International Journal of Mathematics And its Applications
Volume 4, Issue 2–B (2016), 69–84.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Degree of Approximation of Functions by their Fourier Series in the Besov Space by Matrix Mean

Research Article

Madhusmita Mohanty[1]*, Gokulananda Das[2] and Sanghamitra Beuria[3]

1 Department of Mathematics, Utkal University, Vani-Vihar, Bhubaneswar, Odisha, India.
2 177, Dharma Vihar, Bhubaneswar, Odisha, India.
3 Department of Mathematics, College of Basic Science and Humanities, OUAT, Bhubaneswar, Odisha, India.

Abstract: The paper studies the degree of approximation of functions by their Fourier series in the Besov space by matrix mean and this generalizing many known results.

MSC: 41A25, 42A24.

Keywords: Besov space, Hölder space, Lipschitz space, matrix method, Fourier series, Modulus of smoothness.

1. Introduction

Let f be a 2π periodic function and let $f \in L_p[0, 2\pi], p \geq 1$. The Fourier series of f at x is given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1}$$

Let $s_n(x)$ denote the nth partial sums of (1). We know ([6]) that

$$s_n(x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \phi_x(u) D_n(u) du \tag{2}$$

where

$$\phi_x(u) = f(x+u) + f(x-u) - 2f(x) \tag{3}$$

$$D_n(u) = \frac{1}{2} + \sum_{k=0}^{n} \cos ku = \frac{\sin(k+\frac{1}{2})u}{2\sin\frac{u}{2}} \tag{4}$$

$$K_n(u) = \sum_{k=0}^{\infty} a_{n,k} D_k(u) \tag{5}$$

Let $A = (a_{n,k})$ be an infinite matrix. We assume that elements of the matrix $A = (a_{n,k})$ satisfy the following regularity conditions

$$||A|| = \sup_n \sum_{k=0}^{\infty} |a_{n,k}| < \infty \tag{6}$$

* *E-mail: pinku_madhu@yahoo.com*

$$(a_{n,k}) \to 0 \text{ as } n \to \infty \text{ and } k \text{ is fixed} \tag{7}$$

and

$$\sum_{k=0}^{\infty} a_{n,k} = 1 \text{ for each } n = 0, 1, 2 \cdots . \tag{8}$$

2. Definitions and Notations

Definition 2.1 (Modulus of Continuity). *Let $A = R, R + [a, b] \subset R$ or T (which usually taken to be R with identification of points modulo 2π). The modulus of continuity $w(f, t) = w(t)$ of a function f on A can be defined as*

$$w(t) = w(f, t) = \sup_{\substack{|x-y| \le t, \\ x, y \in A}} |f(x) - f(y)|, t \ge 0.$$

Definition 2.2 (Modulus of Smoothness). *The k^{th} order modulus of smoothness [2] of a function $f : A \to R$ is defined by*

$$w_k(f, t) = \sup_{0<h\le t} \{\sup |\Delta_h^k(f, x)| : x, x + kh \in A\},\ t \ge 0 \tag{9}$$

where

$$\Delta_h^k(f, x) = \sum_{i=0}^{k} (-1)^{k-i} \begin{pmatrix} k \\ i \end{pmatrix} f(x + ih),\ k \in N. \tag{10}$$

For $k = 1, w_1(f, t)$ is called the modulus of continuity of f. The function w is continuous at $t = 0$ if and only if f is uniformly continuous on A, that is $f \in \tilde{c}(A)$. The k^{th} order modulus of smoothness of $f \in L_p(A), 0 < p < \infty$ or of $f \in \tilde{c}(A), if p = \infty$ is defined by

$$w_k(f, t)_p = \sup_{0<h\le t} ||\Delta_h^k(f, \cdot)||_p, t \ge 0 \tag{11}$$

if $p \ge 1, k = 1$, then $w_1(f, t)_p = w(f, t)_p$ is a modulus of continuity (or integral modulus of continuity). If $p = \infty, k = 1$ and f is continuous then $w_k(f, t)_p$ reduces to modulus of continuity $w_1(f, t)$ or $w(f, t)$.

Definition 2.3 (Lipschitz Space). *If $f \in \tilde{c}(A)$ and*

$$w(f, t) = O(t^{\alpha}), 0 < \alpha \le 1 \tag{12}$$

then we write $f \in Lip\alpha$. If $w(f, t) = O(t)$ as $t \to 0+$ (in particular (9) holds for $\alpha > 1$) then f reduces to a constant. If $f \in L_p(A),\ 0 < p < \infty$ and

$$w(f, t)_p = O(t^{\alpha}), 0 < \alpha \le 1 \tag{13}$$

then we write $f \in Lip(\alpha, p),\ 0 < p < \infty,\ 0 < \alpha \le 1$.

The case $\alpha > 1$ is of no interest as the function reduces to a constant, whenever

$$w(f, t)_p = O(t) \text{ as t} \to 0+ \tag{14}$$

We note that if $p = \infty$ and $f \in c(A)$, then $Lip(\alpha, p)$ class reduces to $Lip\ \alpha$ class.

Definition 2.4 (Generalized Lipschitz Space). *Let $\alpha > 0$ and suppose that $k = [\alpha] + 1$. For $f \in L_p(A)$, $0 < p < \infty$, if*

$$w_k(f, t) = O(t^{\alpha}), t > 0 \tag{15}$$

then we write

$$f \in Lip^*(\alpha, p),\ \alpha > 0,\ 0 < p \leq \infty \tag{16}$$

and say that f belongs to generalized Lipschitz space. The seminorm is then

$$|f|_{Lip^*(\alpha, L_p)} = \sup_{t>0}(t^{-\alpha} w_k(f, t)_p).$$

It is known [2] that the space $Lip^*(\alpha, L_p)$ contains $Lip(\alpha, L_p)$. For $0 < \alpha < 1$ the spaces coincide, (for $p = \infty$, it is necessary to replace L_∞ by $\tilde{c}$ of uniformly continuous function on A). For $0 < \alpha < 1$ and $p = 1$ the space $Lip^*(\alpha, L_p)$ coincide with $Lip\alpha$.

For $\alpha = 1, p = \infty$, we have

$$Lip(1, \tilde{c}) = Lip\ 1 \tag{17}$$

but

$$Lip^*(1, \tilde{c}) = z \tag{18}$$

is the Zygmund space [5] which is characterized by (13) with $k = 2$.

