_k y^k = (n-1)\, H. \tag{13}$$

The necessary and sufficient condition for a Finsler space F_n $(n > 2)$ to be Finsler space of scalar curvature is given by

$$H^i_h = F^2 R(\delta^i_h - l^i l_h). \tag{14}$$

2. Generalized H-Birecurrent Finsler Space

A Finsler space whose Berwald curvature tensor H^i_{jkh} satisfies the condition

$$\mathcal{B}_n H^i_{jkh} = \lambda_n H^i_{jkh} + \mu_n(\delta^i_j g_{kh} - \delta^i_k g_{jh}),\ \ H^i_{jkh} \neq 0, \tag{A}$$

where $\mathcal{B}_n$ is Berwald's covariant differential operator, λ_l and μ_n are non-zero covariant vector fields, this space introduced by P.N. Pandey, S. Saxena and A. Goswami [8], they called it as a generalized H-recurrent Finsler space. Now, taking the covariant derivative for the condition (A) with respect to x^m in the sense of Berwald and in view of the condition (A) and by using (5b), we get

$$\begin{aligned}\mathcal{B}_m \mathcal{B}_n H^i_{jkh} &= (\mathcal{B}_m \lambda_n) H^i_{jkh} + \lambda_n \left\{\lambda_m H^i_{jkh} + \mu_m(\delta^i_j g_{kh} - \delta^i_k g_{jh})\right\}\\ &\quad + (\mathcal{B}_m \mu_n)(\delta^i_j g_{kh} - \delta^i_k g_{jh}) + \mu_n \left\{\delta^i_j\ (-2y^r \mathcal{B}_r C_{khm}) - \delta^i_k\ (-2y^r \mathcal{B}_r C_{jhm})\right\}\\ &= (\mathcal{B}_m \lambda_n + \lambda_n \lambda_m)\ H^i_{jkh} + (\lambda_n \mu_m + \mathcal{B}_m \mu_n)(\delta^i_j g_{kh} - \delta^i_k g_{jh}) - 2\mu_n y^r \mathcal{B}_r(\delta^i_j C_{khm} - \delta^i_k\ C_{jhm})\end{aligned}$$

which can be written as

$$\mathcal{B}_m \mathcal{B}_n H^i_{jkh} = a_{mn} H^i_{jkh} + b_{mn}\left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right) - 2\mu_n y^r \mathcal{B}_r(\delta^i_j C_{khm} - \delta^i_k C_{jhm}),\ \ H^i_{jkh} \neq\ 0, \tag{15}$$

where $a_{mn} = \mathcal{B}_m \lambda_n + \lambda_n \lambda_m$ and $b_{mn} = \lambda_n \mu_m + \mathcal{B}_m \mu_n$ are non-zero covariant tensors field of second order are non-null covariant tensors field of second order.

Definition 2.1. *A Finsler space F_n whose Berwald curvature tensor H^i_{jkh} satisfies the condition (15), where a_{mn} and b_{mn} are non-zero covariant tensors field of second order. We shall call such Finsler space as a generalized H-birecurrent Finsler space and briefly denoted by $GH - BRF_n$.*

Let us consider a $GH - BRF_n$ which is characterized by the condition (15). Transvecting the condition (15) by y^h , using (11a), (2a), (5a) and (1a), we get

$$\mathcal{B}_m \mathcal{B}_n H^i_{jk} = a_{mn} H^i_{jk} + b_{mn}(\delta^i_j y_k - \delta^i_k y_j). \tag{16}$$

Transvecting the condition (16) by y^k, using (12), (5a) and (2c), we get

$$\mathcal{B}_m \mathcal{B}_n H^i_j = a_{mn} H^i_j\ + b_{mn}(\delta^i_j F^2 - y_j y^i). \tag{17}$$

Thus, we conclude

Theorem 2.2. *In $GH-BRF_n$, Berwald covariant derivative of the second order for the $h(v)$-torsion tensor H^i_{kh} and the deviation tensor H^i_h given by the conditions (16) and (17), respectively.*

