



Comparison of New Integral Transform Aboodh Transform and Domain Decomposition Method

Research Article

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Abstract: In this paper, we present a comparative study between Adomain decomposition method and the new integral transform Aboodh Transform. We use the methods to solve the linear Partial differential equations with constant coefficients.

Keywords: Adomain Decomposition Method, Aboodh Transform, Linear Partial Differential Equation.

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1. Introduction

Linear and Nonlinear partial differential equations can be found in wide variety scientific and engineering applications [1–5]. Many important mathematical models can be expressed in terms of linear and nonlinear partial differential equations. Linear and Nonlinear Partial differential equations are generally difficult to be solved and their exact solution are difficult to be obtained. The exact solution and numerical solutions of this kind of equations play an important role in physical science and in engineering fields; therefore, there have been attempts to develop new techniques for obtaining analytical solutions which reasonably approximate the exact solutions.

In recent years, many research workers have paid attention to find the solutions of linear and nonlinear differential equations by using various methods. Among these are, the variational iteration method [6–10] the homotopy perturbation method, the differential transform method (2008) and Elzaki Transform (Tarig and Salih, (2011), (2012)) [11–16], the Adomain decomposition method, etc. the Adomain decomposition method (ADM) which was introduced by G. Adomian [1–5] in the 1980s in order to solve linear and nonlinear differential equations.

Aboodh Transform is derived from the classical Fourier integral. Based on the mathematical simplicity of the Aboodh Transform and its fundamental properties, Aboodh Transform was introduced by Khalid Aboodh in 2013, to facilitate the process of solving ordinary and partial differential equations in the time domain. This transformation has deeper connection with the Laplace and Elzaki Transform [17, 18]. The main idea of this paper is to introduce a comparative study to solve linear partial differential by using aboodh transform and Adomian decomposition method (ADM).

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2. Aboodh Transform

A new transform called the Aboodh transform defined for function of exponential order we consider functions in the set A, defined by:

$$A = \{f(t) : \exists M, k_1, k_2 > 0, |f(t)| < Me^{-vt}\} \tag{1}$$

For a given function in the set M must be finite number, k_1, k_2 may be finite or infinite. Aboodh transform which is defined by the integral equation

$$A[f(t)] = K(v) = \frac{1}{v} \int_0^\infty f(t) e^{-vt} dt, \quad t \geq 0, \quad k_1 \leq v \leq k_2 \tag{2}$$

2.1. Aboodh Transform of Some Functions

$$\begin{aligned} A(1) &= \frac{1}{v^2}, & A(t^n) &= \frac{n!}{v^{n+2}}, \\ A(e^{at}) &= \frac{1}{v^2 - av}, & A(e^{-at}) &= \frac{1}{v^2 + av}, \\ A(\sin(at)) &= \frac{a}{v(v^2 + a^2)}, & A(\cos(at)) &= \frac{1}{(v^2 + a^2)}, \\ A(\sinh(at)) &= \frac{a}{v(v^2 - a^2)}, & A(\cosh(at)) &= \frac{1}{(v^2 - a^2)}. \end{aligned}$$

2.2. Aboodh Transform of Derivatives

$$\begin{aligned} A[f'(t)] &= vK(v) - \frac{f(0)}{v}, \\ A[f''(t)] &= v^2K(v) - \frac{f'(0)}{v} - f(0), \\ A[f^{(n)}(t)] &= v^n K(v) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{2-n+k}}. \end{aligned}$$

2.3. Aboodh Transform of Some Partial Derivative

$$\begin{aligned} A(u(x, t)) &= K(x, v), & A\left(\frac{\partial u(x, t)}{\partial x}\right) &= K'(x, v), \\ A\left(\frac{\partial^2 u(x, t)}{\partial x^2}\right) &= K''(x, v), & A\left(\frac{\partial^n u(x, t)}{\partial x^n}\right) &= K^{(n)}(x, v), \\ A\left(\frac{\partial u(x, t)}{\partial t}\right) &= v K(x, v) - \frac{u(x, 0)}{v}, & A\left(\frac{\partial^2 u(x, t)}{\partial t^2}\right) &= v^2 K(x, v) - \frac{\partial u(x, 0)}{\partial t} - u(x, 0) \end{aligned}$$

3. Adomain Decomposition Method

Adomain decomposition method [19, 20] define the unknown function by an infinite series

$$u(x) = \sum_{n=0}^\infty u_n(x), \tag{3}$$

Where the components $u_n(x)$ are usually determined recurrently. The nonlinear operator $F(u)$ can be decomposed into an infinite series of polynomials given by

$$F(u) = \sum_{n=0}^{\infty} A_n. \tag{4}$$

Where A_n are the so called Adomain polynomial of $u_0, u_1, u_2, \dots, u_n$ defined by

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} [F(\lambda^i u_i)]_{\lambda=0}, \quad n = 0, 1, 2, \dots \tag{5}$$

It is now well known that these polynomials can be generated for all classes of nonlinear according to specific algorithms defined by (5). Recently, an alternative algorithm for constructing Adomain polynomials has been developed by Wazwaz [22]. This powerful technique handles both linear and nonlinear equations in unified manner without any need for the so called Adomain polynomials .however, Adomin decomposition method provides the component of the exact solution, where these components should follow the summation given in (3), whereas ADM requires the evaluation of the Adomain polynomials that mostly require tedious algebraic work [21].