Definition 2.5 (Hölder (H_α) Space). *For $0 < \alpha \leq 1$, let*

$$H_\alpha = \{f \in C_{2\pi} : w(f, t) = O(t^{\alpha})\}. \tag{19}$$

It is known [3] that H_α is a Banach Space with the norm $||\cdot||_\alpha$ defined by

$$||f||_\alpha = ||f||_c + \sup_{t>0} t^{-\alpha} w(t),\ 0 < \alpha \leq 1 \tag{20}$$

$$||f||_0 = ||f||_c$$

and

$$H_\alpha \subseteq H_\beta \subseteq C_{2\pi},\ 0 < \beta \leq \alpha \leq 1 \tag{21}$$

Definition 2.6 ($\mathrm{H}_{(\alpha,p)}$ Space). *For $0 < \alpha \leq 1$, let*

$$H_{(\alpha,p)} = \{f \in L_p[0, 2\pi] : 0 < p \leq \infty, w(f, t)_p = O(t^{\alpha})\} \tag{22}$$

and introduce the norm $||\cdot||_{(\alpha,p)}$ as follows

$$||f||_{(\alpha,p)} = ||f||_p + \sup_{t>0} t^{-\alpha} w(f, t)_p,\ 0 < \alpha \leq 1. \tag{23}$$

$$||f||_{(0,p)} = ||f||_p.$$

It is known [1] that $H_{(\alpha,p)}$ is a Banach space for $p \geq 1$ and a complete p-normed space for $0 < p < 1$. Also

$$H_{(\alpha,p)} \subseteq H_{(\beta,p)} \subseteq L_p, 0 < \beta \leq \alpha \leq 1. \tag{24}$$

Note that $H_{(\alpha,\infty)}$ is the space H_α defined above. For study of degree of approximation problems the natural way to proceed to consider with some restrictions on some modulus of smoothness as prescribed in H_α and $H_{(\alpha,p)}$ spaces. As we have seen above only a constant function satisfies Lipschitz condition for $\alpha > 1$. However for generalized Lipschitz class there is no such restriction on α. We required a finer scale of smoothness than is provided by Lipschitz class. For each $\alpha > 0$ Besov developed a remarkable technique for restricting modulus of smoothness by introducing a third parameter q (in addition to p on α) and applying $\alpha \cdot q$ norms (rather than α, ∞ norms) to the modulus of smoothness $w_k(f,\cdot)_p$ of f.

Definition 2.7 (Besov space)**.** *Let $\alpha > 0$ be given and let $k = [\alpha] + 1$. For $0 < p, q \leq \infty$, the Besov space ([2]) $B_q^\alpha(L_p)$ is defined as follows:*

$$B_q^\alpha(L_p) = \{f \in L_p : |f|_{B_q^\alpha(L_p)} = ||w_k(f,\cdot)||_{(\alpha,q)} \text{ is finite}\}$$

where

$$||w_k(f,\cdot)||_{(\alpha,q)} = \begin{cases} (\int_0^\infty (t^{-\alpha} w_k(f,t)_p)^q \frac{dt}{t})^{\frac{1}{q}}, & 0 < q < \infty \\ \sup_{t>0} t^{-\alpha} w_k(f,t)_p, & q = \infty. \end{cases} \tag{25}$$

It is known ([2]) that $||w_k(f,\cdot)||_{(\alpha,q)}$ is a seminorm if $1 \leq p, q \leq \infty$ and a quasi-seminorm in other cases.
The Besov norm for $B_q^\alpha(L_p)$ is

$$||f||_{B_q^\alpha(L_p)} = ||f||_p + ||w_k(f,\cdot)||_{(\alpha,q)} \tag{26}$$

It is known ([4]) that for 2π-periodic function f, the integral $(\int_0^\infty (t^{-\alpha} w_k(f,t)_p)^q \frac{dt}{t})^{\frac{1}{q}}$ is replaced by $(\int_0^\pi (t^{-\alpha} w_k(f,t)_p)^q \frac{dt}{t})^{\frac{1}{q}}$.
We know ([2, 4]) the following inclusion relations. For fixed α and p

$$B_q^\alpha(L_p) \subset B_{q_1}^\alpha(L_p), q < q_1.$$

For fixed p and q

$$B_q^\alpha(L_p) \subset B_q^\beta(L_p), \beta < \alpha.$$

For fixed α and q

$$B_q^\alpha(L_p) \subset B_q^\alpha(L_{p_1}), p_1 < p.$$

Definition 2.8 (Special cases of Besov space)**.** *For $q = \infty, B_\infty^\alpha(L_p), \alpha > 0, p \geq 1$ is same as $Lip^*(\alpha, L_p)$ the generalized Lipschitz space and the corresponding norm $||\cdot||_{B_\infty^\alpha(L_p)}$ is given by*

$$||f||_{B_\infty^\alpha(L_p)} = ||f||_p + \sup_{t>0} t^{-\alpha} w_k(f,t)_p \tag{27}$$

for every $\alpha > 0$ with $k = [\alpha] + 1$.

For the special case when $0 < \alpha < 1$, $B_\infty^\alpha(L_p)$ space reduces to $H_{(\alpha,p)}$ space due to Das et al. [1] and the corresponding norm is given by

$$||f||_{B_\infty^\alpha(L_p)} = ||f||_{(\alpha,p)} = ||f||_p + \sup_{t>0} t^{-\alpha} w(f,t)_p, 0 < \alpha < 1. \tag{28}$$

For $\alpha = 1$, the norm is given by

$$||f||_{B_\infty^1(L_p)} = ||f||_p + \sup_{t>0} t^{-\alpha} w_2(f,t)_p. \tag{29}$$

Note that $||f||_{B_\infty^1(L_p)}$ is not same as $||f||_{(1,p)}$ and the space $B_\infty^1(L_p)$ includes the space $H(1,p)$, $p \geq 1$. If we further specialize by taking $p = \infty$, B_∞^α, $0 < \alpha < 1$, coincides with H_α space due to Prossodorf [3] and the norm is given by

$$||f||_{B_\infty^\alpha(L_\infty)} = ||f||_\alpha = ||f||_c + \sup_{t>0} t^\alpha w(f,t), \ 0 < \alpha < 1. \tag{30}$$

For $\alpha = 1,\ p = \infty$, the norm is given by

$$||f||_{B^1_\infty(L_\infty)} = ||f||_c + \sup_{t>0} t^{-1} w_2(f,t),\ \alpha = 1 \tag{31}$$

which is different from $||f||_1$ and $B^1_\infty(L_\infty)$ includes the H_1 space.

3. Main Result

We prove the following theorem.

Theorem 3.1. *Let the matrix $A = (a_{n,k})$ satisfy the following conditions*

(i). $\sup\limits_n \sum_{k=0}^\infty |a_{n,k}| < \infty$

(ii). $\sum\limits_{k=0}^\infty a_{n,k} = 1$ *for all n and*

(iii). $\sum\limits_{k=\mu_n}^\infty k|a_{n,k}| = O(\mu_n)$.

where (μ_n) is a positive non-decreasing sequence $\mu_1 = 1$.