Contracting the indices i and j in the conditions (15), (16) and (17), using (13a), (13b), (13c) and (2c), we get

$$\mathcal{B}_m\mathcal{B}_nH_{kh} = a_{mn}H_{kh} + (n-1)\,b_{mn}g_{kh} - 2\,(n-1)\,\mu_ny^r\mathcal{B}_rC_{khm}, \tag{18}$$

$$\mathcal{B}_m\mathcal{B}_nH_k = a_{mn}H_k + (n-1)b_{mn}y_k \tag{19}$$

and

$$\mathcal{B}_m\mathcal{B}_nH = a_{mn}H + b_{mn}F^2. \tag{20}$$

The conditions (18), (19) and (20) show that Ricci tensor H_{kh}, the curvature vector H_k and the curvature scalar H can't vanish because the vanishing of any one of these would imply $a_{mn}=0$, $b_{mn}=0$ and $\mu_n=0$, that is a contradiction. Thus, we conclude

Theorem 2.3. *In $GH-BRF_n$, Ricci tensor H_{kh}, the curvature vector H_k and the curvature scalar H are non-vanishing.*

Let us consider a $GH-BRF_n$. Differentiating the condition (19) partially with respect to y^h, using ($\dot\partial_hH_k = H_{kh}$) and (2b), we get

$$\dot\partial_h\mathcal{B}_m\left(\mathcal{B}_nH_k\right) = \left(\dot\partial_ha_{mn}\right)H_k + a_{mn}H_{kh} + (n-1)(\dot\partial_hb_{mn})y_k + (n-1)b_{mn}g_{kh}. \tag{21}$$

Using the commutation formula (7) for $(\mathcal{B}_nH_k)$ in (21), we get

$$\mathcal{B}_m\dot\partial_h\left(\mathcal{B}_nH_k\right) - \left(\mathcal{B}_rH_k\right)G^r_{hmn} - \left(\mathcal{B}_nH_r\right)G^r_{hmk} = (\dot\partial_ha_{mn})H_k + a_{mn}H_{kh} + (n-1)\left(\dot\partial_hb_{mn}\right)y_k + (n-1)\,b_{mn}g_{kh}. \tag{22}$$

Again applying the commutation formula (7) for (H_k) in (22) and ($\dot\partial_hH_k = H_{kh}$), we get

$$\mathcal{B}_m\mathcal{B}_nH_{kh} - \mathcal{B}_m(H_rG^r_{hnk}) - \left(\mathcal{B}_rH_k\right)G^r_{hmn} - \left(\mathcal{B}_nH_r\right)G^r_{hmk} = \left(\dot\partial_ha_{mn}\right)H_k + a_{mn}H_{kh} + (n-1)\left(\dot\partial_hb_{mn}\right)y_k + (n-1)\,b_{mn}g_{kh}. \tag{23}$$

Using the condition (18) in (23), we get

$$-\,2\,(n-1)\,\mu_n\;y^r\mathcal{B}_rC_{khm} - \mathcal{B}_m(H_rG^r_{hnk}) - \left(\mathcal{B}_rH_k\right)G^r_{hmn} - \left(\mathcal{B}_nH_r\right)G^r_{hmk} = (\dot\partial_ha_{mn})H_k + (n-1)\left(\dot\partial_hb_{mn}\right)y_k. \tag{24}$$

Transvecting (24) by y^k, using (5a), (1a), (1b), (13d) and (2c), we get

$$-\left(\mathcal{B}_rH\right)G^r_{hmn} = (\dot\partial_ha_{mn})H + (\dot\partial_hb_{mn})F^2. \tag{25}$$