4. Applications

In this section we introduce some examples to explain the method

Example 4.1. Find the solution of the first order initial value problem:

$$\frac{\partial u(x, t)}{\partial t} + u(x, t) = 0, \quad u(x, 0) = 1 \tag{6}$$

And $u(x, t)$ is bounded for $x > 0, \quad t > 0$.

4.1. Use Aboodh Transform

Let $K(x, v)$ be the Aboodh transform of $u(x, t)$. Then, taking the Aboodh transform of Equation (6) we have,

$$vK(x, v) - \frac{u(x, 0)}{v} + K(x, v) = 0 \tag{7}$$

by applying the initial condition, we get

$$vK(x, v) - \frac{1}{v} + K(x, v) = 0 \tag{8}$$

and

$$K(x, v) = \frac{\frac{1}{v}}{(v+1)} = \frac{1}{v(v+1)} \tag{9}$$

If we take the inverse Aboodh transform for Equation (9), we obtain solution of Equation (6) in the form.

$$u(x, t) = e^{-t}. \tag{10}$$

4.2. Use Adomain Decomposition Method

We first rewrite Equation (6) in an operator is

$$L_t u = -u \tag{11}$$

Where the differential operators L_t , is

$$L_t(.) = \frac{\partial}{\partial t} (.) \tag{12}$$

The inverse L_t^{-1} are assumed as an integral operator given by

$$L_t^{-1}(\cdot) = \int_0^t (\cdot) dt \tag{13}$$

Applying the inverse operator L_t^{-1} on both sides of Equation (11) and using initial condition we find

$$\begin{aligned} u_0(x, t) &= 1 \\ u_{n+1}(x, t) &= -L_t^{-1}[u_n], \quad n \geq 0 \end{aligned} \tag{14}$$

Evaluating the components $u_n(x, t)$, $n = 0, 1, 2, \dots$

$$\begin{aligned} u_1 &= -L_t^{-1}[u_0(x, t)] = -\int_0^t (1) dt = -t \\ u_2 &= -L_t^{-1}[u_1(x, t)] = -\int_0^t -t dt = \frac{t^2}{2!} \\ u_3 &= -L_t^{-1}[u_2(x, t)] = -\int_0^t \frac{t^2}{2} dt = -\frac{t^3}{3!} \dots \end{aligned} \tag{15}$$

Finally, using Equation (3) we obtain the solution in series form:

$$u(x, t) = u_0 + u_1 + u_2 + \dots$$

That is

$$u(x, t) = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots$$

The exact solution is given by

$$u(x, t) = e^{-t} \tag{16}$$

Example 4.2. Consider

$$u_{xx} - u_{tt} = 0 \quad \text{with } 0 \leq x \leq \pi \text{ and } t \geq 0 \tag{17}$$

$$u(x, 0) = \sin x, \quad u(0, t) = 0, \quad u_t(x, 0) = 0, \quad u(\pi, t) = 0$$

4.3. Use Aboodh Transform

Let $k(v)$ be the Aboodh transform of $u(x, t)$. Then, taking the Aboodh transform of Equation (17) we have:

$$v^2 k(x, v) - \frac{1}{v} u_t(x, 0) - u(x, 0) - k''(x, v) = 0 \tag{18}$$

by applying the initial conditions we get

$$v^2 k(x, v) - \sin x - k''(x, v) = 0$$

This is the second order differential equation have the particular, solution in the form

$$k(x, v) = \frac{-\sin x}{D^2 - v^2} = \frac{-\sin x}{-1 - v^2} = \frac{\sin x}{1 + v^2}, \quad \text{where } D^2 = \frac{d^2}{dx^2} \tag{19}$$

If we take the inverse Aboodh transform for Equation (19), we obtain solution of Equation (17) in the form

$$u(x, t) = \cos t \sin x.$$

4.4. Use Adomain Decomposition Method

We first rewrite Equation (17) in an operator is

$$L_x u = L_t u \tag{20}$$

Where the differential operators L_t, L_x are

$$L_t (\cdot) = \frac{\partial^2}{\partial t^2} (\cdot), \quad L_x (\cdot) = \frac{\partial^2}{\partial x^2} (\cdot) \tag{21}$$