Let $\psi(n) = \sum\limits_{k=0}^\infty |a_{n,k} - a_{n,k+1}|$ and $0 < \alpha < 2$ and $0 \le \beta < \alpha$. If $f \in B^\alpha_q(L_p),\ p \ge 1$ and $1 < q \le \infty$ and let $t_n(x)$ be the A-transform of the Fourier series of f, that is,

$$t_n(f) = t_n(f;x) = \sum_{k=0}^\infty a_{n,k} s_k(x)$$

Then

Case 1 $(1 < q < \infty)$

$$||T_n(\cdot)||_{B^\beta_q(L_p)} = O\left(\frac{1}{\mu_n^{\alpha-\beta-\frac{1}{q}}}\right) + O(\psi(n)) \left\{\sum_{k=1}^n \left(\frac{(\mu_{k+1}-\mu_k)^{1-\frac{1}{q}}}{\mu_k^{\alpha-\beta-\frac{2}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}}$$

Case 2 $(q = \infty)$

$$||T_n(\cdot)||_{B^\beta_q(L_p)} = O\left(\frac{1}{\mu_n^{\alpha-\beta}}\right) + O(\psi(n)) \sum_{k=1}^n \left(\frac{(\mu_{k+1}-\mu_k)}{\mu_k^{\alpha-\beta}}\right)$$

4. Additional Notations and Lemmas

We need the following additional notations

$$\phi(x,t,u) = \begin{cases} \phi_{x+t}(u) - \phi_x(u), & 0 < \alpha < 1 \\ \phi_{x+t}(u) + \phi_{x-t}(u) - 2\phi_x(u), & 1 \le \alpha < 2 \end{cases}$$

For $k = [\alpha] + 1$, we have for $p \ge 1$

$$w_k(f,t)_p = \begin{cases} w_1(f,t)_p, & 0 < \alpha < 1 \\ w_2(f,t)_p, & 1 \le \alpha < 2 \end{cases}$$

Let

$$T_n(x,t) = \begin{cases} T_n(x+t) - T_n(x), & 0 < \alpha < 1 \\ T_n(x+t) + T_n(x-t) - 2T_n(x), & 1 \leq \alpha < 2 \end{cases}$$

Using above equation and definition of $w_k(f,t)_p$, we have

$$w_k(T_n, t)_p = ||T_n(\cdot, t)||_p$$

We require the following lemmas for the proof of the theorem.

Lemma 4.1. *Let $1 \leq p \leq \infty$ and $0 < \alpha < 2$. If $f \in L_p[0, 2\pi]$, then for $0 < t, u \leq \pi$*

(i). $||\phi(\cdot, t, u)||_p \leq 4w_k(f,t)_p$

(ii). $||\phi(\cdot, t, u)||_p \leq 4w_k(f,u)_p$

(iii). $||\phi_\cdot(u)||_p \leq 2w_k(f,u)_p$,

where $k = [\alpha] + 1$.

Proof. Case $0 < \alpha < 1$.

Clearly $k = [\alpha] + 1 = 1$. By virtue of (3), $\phi(x,t,u) = \phi_{x+t}(u) - \phi_x(u)$, can be written as

$$\phi(x,t,u) = \begin{cases} \{f(x+t+u) - f(x+u)\} + \{f(x+t-u) - f(x-u)\} - 2\{f(x+t) - f(x)\} \\ \{f(x+t+u) - f(x+t)\} + \{f(x-u+t) - f(x+t)\} - \{f(x+u) - f(x)\} - \{f(x-u) - f(x)\} \end{cases} \tag{32}$$

Applying Minkowski's inequality to (32), we get for $p \geq 1$

$$\begin{aligned} ||\phi(\cdot, t, u)||_p &\leq ||f(\cdot + t + u) - f(\cdot + u)||_p + ||f(\cdot + t - u) - f(\cdot - u)||_p + 2||f(\cdot + t) - f(\cdot)||_p \\ &\leq 4w_1(f,t)_p, \quad 0 < \alpha < 1 \end{aligned}$$

Similarly applying Minkowski's inequality to (32), we get for $p \geq 1$

$$||\phi(\cdot, t, u)||_p \leq 4w_1(f,u)_p.$$

Case $1 \leq \alpha < 2$.

Clearly $k = [\alpha] + 1 = 2$. By virtue of (3), $\phi(x,t,u) = \phi_{x+t}(u) + \phi_{x-t}(u) - 2\phi_x(u)$, can be written as

$$\phi(x,t,u) = \begin{cases} \{f(x+t+u) + f(x+t-u) - 2f(x+t)\} + \{f(x-t+u) + f(x-t-u) \\ -2f(x-t)\} - 2\{f(x+u) + f(x-u) - 2f(x)\} \\ \{f(x+t+u) + f(x-t+u) - 2f(x+u)\} + \{f(x+t-u) + f(x-t-u) \\ -2f(x-u)\} - 2\{f(x+t) + f(x-t) - 2f(x)\} \end{cases} \tag{33}$$

Applying Minkowski's inequality to (33), we obtain for $p \geq 1$

$$\begin{aligned} ||\phi(\cdot, t, u)||_p &\leq ||f(\cdot + t + u) + f(\cdot + t - u) - 2f(\cdot + t)||_p \\ &+ ||f(\cdot - t + u) + f(\cdot - t - u) - 2f(\cdot - t)||_p \\ &+ 2||f(\cdot + u) + f(\cdot - u) - 2f(\cdot)||_p \\ &\leq 4w_2(f,u)_p \end{aligned}$$

Using (33) and proceeding as above, we get

$$||\phi(\cdot,t,u)||_p \leq 4w_2(f,t)_p$$

this completes the proof of part (i) and (ii). We omit the proof of (iii) as it is trivial. □

Lemma 4.2. *Let* $0<\alpha<2$*. Suppose that* $0\leq\beta<\alpha$*. If* $f\in B_q^\alpha(L_p), p\geq 1,\ 1<q<\infty$*, then*

(i). $\int_0^\pi |K_n(u)|\left(\int_0^u \frac{||\phi(\cdot,t,u)||_p^q}{t^{\beta q}}\frac{dt}{t}\right)^{\frac{1}{q}}du = O(1)\left\{\int_0^\pi \left(u^{\alpha-\beta}|K_n(u)|\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}$