If $(\mathcal{B}_rH)G^r_{hmn}=0$, the equation (25) can be written as

$$\dot\partial_hb_{mn} = -\frac{\dot\partial_ha_{mn}}{F^2}H. \tag{26}$$

If the covariant tensor field a_{mn} is independent of the directional argument, the equation (26) shows that the covariant tensor field b_{mn} is also independent of the directional argument. Conversely, if the covariant tensor field b_{mn} is independent of the directional argument, we get $\left(\dot\partial_ha_{mn}\right)H=0$. In view of Theorem 2.2, the condition $\left(\dot\partial_ha_{mn}\right)H=0$ implies $\dot\partial_ha_{mn}=0$, i.e. the covariant tensor field a_{mn} is also independent of the directional argument. Thus, we conclude

Theorem 2.4. *In $GH-BRF_n$, the covariant tensor field b_{mn} is independent of the directional argument if and only if the covariant tensor field a_{mn} is independent of the directional argument provided $(\mathcal{B}_r H)\, G^r_{hmn} = 0$.*

Transvecting (25) by y^m and using (1b), we get

$$\dot{\partial}_h b_n - b_{hn} = -\frac{(\dot{\partial}_h a_n - a_{hn})}{F^2} H, \tag{27}$$

where $a_{mn} y^m = a_n$ and $b_{mn} y^m = b_n$. Since the covariant vector field a_n is not independent of the directional argument, the equation (27) shows that the covariant vector field b_n is also not independent of the directional argument. Conversely, if the covariant vector field b_n is not independent of the directional argument, we have $\left(\dot{\partial}_h a_n - a_{hn}\right) H = 0$. In view of Theorem 2.2, the condition $\left(\dot{\partial}_h a_n - a_{hn}\right) H = 0$ implies $\dot{\partial}_h a_n = a_{hn}$, i.e. the covariant vector field a_n also is not independent of the directional argument. Thus, we conclude

Theorem 2.5. *In $GH-BRF_n$, the covariant vector field a_n is not independent of the directional argument if and only if the covariant vector field b_n is not independent of the directional argument.*

Suppose that the covariant tensor field a_{mn} is not independent of the directional argument, then by using (26) in (24), we have

$$-2\,(n-1)\,\mu_n\; y^r \mathcal{B}_r C_{khm} - \mathcal{B}_m\left(H_r G^r_{hnk}\right) - \left(\mathcal{B}_r H_k\right) G^r_{hmn} - \left(\mathcal{B}_n H_r\right) G^r_{hmk} = \left(\dot{\partial}_h a_{mn}\right)\left\{H_k - \frac{(n-1)H}{F^2} y_k\right\}. \tag{28}$$

Transvecting (28) by y^k, using (5a), (1a), (1b), (13d) and (2c), we get

$$(\mathcal{B}_r H)\, G^r_{hmn} = 0.$$

Thus, we have

Theorem 2.6. *In $GH-BRF_n$, we have the identity $(\mathcal{B}_r H)\, G^r_{hmn} = 0$.*

Transvecting (28) by y^m and using (1a) and (1b), we get

$$-\mathcal{B}_m\left(H_r G^r_{hnk}\right) y^m = \left(\dot{\partial}_h a_n - a_{hn}\right)\left\{H_k - \frac{(n-1)H}{F^2} y_k\right\}, \tag{29}$$

where $a_{mn} y^m = a_n$. If $\mathcal{B}_m\left(H_r G^r_{hnk}\right) y^m = 0$, the equation (29) implies at least one of the following conditions

$$a)\;\; a_{hn} = \dot{\partial}_h a_n \;\;\text{or}\;\; b)\;\; H_k = \tfrac{(n-1)H}{F^2} y_k. \tag{30}$$

Thus, we conclude

Theorem 2.7. *In $GH-BRF_n$, for which the covariant tensor field a_{hn} is not independent of the directional argument at least one of the conditions (30a) or (30b) holds provided $\mathcal{B}_m\left(H_r G^r_{hnk}\right) y^m = 0$.*

Suppose that (30b) holds. Then (28) implies

$$-2\mu_n y^r \mathcal{B}_r C_{khm} - \mathcal{B}_m\left(\frac{H}{F^2} y_r G^r_{hnk}\right) - \left(\mathcal{B}_r \frac{H}{F^2} y_k\right) G^r_{hmn} - \left(\mathcal{B}_n \frac{H}{F^2} y_r\right) G^r_{hmk} = 0. \tag{31}$$