The inverse L_t^{-1} are assumed as an integral operator given by

$$L_t^{-1} (\cdot) = \int_0^t \int_0^t (\cdot) dt dt \tag{22}$$

Applying the inverse operator L_t^{-1} on both sides of (20) and using initial condition we find

$$\begin{aligned} u_0(x, t) &= \sin x \\ u_{n+1}(x, t) &= \int_0^t \int_0^t \frac{\partial^2 u_n(x, t)}{\partial x^2} dt dt \end{aligned} \tag{23}$$

Evaluating the components $u_n(x, t), n = 0, 1, 2, \dots$, then

$$u_1(x, t) = \int_0^t \int_0^t \frac{\partial^2 u_0(x, t)}{\partial x^2} dt dt = \int_0^t \int_0^t \sin x dt dt = -\frac{t^2}{2!} \sin x$$

and

$$u_2(x, t) = \int_0^t \int_0^t \frac{\partial^2 u_1(x, t)}{\partial x^2} dt dt = \int_0^t \int_0^t \frac{\partial^2}{\partial x^2} \left(-\frac{t^2}{2!} \sin x \right) dt dt = \frac{t^4}{4!} \sin x$$

and

$$u_3(x, t) = -\frac{t^6}{6!} \sin x, \quad u_4(x, t) = \frac{t^8}{8!} \sin x \dots \tag{24}$$

Finally, using Equation (3) we obtain the solution in series form:

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} u_n(x, t) \\ &= \sin x - \frac{t^2}{2!} \sin x + \frac{t^4}{4!} \sin x - \frac{t^6}{6!} \sin x + \frac{t^8}{8!} \sin x - \dots \\ &= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - \dots \right) \sin x \end{aligned}$$

The exact solution is given by

$$u(x, t) = \cos t \sin x \tag{25}$$

Example 4.3. Consider the homogeneous heat equation in one dimension in a normalized form:

$$u_t = u_{xx}, \tag{26}$$

$$u(x, 0) = \sin \frac{\pi}{l} x, \quad u(0, t) = u(l, t) = 0$$

4.5. Use Aboodh Transform

Let $k(v)$ be the Aboodh transform of $u(x,t)$. Then, taking the Aboodh transform of Equation (26) we have:

$$\begin{aligned}
 K''(x, v) - \left[vK(x, v) - \frac{u(x, 0)}{v} \right] &= 0 \\
 K''(x, v) - vK(x, v) &= -\frac{1}{v} \sin \frac{\pi}{l} x
 \end{aligned}
 \tag{27}$$

Solve for $K(x, v)$ we find that the particular solution is

$$K(x, v) = \frac{-\frac{1}{v} \sin \frac{\pi}{l} x}{D^2 - v} = \frac{-\frac{1}{v} \sin \frac{\pi}{l} x}{-\left(\frac{\pi}{l}\right)^2 - v} = \frac{1}{v^2 + \frac{\pi^2}{l^2} v} \sin \frac{\pi}{l} x
 \tag{28}$$

And similarly if we take the inverse Aboodh transform for Equation (28), we obtain the Solution of Equation (26) in the form

$$u(x, t) = \sin \frac{\pi}{l} x e^{\frac{\pi^2}{l^2} t}
 \tag{29}$$

4.6. Use Adomain Decomposition Method

We first rewrite Equation (26) in an operator is

$$L_t u = L_x u
 \tag{30}$$

Where the differential operators L_t, L_x are

$$L_t(\cdot) = \frac{\partial}{\partial t}(\cdot), \quad L_x(\cdot) = \frac{\partial^2}{\partial x^2}(\cdot)
 \tag{31}$$

The inverse L_t^{-1} are assumed as an integral operator given by

$$L_t^{-1}(\cdot) = \int_0^t (\cdot) dt
 \tag{32}$$

Applying the inverse operator L_t^{-1} on both sides of Equation (30) and using initial condition we find

$$\begin{aligned}
 u_0(x, t) &= \sin \frac{\pi}{l} x \\
 u_{n+1}(x, t) &= \int_0^t \frac{\partial^2 u_n(x, t)}{\partial x^2} dt
 \end{aligned}
 \tag{33}$$

Evaluating the components $u_n(x, t), n = 0, 1, 2, \dots$, then

$$u_1(x, t) = \int_0^t \frac{\partial^2 u_0(x, t)}{\partial x^2} dt = \int_0^t \frac{\partial^2 (\sin \frac{\pi}{l} x)}{\partial x^2} dt = -\frac{\pi^2}{l^2} t (\sin \frac{\pi}{l} x)$$

and

$$u_2(x, t) = \int_0^t \frac{\partial^2 (-\frac{\pi^2}{l^2} t (\sin \frac{\pi}{l} x))}{\partial x^2} dt = \frac{\pi^4}{l^4} \frac{t^2}{2!} (\sin \frac{\pi}{l} x)$$

and

$$u_3(x, t) = -\frac{\pi^6}{l^6} \frac{t^3}{3!} (\sin \frac{\pi}{l} x)
 \tag{34}$$