(ii). $\int_0^\pi |K_n(u)|\left(\int_u^\pi \frac{||\phi(\cdot,t,u)||_p^q}{t^{\beta q}}\frac{dt}{t}\right)^{\frac{1}{q}}du = O(1)\left\{\int_0^\pi \left(u^{\alpha-\beta+\frac{1}{q}}|K_n(u)|\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}$

where $K_n(u)$ *is defined as in (5).*

Proof. Applying Lemma 4.1(i), we have

$$\begin{aligned}\int_0^\pi |K_n(u)|\left(\int_0^u \frac{||\phi(\cdot,t,u)||_p^q}{t^{\beta q+1}}dt\right)^{\frac{1}{q}}du &= O(1)\int_0^\pi |K_n(u)|\left(\int_0^u\left(\frac{w_k(f,t)_p}{t^\alpha}\right)^q t^{(\alpha-\beta)q}\frac{dt}{t}\right)^{\frac{1}{q}}du\\ &= O(1)\int_0^\pi |K_n(u)|u^{(\alpha-\beta)}du\left(\int_0^u \frac{w_k(f,t)_p}{t^\alpha}\frac{dt}{t}\right)^{\frac{1}{q}}\\ &= O(1)\int_0^\pi |K_n(u)|u^{(\alpha-\beta)}du\end{aligned}$$

by Second Mean value theorem and by the definition of Besov space. Applying Holders inequality

$$\begin{aligned}&= O(1)\left\{\int_0^\pi\left(|K_n(u)|u^{(\alpha-\beta)}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\left(\int_0^\pi 1^q du\right)^{\frac{1}{q}}\\ &= O(1)\left\{\int_0^\pi\left(|K_n(u)|u^{(\alpha-\beta)}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\end{aligned}$$

For the second part, applying Lemma 4.1(ii), we get

$$\begin{aligned}\int_0^\pi |K_n(u)|du\left(\int_u^\pi \frac{||\phi(\cdot,t,u)||_p^q}{t^{\beta q+1}}dt\right)^{\frac{1}{q}} &= O(1)\int_0^\pi |K_n(u)|w_k(f,u)_p du\left(\int_u^\pi \frac{dt}{t^{\beta q+1}}\right)^{\frac{1}{q}}\\ &= O(1)\int_0^\pi |K_n(u)|w_k(f,u)_p u^{-\beta}du\\ &= O(1)\int_0^\pi\left(\frac{w_k(f,u)_p}{u^{\alpha+\frac{1}{q}}}\right)u^{\alpha-\beta+\frac{1}{q}}|K_n(u)|du\end{aligned}$$

Applying Hölder's inequality

$$\begin{aligned}&= O(1)\left\{\int_0^\pi\left(\frac{w_k(f,u)_p}{u^\alpha}\right)^q\frac{du}{u}\right\}^{\frac{1}{q}}\left\{\int_0^u\left(u^{\alpha-\beta+\frac{1}{q}}|K_n(u)|\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\ &= O(1)\left\{\int_0^\pi\left(u^{\alpha-\beta+\frac{1}{q}}|K_n(u)|\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\end{aligned}$$

by definition of Besov space. □

Lemma 4.3. *Let* $0<\alpha<2$*. Suppose that* $0\leq\beta<\alpha$*. If* $f\in B_q^\alpha(L_p),\ p\geq 1$ *and* $q=\infty$*, then*

$$\sup_{0<t,u\leq\pi} t^{-\beta}||\phi(\cdot,t,u)||_p = O(u^{\alpha-\beta})$$

Proof. For $0 < t \leq u \leq \pi$, applying Lemma 4.1(i), we have

$$\begin{aligned}
\sup_{\substack{t,\\ 0<t\leq u\leq\pi}} t^{-\beta}||\phi(\cdot,t,u)||_p &= \sup_{\substack{t,\\ 0<t\leq u\leq\pi}} t^{\alpha-\beta}(t^{-\alpha}||\phi(\cdot,t,u)||_p) \\
&\leq 4u^{\alpha-\beta}\sup_t(t^{-\alpha}w_k(f,t)_p) \\
&= O(u^{\alpha-\beta}), \qquad \text{by the hypothesis.}
\end{aligned}$$

Next for $0 < u \leq t \leq \pi$, applying Lemma 4.1(ii), we get

$$\begin{aligned}
\sup_{\substack{t,\\ 0<u\leq t\leq\pi}} t^{-\beta}||\phi(\cdot,t,u)||_p &\leq 4w_k(f,u)_p \sup_{\substack{t,\\ 0<u\leq t\leq\pi}} t^{-\beta} \\
&\leq 4u^{\alpha-\beta}\sup_u(u^{-\alpha}w_k(f,u)_p) \\
&= O(u^{\alpha-\beta}), \qquad \text{by the hypothesis}
\end{aligned}$$

and this completes the proof. □

Lemma 4.4.

(a) Let $K_n(u)$ be defined as in (6). Let there exist a positive non-decreasing sequence (μ_n) with $\mu_1 = 1$, then for $0 < u \leq \pi$

$$K_n(u) = O(\mu_n).$$

(b) Let $\psi(n) = \sup \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+1}|$. Then for $0 < u \leq \pi$,

$$K_n(u) = O(u^{-2}\psi(n)).$$

Proof.

(a) From (4), we have

$$\begin{aligned}
|D_k(u)| &= |\frac{1}{2} + \sum_{v=0}^{k}\cos vu| \\
&\leq \frac{1}{2} + \sum_{v=0}^{k}|\cos vu| \\
&\leq k+1 \qquad (34)
\end{aligned}$$

Then

$$\begin{aligned}
|K_n(u)| &\leq \sum_{k=0}^{\infty}|a_{n,k}D_k(u)| \\
&\leq \sum_{k=0}^{\mu_n}|a_{n,k}|(k+1) + \sum_{k=\mu_n+1}^{\infty}|a_{n,k}|(k+1) \quad (\text{ by using } (34)) \\
&\leq \mu_n\sum_{k=0}^{\mu_n}|a_{n,k}| + \sum_{k=\mu_n+1}^{\infty}|a_{n,k}|(k+1) \\
&\leq \mu_n\sum_{k=0}^{\mu_n}|a_{n,k}| + O(\mu_n) \\
&= O(\mu_n) + O(\mu_n) \\
&= O(\mu_n), \qquad (\text{by hypothesis (iii)})
\end{aligned}$$

(b) Applying Abel's transformation, we have

$$\begin{aligned}\sum_{k=0}^{\infty} a_{n,k}\sin(k+\frac{1}{2})u &= O(u^{-1})\sum_{k=0}^{\infty}|a_{n,k}-a_{n,k+1}| \\ &= O(u^{-1}\psi(n))\end{aligned}$$

from which it follows that

$$K_n(u) = O(u^{-2}\psi(n))$$

□

5. Proof of Theorem

Case 1 ($1 < q < \infty$)