Transvecting (31) by y^m, using (5a), (1a) and (1b), we get

$$\mathcal{B}_m\left(\frac{H}{F^2} y_r G^r_{hnk}\right) y^m = 0$$

which can be written as

$$F^{-2}\left\{(\mathcal{B}_m H)y_r G^r_{hnk} + H\mathcal{B}_m\left(y_r G^r_{hnk}\right)\right\} y^m = 0.$$

If $H\mathcal{B}_m\left(y_r G^r_{hnk}\right) y^m = 0$, the above equation implies $(\mathcal{B}_m H)y^m y_r G^r_{hnk} = 0$. Since $(\mathcal{B}_m H)\ y^m \neq 0$, so $y_r G^r_{hnk} = 0$,therefore the space is a Landsberg space. Thus, we conclude

Theorem 2.8. *The $GH-BRF_n$ is a Landsberg space if the condition (30b) holds provided $\mathcal{B}_m\left(H_r G^r_{hnk}\right) y^m = 0$ and $(\mathcal{B}_m H)y^m y_r G^r_{hnk} = 0$.*

If the covariant tensor field $a_{hn} \neq \dot{\partial}_h a_n$, in view of Theorem 2.4 (30b) holds good. In view of this fact, we may rewrite Theorem 2.7 in the following form

Theorem 2.9. *The $GH-BRF_n$ is necessarily a Landsberg space provided $a_{hn} \neq \dot{\partial}_h a_n$, $\mathcal{B}_m\left(H_r G^r_{hnk}\right) y^m = 0$ and $(\mathcal{B}_m H)y^m y_r G^r_{hnk} = 0$.*

Differentiating the condition (16) partially with respect to y^h, using (11b) and (2b), we get

$$\dot{\partial}_h \mathcal{B}_m\left(\mathcal{B}_n H^i_{jk}\right) = \left(\dot{\partial}_h a_{mn}\right) H^i_{jk} + a_{mn} H^i_{jkh} + (\dot{\partial}_h b_{mn})(\delta^i_j y_k - \delta^i_k y_j) + b_{mn}(\delta^i_j g_{kh} - \delta^i_k g_{jh}). \tag{32}$$

Using the commutation formula (7) for $\left(\mathcal{B}_n H^i_{jk}\right)$ in (32), we get

$$\begin{aligned}\mathcal{B}_m(\dot{\partial}_h \mathcal{B}_n H^i_{jk}) - \left(\mathcal{B}_r\ H^i_{jk}\right) G^r_{hmn} + \left(\mathcal{B}_n\ H^r_{jk}\right) G^i_{hmr} - \left(\mathcal{B}_n H^i_{rk}\right) G^r_{hmj} - \left(\mathcal{B}_n H^i_{jr}\right) G^r_{hmk} \\ = \left(\dot{\partial}_h a_{mn}\right) H^i_{jk} + a_{mn} H^i_{jkh} + \left(\dot{\partial}_h b_{mn}\right) (\delta^i_j\ y_k - \delta^i_k\ y_j\ ) + b_{mn}\ (\delta^i_j\ g_{kh} - \delta^i_k\ g_{jh}).\end{aligned} \tag{33}$$

Again using the commutation formula (7) for (H^i_{jk}) in (33) and using (11b), we get