Finally, using Equation (3) we obtain the solution in series form:

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} u_n(x, t) \\ &= \sin \frac{\pi}{l} x - \frac{\pi^2}{l^2} t (\sin \frac{\pi}{l} x) + \frac{\pi^4}{l^4} \frac{t^2}{2!} (\sin \frac{\pi}{l} x) - \frac{\pi^6}{l^6} \frac{t^3}{3!} (\sin \frac{\pi}{l} x) + \dots \\ &= \sin \frac{\pi}{l} x \left[1 - \frac{\pi^2}{l^2} t + \frac{\pi^4}{l^4} \frac{t^2}{2!} - \frac{\pi^6}{l^6} \frac{t^3}{3!} + \dots \right] \end{aligned}$$

The exact solution is given by

$$u(x, t) = \sin \frac{\pi}{l} x e^{\frac{\pi^2}{l^2} t} \tag{35}$$

Example 4.4. Consider the homogeneous heat equation in one dimension in a normalized form:

$$u_{tt} + u_{xx} = 0 \tag{36}$$

$$u(x, 0) = 0, \quad u_t(x, 0) = \cos x; \quad x, t > 0$$

4.7. Use Aboodh Transform

Equation (36) we can obtain

$$\begin{aligned} K''(x, v) + \left[v^2 K(x, v) - \frac{u_t(x, 0)}{v} - u(x, 0) \right] &= 0 \\ K''(x, v) + v^2 K(x, v) &= \frac{1}{v} \cos x \end{aligned} \tag{37}$$

Solve for $K(x, v)$ we find that the particular solution is

$$K(x, v) = \frac{\frac{1}{v} \cos x}{D^2 + v^2} = \frac{\frac{1}{v} \cos x}{-1 + v^2} = \frac{\cos x}{v(v^2 - 1)} \tag{38}$$

And similarly if we take the inverse Aboodh transform for Equation (38), we obtain the Solution of Equation (36) in the form

$$u(x, t) = \sinh x \cos x \tag{39}$$

4.8. Use Adomain Decomposition Method

We first rewrite Equation (36) in an operator is

$$L_t u = -L_x u \tag{40}$$

Where the differential operators L_t, L_x are

$$L_t(\cdot) = \frac{\partial^2}{\partial t^2}(\cdot), \quad L_x(\cdot) = \frac{\partial^2}{\partial x^2}(\cdot)$$

The inverse L_t^{-1} are assumed as an integral operator given by

$$L_t^{-1}(\cdot) = \int_0^t (\cdot) dt \tag{41}$$

Applying the inverse operator L_t^{-1} on both sides of (40) and using initial condition we find

$$u_0(x, t) = t \cos x$$

$$u_{n+1}(x, t) = - \int_0^t \int_0^t \frac{\partial^2 u_n(x, t)}{\partial x^2} dt dt \quad (42)$$

Evaluating the components $u_n(x, t)$, $n = 0, 1, 2, \dots$, then

$$\begin{aligned} u_1(x, t) &= - \int_0^t \int_0^t \frac{\partial^2 u_0(x, t)}{\partial x^2} dt dt \\ &= - \int_0^t \int_0^t \frac{\partial^2 (t \cos x)}{\partial x^2} dt dt = - \frac{t^3}{3!} \cos x \quad \text{and} \\ u_2(x, t) &= - \int_0^t \int_0^t \frac{\partial^2 (-\frac{t^3}{3!} \cos x)}{\partial x^2} dt dt = \frac{t^5}{5!} \cos x \end{aligned}$$

and

$$u_3(x, t) = - \frac{t^7}{7!} \cos x \quad (43)$$

Finally, using Equation (3) we obtain the solution in series form:

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} u_n(x, t) = t \cos x - \frac{t^3}{3!} \cos x + \frac{t^5}{5!} \cos x - \frac{t^7}{7!} \cos x + \dots \\ &= \left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots \right] \cos x \end{aligned}$$

The exact solution is given by

$$u(x, t) = \sinh x \cos x \quad (44)$$

5. Conclusion

In this paper, solved linear partial differential equations by two different transformations, one is Aboodh Integral Transformations and the other one is Adomain decomposition method and solutions obtained by these two transformations are compared, An important conclusion can made here. Adomain decomposition methods for solving linear partial differential equations, the same problems are solved by Aboodh Transform. Adomain decomposition method provides the components of exact solution, However, Application of the new transform Aboodh Transform to Solutions of linear PDEs has been demonstrated.

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