Since $t_n(x)$ denote the transformations of the Fourier series f, we have

$$\begin{aligned} t_n(x) &= \sum_{k=0}^{\infty} a_{n,k}s_k(x) & (35)\\ &= \sum_{k=0}^{\infty} a_{n,k}\left[\frac{1}{\pi}\int_0^{\pi}\phi_x(u)D_k(u)du + f(x)\right] \text{ (by (2))} \\ &= \frac{1}{\pi}\sum_{k=0}^{\infty} a_{n,k}\int_0^{\pi}\phi_x(u)D_k(u)du + \sum_{k=0}^{\infty}a_{n,k}f(x) \\ &= \frac{1}{\pi}\int_0^{\pi}\left(\sum_{k=0}^{\infty}a_{n,k}D_k(u)\right)\phi_x(u)du + f(x)\sum_{k=0}^{\infty}a_{n,k} \\ \text{Now, } T_n(x) &= \frac{1}{\pi}\int_0^{\pi}\phi_x(u)K_n(u)du & (36)\\ \text{where we write } T_n(x) &= t_n(x) - f(x). & (37)\end{aligned}$$

We first consider the case $1 < q < \infty$. We have for $p \geq 1$ and $0 \leq \beta < \alpha < 2$, by use of Besov norm defined in (26) for $B_q^{\beta}(L_p)$ is

$$||f||_{B_q^{\alpha}(L_p)} = ||f||_p + ||w_k(f,\cdot)||_{\alpha,q} \tag{38}$$

$$||T_n(\cdot)||_{B_q^{\beta}(L_p)} = ||T_n(\cdot)||_p + ||w_k(T_n,\cdot)||_{\beta,q} \tag{39}$$

Applying Lemma 4.1(iii) in equation (39), we have

$$\begin{aligned}||T_n(\cdot)||_p &\leq \frac{1}{\pi}\int_0^{\pi}||\phi_\cdot(u)||_p|K_n(u)|du \\ &\leq \frac{1}{\pi}\int_0^{\pi}2w_k(f,u)_p|K_n(u)|du \\ &= \frac{2}{\pi}\int_0^{\pi}|K_n(u)|w_k(f,u)_p du\end{aligned}$$

Applying Hölder's inequality, we have

$$||T_n(\cdot)||_p \leq \frac{2}{\pi}\left\{\int_0^{\pi}\left(|K_n(u)|u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\left\{\int_0^{\pi}\left(\frac{w_k(f,u)_p}{u^{\alpha+\frac{1}{q}}}\right)^q du\right\}^{\frac{1}{q}}$$

By definition of Besov Space, we have

$$\begin{aligned}
||T_n(\cdot)||_p &\le O(1)\left\{\int_0^{\pi}\left(|K_n(u)|u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\
&= O(1)\left[\left\{\int_0^{\mu_n}\left(|K_n(u)|u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}+\left\{\int_{\mu_n}^{\pi}\left(|K_n(u)|u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\right]\\
&= O(1)\left[I+J\right], \quad (say) \qquad (40)
\end{aligned}$$

By using Lemma 4.4(a) in I of (40), we have

$$\begin{aligned}
I &= \left\{\int_0^{\frac{\pi}{\mu_n}}(|K_n(u)|u^{\alpha+\frac{1}{q}})^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\
&= O(\mu_n)\left\{\int_0^{\frac{\pi}{\mu_n}}u^{(\alpha+\frac{1}{q})\cdot\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\
&= O(\mu_n)\left\{\int_0^{\frac{\pi}{\mu_n}}u^{\frac{q}{q-1}(\alpha+1)-1}du\right\}^{1-\frac{1}{q}}\\
&= O\left(\frac{1}{\mu_n^{\alpha}}\right) \qquad (41)
\end{aligned}$$

Applying Lemma 4.4(b) in J of (40), we have

$$\begin{aligned}
J &= \left\{\int_{\frac{\pi}{\mu_n}}^{\pi}\left(|K_n(u)|u^{\alpha+\frac{1}{q}}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\
&= O\left(\psi(n)\right)\left\{\int_{\frac{\pi}{\mu_n}}^{\pi}\left(u^{\alpha+\frac{1}{q}-2}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\
&= O\left(\psi(n)\right)\left\{\sum_{k=1}^{n-1}\int_{\frac{\pi}{\mu_{k+1}}}^{\frac{\pi}{\mu_k}}\left(u^{\alpha+\frac{1}{q}-2}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\
&= O\left(\psi(n)\right)\left\{\sum_{k=1}^{n}\int_{\frac{\pi}{\mu_{k+1}}}^{\frac{\pi}{\mu_k}}u^{(\alpha+\frac{1}{q}-2)\cdot\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\
&= O\left(\psi(n)\right)\left\{\sum_{k=1}^{n}\frac{\mu_{k+1}-\mu_k}{\mu_k^2\mu_k^{\frac{q}{q-1}(\alpha+\frac{1}{q}-2)}}\right\}^{1-\frac{1}{q}}\\
&= O\left(\psi(n)\right)\left\{\sum_{k=1}^{n}\left(\frac{(\mu_{k+1}-\mu_k)^{1-\frac{1}{q}}}{\mu_k^{(\alpha-\frac{1}{q})}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \qquad (42)
\end{aligned}$$

Using (41) and (42) and we have from (40),

$$||T_n(\cdot)||_p = O\left(\frac{1}{\mu_n^{\alpha}}\right)+O\left(\psi(n)\right)\left\{\sum_{k=1}^{n}\left(\frac{(\mu_{k+1}-\mu_k)^{1-\frac{1}{q}}}{\mu_k^{(\alpha-\frac{1}{q})}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \qquad (43)$$

By using Besov space, we have

$$\begin{aligned}
||w_k(T_n,\cdot)||_{\beta,q} &= \left\{\int_0^{\pi}\left(t^{-\beta}w_k(T_n,t)_p\right)^q\frac{dt}{t}\right\}^{\frac{1}{q}}\\
&= \int_0^{\pi}\left\{\left(\frac{w_k(T_n,t)_p}{t^{\beta}}\right)^q\frac{dt}{t}\right\}^{\frac{1}{q}}
\end{aligned}$$

From definition of $w_k(T_n,t)_p$, we have

$$\begin{aligned} w_k(T_n,t)_p &= ||T_n(\cdot,t)||_p \\ &\leq \left\{\int_0^\pi \left(\frac{||T_n(\cdot,t)||_p}{t^\beta}\right)^q \frac{dt}{t}\right\}^{\frac{1}{q}} \\ &= \left[\int_0^\pi \left\{\int_0^\pi |T_n(x,t)|^p dx\right\}^{\frac{q}{p}} \frac{dt}{t^{\beta q+1}}\right]^{\frac{1}{q}} \\ &= \left[\int_0^\pi \left\{\int_0^\pi \left|\int_0^\pi \phi(x,t,u)K_n(u)du\right|^p dx\right\}^{\frac{q}{p}} \frac{dt}{t^{\beta q+1}}\right]^{\frac{1}{q}} \end{aligned}$$