$$\begin{aligned}&\mathcal{B}_m \mathcal{B}_n\ H^i_{jkh} + (\mathcal{B}_m H^r_{jk})G^i_{hnr} + H^r_{jk}\ (\mathcal{B}_m G^i_{hnr}) - (\mathcal{B}_m\ H^i_{rk})G^r_{hnj} - H^i_{rk}(\mathcal{B}_m\ G^r_{hnj}) - (\mathcal{B}_m H^i_{jr})G^r_{hnk} - H^i_{jr}(\mathcal{B}_m G^r_{hnk}) \\ &\quad - (\mathcal{B}_r H^i_{jk})G^r_{hmn} + (\mathcal{B}_n H^r_{jk})G^i_{hmr} - \left(\mathcal{B}_n H^i_{jr}\right) G^r_{hmk} = \left(\dot{\partial}_h a_{mn}\right) H^i_{jk} + a_{mn} H^i_{jkh} - (\mathcal{B}_n H^i_{rk})G^r_{hmj} \\ &\quad + \left(\dot{\partial}_h b_{mn}\right) \left(\delta^i_j\ y_k - \delta^i_k y_j\right) + b_{mn}(\delta^i_j\ g_{kh} - \delta^i_k\ g_{jh}).\end{aligned}$$

By using the condition (15), the above equation can be written as

$$\begin{aligned}&(\mathcal{B}_m H^r_{jk})G^i_{hnr} + H^r_{jk}\ (\mathcal{B}_m G^i_{hnr}) - (\mathcal{B}_m H^i_{rk})G^r_{hnj} - H^i_{rk}(\mathcal{B}_m G^r_{hnj}) - (\mathcal{B}_m H^i_{jr})G^r_{hnk} - H^i_{jr}\,(\mathcal{B}_m G^r_{hnk}) - \left(\mathcal{B}_r H^i_{jk}\right) G^r_{hmn} \\ &\quad + \left(\mathcal{B}_n H^r_{jk}\right) G^i_{hmr} - \left(\mathcal{B}_n H^i_{rk}\right) G^r_{hmj} - \left(\mathcal{B}_n H^i_{jr}\right) G^r_{hmk} \\ &\quad = (\dot{\partial}_h\ a_{mn})H^i_{jk} + \left(\dot{\partial}_h b_{mn}\right) \left(\delta^i_j\ y_k - \delta^i_k\ y_j\ \right) + 2\ \mu_n\ y^r \mathcal{B}_r(\delta^i_j\ C_{khm} - \delta^i_k\ C_{jhm}).\end{aligned} \tag{34}$$

Transvecting (34) by y_i, using the identity $(y_i H^i_{jk} = 0)$ which established by [7] and (5b), we get

$$\left(\mathcal{B}_m H^r_{jk}\right) y_i G^i_{hnr} + H^r_{jk}\{\mathcal{B}_m\left(y_i G^i_{hnr}\right)\} + \left(\mathcal{B}_n H^r_{jk}\right) y_i G^i_{hmr} = 2\mu_n y^r \mathcal{B}_r(y_j C_{khm} - y_k C_{jhm}). \tag{35}$$

Transvecting (35) by y^m, using (1b), (5a) and (1a), we get

$$\left[\left(\mathcal{B}_m H^r_{jk}\right) y_i G^i_{hnr} + H^r_{jk}\left\{\mathcal{B}_m\left(y_i\ G^i_{hnr}\right)\right\}\right] y^m = 0$$

which can be written as

$$\left[\mathcal{B}_m\left(H^r_{jk} y_i G^i_{hnr}\right)\right] y^m = 0.$$

Thus, we conclude

Theorem 2.10. *In $GH-BRF_n$, we have the identity $\left[\mathcal{B}_m\left(H^r_{jk}y_iG^i_{hnr}\right)\right]y^m=0$.*

Transvecting (34) by y^k, using (12), (1b), (5a), (2c) and (1a), we get

$$\left(\mathcal{B}_mH^r_j\right)\ G^i_{hnr}+H^r_j\left(\mathcal{B}_mG^i_{hnr}\right)-\left(\mathcal{B}_m\ H^i_r\right)\ G^r_{hnj}-H^i_r(\mathcal{B}_mG^r_{hnj})-\left(\mathcal{B}_r\ H^i_j\right)\ G^r_{hmn}+\ \left(\mathcal{B}_nH^r_j\right)\ G^i_{hmr}-\left(\mathcal{B}_nH^i_r\right)\ G^r_{hmj}$$
$$=\left(\dot{\partial}_ha_{mn}\ \right)\ H^i_j+\ \left(\dot{\partial}_hb_{mn}\right)\left(\delta^i_j\ F^2-y^i\ y_j\ \right)+2\ \mu_n\ y^r\mathcal{B}_r\left(\ y^iC_{jhm}\right). \tag{36}$$

Substituting the value of $(\dot{\partial}_hb_{mn})$ from (26) in (36), using (3a) and (3b), we get

$$\mathcal{B}_m(\ H^r_jG^i_{hnr})-\mathcal{B}_m\ (\ H^i_r\ G^r_{hnj})\ -(\ \mathcal{B}_r\ H^i_j)G^r_{hmn}\ +(\mathcal{B}_nH^r_j)G^i_{hmr}-(\mathcal{B}_nH^i_r)G^r_{hmj}$$
$$=\left(\dot{\partial}_h\ a_{mn}\right)\{H^i_j-H\left(\ \delta^i_j\ -l^il_j\right)\}+2\ \mu_n\ y^r\mathcal{B}_r\left(\ y^iC_{jhm}\right).$$

If $\ \mathcal{B}_m(H^r_jG^i_{hnr})-\mathcal{B}_m\left(H^i_rG^r_{hnj}\right)\mathcal{B}_m-(\mathcal{B}_rH^i_j)G^r_{hmn}+\left(\mathcal{B}_nH^r_j\right)G^i_{hmr}-\left(\mathcal{B}_nH^i_r\right)G^r_{hmi}-2\mu_ny^r\mathcal{B}_r\left(y^iC_{jhm}\right)=0.$ (37)

Then, we have at least one of the following conditions

$$a)\ \ \dot{\partial}_ha_{hn}=0\ \ \text{or}\ \ b)\ \ H^i_j=H(\ \delta^i_j\ -l^il_j). \tag{38}$$

Putting $H=F^2R$, (2.24b) may be written as

$$H^i_j=F^2\ R\ (\ \delta^i_j\ -l^il_j).$$

Therefore, the space is a Finsler space of scalar curvature. Thus, we conclude

Theorem 2.11. *In $GH-BRF_n$, for $n>2$ admitting the condition (37) is a Finsler space of scalar curvature provided $R\neq0$ and the covariant tensor field a_{mn} is not independent of the directional argument.*

References

[1] F.Y.A.Qasem, *On transformations in Finsler spaces*, D. Phil., Thesis, University of Allahabad, India, (2000).

[2] H.Rund, *The differential geometry of Finsler spaces*, Springer-Verlag, Göttingen-Heidelberg, Berlin, (1959); 2^{nd} Edit. (in Russian), Nauka, (Moscow), (1981).

[3] M.L.Youssef and A.Soleiman, *On concircularly recurrent Finsler manifolds*, Balkan Journal of geometry and its Applications, 18(1)(2013), 101-113.

[4] N.S.H.Hussien, *On K^h recurrent Finsler spaces*, M.Sc. Thesis, University of Aden, Aden, Yemen, (2009).

[5] P.N.Pandey, A notes on recurrence vector, Proc. Nat. Acad. Sci., 51A(I)(1981), 6-8.

[6] P.N.Pandey, On some Finsler spaces of scalar curvature, Proc. of Maths., I8(1)(1984), 41-48.

[7] P.N. Pandey and V.J.Dwivedi, *On T-recurrent Finsler space*, Prog. Math. (Varanasi), 21(2)(1987), 101-112.

[8] P.N. Pandey, S.Saxena and A.Goswami, *On a generalized H-recurrent Finsler space*, Journal of International Academy of Physical Sciences, 15(Special Issue 1)(2011), 201-211.

[9] R.Verma, *Some transformations in Finsler spaces*, D. Phil., Thesis, University of Allahabad, Allahabad, India, (1991).

[10] S.Dikshit, *Certain types of recurrences in Finsler spaces*, D.Phil., Thesis, University of Allahabad, Allahabad, India, (1992).