By repeated application of generalized Minkowski's inequality, we have

$$\begin{aligned} ||w_k(T_n,\cdot)||_{\beta,p} &\leq \frac{1}{\pi}\left[\int_0^\pi \left\{\int_0^\pi \left(\int_0^\pi |\phi(x,t,u)|^p|K_n(u)|^p dx\right)^{\frac{1}{p}} du\right\}^q \frac{dt}{t^{\beta q+1}}\right]^{\frac{1}{q}} \\ &= \frac{1}{\pi}\left[\int_0^\pi \left\{\int_0^\pi |K_n(u)|||\phi(\cdot,t,u)||_p du\right\}^q \frac{dt}{t^{\beta q+1}}\right]^{\frac{1}{q}} \\ &\leq \frac{1}{\pi}\int_0^\pi |K_n(u)|du\left(\int_0^\pi \frac{||\phi(\cdot,t,u)||_p^q}{t^{\beta q}}\frac{dt}{t}\right)^{\frac{1}{q}} \\ &= \frac{1}{\pi}\int_0^\pi |K_n(u)|du\left\{\left(\int_0^u + \int_u^\pi\right) \frac{||\phi(\cdot,t,u)||_p^q}{t^{\beta q}}\frac{dt}{t}\right\}^{\frac{1}{q}} \\ &\leq \frac{1}{\pi}\int_0^\pi |K_n(u)|du\left\{\int_0^u \frac{||\phi(\cdot,t,u)||_p^q}{t^{\beta q}}\frac{dt}{t}\right\}^{\frac{1}{q}} \\ &\quad +\frac{1}{\pi}\int_0^\pi |K_n(u)|du\left\{\int_u^\pi \frac{||\phi(\cdot,t,u)||_p^q}{t^{\beta q}}\frac{dt}{t}\right\}^{\frac{1}{q}} \\ &= O(1)\left[\left\{\int_0^\pi \left(|K_n(u)|u^{\alpha-\beta}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \left\{\int_0^\pi \left(|K_n(u)|u^{\alpha-\beta+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}\right] (\text{using Lemma 4.2}) \\ &= O(1)[I'+J'], \quad (say) \end{aligned} \tag{44}$$

$$\begin{aligned} I' &= \left\{\int_0^\pi \left(|K_n(u)|u^{\alpha-\beta}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \\ &= \left\{\left(\int_0^{\mu_n} + \int_{\mu_n}^\pi\right)\left(|K_n(u)|u^{\alpha-\beta}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \\ &\leq \left\{\int_0^{\mu_n} \left(|K_n(u)|u^{\alpha-\beta}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} + \left\{\int_{\mu_n}^\pi \left(|K_n(u)|u^{\alpha-\beta}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \\ &= I_1'+I_2', \quad (say) \end{aligned} \tag{45}$$

Applying Lemma 4.4(a) in I_1', we have

$$\begin{aligned} I_1' &= \left\{\int_0^{\frac{\pi}{\mu_n}} \left(|K_n(u)|u^{\alpha-\beta}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \\ &= O(\mu_n)\left\{\int_0^{\frac{\pi}{\mu_n}} u^{\alpha-\beta(\frac{q}{q-1})}du\right\}^{1-\frac{1}{q}} \\ &= O(\mu_n)\left\{\int_0^{\frac{\pi}{\mu_n}} u^{\frac{q}{q-1}(\alpha-\beta+1-\frac{1}{q})-1}du\right\}^{1-\frac{1}{q}} \\ &= O\left(\frac{1}{\mu_n^{\alpha-\beta-\frac{1}{q}}}\right) \end{aligned} \tag{46}$$

Applying Lemma 4.4(b) in I_2', we have

$$\begin{aligned} I_2' &= \left\{\int_{\frac{\pi}{\mu_n}}^{\pi} \left(|K_n(u)|u^{\alpha-\beta}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \\ &= O(\psi(n))\left\{\int_{\frac{\pi}{\mu_n}}^{\pi} \left(u^{\alpha-\beta-2}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \\ &= O(\psi(n))\left\{\sum_{k=1}^{n-1}\int_{\frac{\pi}{\mu_{k+1}}}^{\frac{\pi}{\mu_k}} u^{(\alpha-\beta-2)\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \\ &= O(\psi(n))\left\{\sum_{k=1}^{n}\int_{\frac{\pi}{\mu_{k+1}}}^{\frac{\pi}{\mu_k}} u^{(\alpha-\beta-2)\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \end{aligned}$$

Let $h(u) = (u^{\alpha-\beta})^{\frac{q}{q-1}}$ and $H(u)$ be a primitive of $h(u)$, then

$$\begin{aligned} \int_{\frac{\pi}{\mu_{k+1}}}^{\frac{\pi}{\mu_k}} \left(u^{\alpha-\beta-2}\right)^{\frac{q}{q-1}} du &= \int_{\frac{\pi}{\mu_{k+1}}}^{\frac{\pi}{\mu_k}} h(u)du \\ &= H\left(\frac{\pi}{\mu_k}\right) - H\left(\frac{\pi}{\mu_{k+1}}\right) \\ &= \left(\frac{\pi}{\mu_k} - \frac{\pi}{\mu_{k+1}}\right) h(c), \text{ for some } \frac{\pi}{\mu_{k+1}} < c < \frac{\pi}{\mu_k} \\ &= O(1)\frac{(\mu_{k+1}-\mu_k)}{\mu_k^2}\left(\frac{1}{\mu_k^{\alpha-\beta-2}}\right)^{\frac{q}{q-1}} \\ &= O(1)\left(\frac{(\mu_{k+1}-\mu_k)^{1-\frac{1}{q}}}{\mu_k^{\alpha-\beta-\frac{2}{q}}}\right)^{\frac{q}{q-1}} \end{aligned}$$

$$I_2' = O\left(\psi(n)\right)\left\{\sum_{k=1}^{n}\left(\frac{(\mu_{k+1}-\mu_k)^{1-\frac{1}{q}}}{\mu_k^{(\alpha-\beta-\frac{2}{q})}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \tag{47}$$

From (46), (47) and (45), we have

$$I' = O\left(\frac{1}{\mu_n^{\alpha-\beta-\frac{1}{q}}}\right) + O\left(\psi(n)\right)\left\{\sum_{k=1}^{n}\left(\frac{(\mu_{k+1}-\mu_k)^{1-\frac{1}{q}}}{\mu_k^{(\alpha-\beta-\frac{2}{q})}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \tag{48}$$

$$\begin{aligned} J' &= \left\{\int_0^{\pi} \left(|K_n(u)|u^{\alpha-\beta+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \\ &= \left\{\left(\int_0^{\frac{\pi}{\mu_n}} + \int_{\frac{\pi}{\mu_n}}^{\pi}\right) \left(|K_n(u)|u^{\alpha-\beta+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \\ &\le \left\{\int_0^{\frac{\pi}{\mu_n}} \left(|K_n(u)|u^{\alpha-\beta+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \\ &\quad + \left\{\int_{\frac{\pi}{\mu_n}}^{\pi} \left(|K_n(u)|u^{\alpha-\beta+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \\ &= \left(J_1^1 + J_2^1\right), \quad (say) \end{aligned} \tag{49}$$

Applying Lemma 4.4(a) in J_1^1, we have

$$\begin{aligned} J_1^1 &= \left\{\int_0^{\frac{\pi}{\mu_n}} \left(|K_n(u)|u^{\alpha-\beta+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \\ &= O\left(\frac{1}{\mu_n}\right)\left\{\int_0^{\frac{\pi}{\mu_n}} u^{\frac{q}{q-1}(\alpha-\beta+\frac{1}{q})} du\right\}^{1-\frac{1}{q}} \\ &= O\left(\frac{1}{\mu_n}\right)\left\{\int_0^{\frac{\pi}{\mu_n}} u^{\frac{q}{q-1}(\alpha-\beta+1)-1} du\right\}^{1-\frac{1}{q}} \\ &= O\left(\frac{1}{\mu_n^{\alpha-\beta}}\right) \end{aligned} \tag{50}$$

Applying Lemma 4.4(b) in J_2^1, we have

$$\begin{aligned} J_2^1 &= \left\{\int_{\frac{\pi}{\mu_n}}^{\pi} \left(|K_n(u)|u^{\alpha-\beta+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \\ &= O(\psi(n))\left\{\int_{\frac{\pi}{\mu_n}}^{\pi} \left(u^{\alpha-\beta-2+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \\ &= O(\psi(n))\left\{\sum_{k=1}^{n-1}\int_{\frac{\pi}{\mu_{k+1}}}^{\frac{\pi}{\mu_k}} \left(u^{\alpha-\beta-2+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \\ &= O(\psi(n))\left\{\sum_{k=1}^{n}\int_{\frac{\pi}{\mu_{k+1}}}^{\frac{\pi}{\mu_k}} \left(u^{\alpha-\beta-2+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \end{aligned}$$

Proceeding as in I_2^1, we have

$$J_2^1 = O(\psi(n))\left\{\sum_{k=1}^{n}\left(\frac{(\mu_{k+1}-\mu_k)^{1-\frac{1}{q}}}{\mu_k^{\alpha-\beta-\frac{1}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \tag{51}$$

From (51), (50), (49), we have

$$J^1 = O\left(\frac{1}{\mu_n^{\alpha-\beta}}\right) + O(\psi(n))\left\{\sum_{k=1}^{n}\left(\frac{(\mu_{k+1}-\mu_k)^{1-\frac{1}{q}}}{\mu_k^{\alpha-\beta-\frac{1}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \tag{52}$$

From (44), (48) and (52), we have

$$\begin{aligned} ||w_k(T_{n,\cdot})||_{\beta,q} &= O(1)\left(I' + J'\right) \\ &= O\left(\frac{1}{\mu_n^{\alpha-\beta-\frac{1}{q}}}\right) + O(\psi(n))\left\{\sum_{k=1}^{n}\left(\frac{(\mu_{k+1}-\mu_k)^{1-\frac{1}{q}}}{\mu_k^{(\alpha-\beta-\frac{2}{q})}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \\ &\quad + O\left(\frac{1}{\mu_n^{\alpha-\beta}}\right) + O(\psi(n))\left\{\sum_{k=1}^{n}\left(\frac{(\mu_{k+1}-\mu_k)^{1-\frac{1}{q}}}{\mu_k^{\alpha-\beta-\frac{1}{q}}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \\ &= O\left(\frac{1}{\mu_n^{\alpha-\beta-\frac{1}{q}}}\right) + O(\psi(n))\left\{\sum_{k=1}^{n}\left(\frac{(\mu_{k+1}-\mu_k)^{1-\frac{1}{q}}}{\mu_k^{(\alpha-\beta-\frac{2}{q})}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \end{aligned} \tag{53}$$

From (53), (43) and (39), we have

$$||T_n(\cdot)||_{B_q^\beta(L_p)} = O\left(\frac{1}{\mu_n^{\alpha-\beta-\frac{1}{q}}}\right) + O(\psi(n))\left\{\sum_{k=1}^{n}\left(\frac{(\mu_{k+1}-\mu_k)^{1-\frac{1}{q}}}{\mu_k^{(\alpha-\beta-\frac{2}{q})}}\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}} \tag{54}$$

This complete the proof of Case 1.

Case 2 ($q = \infty$)

Now, we consider the case $q = \infty$

$$||T_n(\cdot)||_{B^{\beta}_{\infty}(L_p)} = ||T_n(\cdot)||_p + ||w_k(T_n, \cdot)||_{\beta,\infty} \tag{55}$$

We know $T_n(x) = \frac{1}{\pi}\int_0^{\pi} \phi_x(u)K_n(u)du$.

Applying Lemma 4.1(iii), we have

$$\begin{aligned}
||T_n(\cdot)||_p &\leq \frac{1}{\pi}\int_0^{\pi} ||\phi.(u)||_p K_n(u)du \\
&\leq \frac{2}{\pi}\int_0^{\pi} |K_n(u)|w_k(f,u)_p du \\
&= O(1)\int_0^{\pi} |K_n(u)|u^{\alpha}du \quad (\textit{by the hypothesis}) \\
&= O(1)\left[\int_0^{\frac{\pi}{\mu_n}} |K_n(u)|u^{\alpha}du + \int_{\frac{\pi}{\mu_n}}^{\pi} |K_n(u)|u^{\alpha}du\right] \\
&= O(1)[I^{II} + J^{II}], \quad (say)
\end{aligned} \tag{56}$$

Applying Lemma 4.4(a) in I^{II}, we have

$$\begin{aligned}
I^{II} &= \int_0^{\frac{\pi}{\mu_n}} |K_n(u)|u^{\alpha}du \\
&= O(\mu_n)\int_0^{\frac{\pi}{\mu_n}} u^{\alpha}du \\
&= O\left(\frac{1}{\mu_n^{\alpha}}\right)
\end{aligned} \tag{57}$$

Applying Lemma 4.4(b) in J^{II}, we have

$$\begin{aligned}
J^{II} &= \int_{\frac{\pi}{\mu_n}}^{\pi} |K_n(u)|u^{\alpha}du \\
&= O(\psi(n))\int_{\frac{\pi}{\mu_n}}^{\pi} u^{\alpha-2}du \\
&= O(\psi(n))\sum_{k=1}^{n-1}\int_{\frac{\pi}{\mu_{k+1}}}^{\frac{\pi}{\mu_k}} u^{\alpha-2}du \\
&= O(\psi(n))\sum_{k=1}^{n}\int_{\frac{\pi}{\mu_{k+1}}}^{\frac{\pi}{\mu_k}} u^{\alpha-2}du
\end{aligned}$$

Proceeding as in I'_2, we have

$$= O(\psi(n))\sum_{k=1}^{n}\left(\frac{\mu_{k+1}-\mu_k}{\mu_k^{\alpha}}\right) \tag{58}$$

From (56), (57) and (58), we have

$$||T_n(\cdot)||_p = O\left(\frac{1}{\mu_n^{\alpha}}\right) + O(\psi(n))\sum_{k=1}^{n}\left(\frac{\mu_{k+1}-\mu_k}{\mu_k^{\alpha}}\right) \tag{59}$$

Again,

$$\begin{aligned}
||w_k(T_n,\cdot)||_{\beta,q} &= \sup_{t>0}\frac{||T_n(\cdot,t)||_p}{t^{\beta}} \\
&= \sup_{t>0}\frac{t^{-\beta}}{\pi}\left[\int_0^{\pi}\left|\int_0^{\pi}\phi(x,t,u)K_n(u)du\right|^p dx\right]^{\frac{1}{p}}
\end{aligned} \tag{60}$$

Applying generalised Minkowski's inequality, we have

$$\begin{aligned} ||w_k(T_n,\cdot)||_{\beta,q} &= \sup_{t>0}\frac{t^{-\beta}}{\pi}\int_0^\pi du\left\{\int_0^\pi |\phi(x,t,u)|^p|K_n(u)|^p dx\right\}^{\frac{1}{p}} \\ &= \sup_{t>0}\frac{t^{-\beta}}{\pi}\int_0^\pi |K_n(u)|||\phi(\cdot,t,u)||_p du \\ &\leq \frac{1}{\pi}\int_0^\pi |K_n(u)|du \sup_{t>0} t^{-\beta}||\phi(\cdot,t,u)||_p \end{aligned} \tag{61}$$

Using Lemma 4.3, we have

$$\begin{aligned} ||w_k(T_n,\cdot)||_{\beta,\infty} &\leq O(1)\int_0^\pi u^{\alpha-\beta}|K_n(u)|du \\ &= O(1)\left(\int_0^{\frac{\pi}{\mu_n}} + \int_{\frac{\pi}{\mu_n}}^\pi\right) u^{\alpha-\beta}|K_n(u)|du \\ &= O(1)\left[\int_0^{\frac{\pi}{\mu_n}} u^{\alpha-\beta}|K_n(u)|du + \int_{\frac{\pi}{\mu_n}}^\pi u^{\alpha-\beta}|K_n(u)|du\right] \\ &= O(1)[I^{III} + J^{III}], \quad (say) \end{aligned} \tag{62}$$

Using Lemma 4.4(a) in I^{III}, we have

$$\begin{aligned} I^{III} &= \int_0^{\frac{\pi}{\mu_n}} |K_n(u)|u^{\alpha-\beta}du \\ &= O(\mu_n)\int_0^{\frac{\pi}{\mu_n}} u^{\alpha-\beta}du \\ &= O\left(\frac{1}{\mu_n^{\alpha-\beta}}\right) \end{aligned} \tag{63}$$

Using Lemma 4.4(b) in J^{III}, we have

$$\begin{aligned} J^{III} &= \int_{\frac{\pi}{\mu_n}}^\pi u^{\alpha-\beta}|K_n(u)|du \\ &= O(\psi(n))\int_{\frac{\pi}{\mu_n}}^\pi u^{\alpha-\beta-2}du \\ &= O(\psi(n))\sum_{k=1}^{n-1}\int_{\frac{\pi}{\mu_{k+1}}}^{\frac{\pi}{\mu_k}} u^{\alpha-\beta-2}du \\ &= O(\psi(n))\sum_{k=1}^{n}\int_{\frac{\pi}{\mu_{k+1}}}^{\frac{\pi}{\mu_k}} u^{\alpha-\beta-2}du \\ &= O(\psi(n))\sum_{k=1}^{n}\left(\frac{\mu_{k+1}-\mu_k}{\mu_k^{\alpha-\beta}}\right) \end{aligned} \tag{64}$$

From (62), (63) and (64), we have

$$||w_k(T_n,\cdot)||_{\beta,\infty} = O\left(\frac{1}{\mu_n^{\alpha-\beta}}\right) + O(\psi(n))\sum_{k=1}^{n}\left(\frac{\mu_{k+1}-\mu_k}{\mu_k^{\alpha-\beta}}\right) \tag{65}$$

From (55),(59) and (65), we have

$$||T_n(\cdot)||_{B^\beta_\infty(L_p)} = O\left(\frac{1}{\mu_n^{\alpha-\beta}}\right) + O(\psi(n))\sum_{k=1}^{n}\left(\frac{\mu_{k+1}-\mu_k}{\mu_k^{\alpha-\beta}}\right) \tag{66}$$

This completes the Case 2.

Combining the Case 1 and Case 2, we obtain the proof of the theorem.

Acknowledgement

We thank Dr.B.K.Ray for his valuable suggestions.

References

[1] G.Das, T.Ghosh and B,K.Ray, *Degree of Approximation of function by their Fourier series in the generalized Holder metric*, Proc. Indian Acad. Sci. (Math.Sci), 106(1996), 139-153.

[2] A.Devore Ronald and G.Lorentz, *Constructive approximation*, Springler- Verlay, Berlin Heidelberg New York, (1993).

[3] S.Prossdorf, *Zur Konvergenz der Fourier richen Holder stetiger Funktionen*, Math. Nachar, 69(1975), 7-14.

[4] P.Wojtaszczyk, *A Mathematical Introduction to Wevlets*, London Mathematical Society students texts 37, Cambridge University Press, New York, (1997).

[5] A.Zygmund, *Smooth fuctions*, Duke Math. Jounal, 12(1945), 47-56.

[6] A.Zygmund, *Trigonometric series*, vols I & II combined, Cambridge Univ. Press, New York, (1993).