



On the Notion of Uniform Integrability and Mean Convergence Theorem for Fuzzy Random Variables

Research Article

J.Earnest Lazarus Piriya Kumar¹ and R.Deepa^{2*}

1 Department of Mathematics, Tranquebar Bishop Manickam Lutheran College, Porayar, Tamilnadu, India.

2 Department of Mathematics, E.G.S.Pillay Engineering College, Nagapattinam, Tamilnadu, India.

Abstract: In this paper the convergence criterion of fuzzy random variable is investigated. An attempt is made to study the equivalence relation of uniform integrability of fuzzy random variables. Mean convergence theorem, Lebesgue dominated convergence theorem and Mean Ergodic theorem for the case of fuzzy random variable are introduced.

Keywords: Fuzzy random variables, uniform integrability, Fuzzy numbers, Fuzzy valued functions.

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1. Introduction

Convergence concepts are the foundation of mathematical analysis while the convergence of fuzzy sets is the foundation of fuzzy analysis. Since the introduction of fuzzy sets by Zadeh [5] researchers have been concerned with the calculus of functions and the definition and generalization of convergence concepts in the domain of fuzzy sets and systems. We have defined the concept of uniform integrability for the case of fuzzy random variables to derive. Mean convergence theorem for fuzzy random variables.

The notion of a fuzzy random variables was introduced as a natural generalization of random set in order to represent associations between the outcomes of random experiment and non-statistical in exact data.

Limit theorems for random sets and fuzzy random variables have received much attention in recent years because of its application in several applied fields such as Mathematical economics system analysis and Stochastic Control theory.

Klementet.al., [2] provided a good intuition about the central limit theorem for fuzzy random variables which generalizes the central limit theorem for random sets. Fuzzy random variables have been designed to deal with situations in which both random performance and fuzzy perception must be considered.

The purpose of this paper is to generalize the convergence theorems of the basic probability theory to fuzzy random variables as well as some fundamental theorems in the light of uniform integrability. The study of the theory of fuzzy random variables was proposed by Kwakernack [1] and Puri and Relescu [3, 4].

* E-mail: deeparajan_1983@yahoo.co.in

2. Fuzzy Random Variables

The concepts of a fuzzy random variables and its expectation were introduced by Puri and Ralescu [? ?]. Let (Ω, \mathcal{A}, P) be a complete probability space, and fuzzy random variable is a Borel measurable function. If $X : \Omega \rightarrow F(\mathbb{R})$ is a fuzzy number valued function, where $F(\mathbb{R})$ is a family of all fuzzy numbers, and B is the subset of \mathbb{R} then $X^{-1}(B)$ denotes the fuzzy subset of Ω defined by

$$X^{-1}(B)(W) = \sup_{x \in B} X(w)(x) \text{ for each } w \in \Omega$$

The function $X : \Omega \rightarrow F(\mathbb{R})$ is called a fuzzy random variable. If for every closed subset B of \mathbb{R} , the fuzzy set $X^{-1}(B)$ is measurable when considered as a function from Ω to $[0, 1]$. If we denote

$$X(w) = \{(X_{\alpha}^{-}(w), X_{\alpha}^{+}(w)) | 0 \leq \alpha \leq 1\}$$

then it is well known that X is a fuzzy random variable if and only if for each $\alpha \in [0, 1]$, X_{α}^{-} and X_{α}^{+} are random variables.

3. Uniform Integrability and Mean Convergence Theorem

Let $\{x_n, n \geq 1\}$ and x are fuzzy random variables on a probability space (Ω, \mathcal{A}, P)

Definition 3.1. A sequence of fuzzy random variables $\{x_n, n \geq 1\}$ is called uniformly integrable if for every $\epsilon > 0$ there exists a $\gamma > 0$ such that $\delta > 0$ such that

$$\sup_n \int_A [(x_n)_{\alpha}^{-} V(x_n)_{\alpha}^{+}] dP < \epsilon \tag{1}$$

$$\text{Whenever } P(A) < \delta \text{ and } \sup_{n \geq 1} E(\alpha((x_n)_{\alpha}^{-} V(x_n)_{\alpha}^{+})) \leq C < \infty \tag{2}$$

are satisfied, where $A \in \mathcal{A}$.

Theorem 3.2 (Equivalence Relation of Uniform Integrability). Let $\{x_n\}$ be a sequence of a fuzzy random variables. Then $\{x_n\}$ is uniformly integrable if and only if

$$\lim_{b \rightarrow \infty} \int_{(|(x_n)_{\alpha}^{-} V(x_n)_{\alpha}^{+}|) \geq b} \alpha(|(x_n)_{\alpha}^{-} V(x_n)_{\alpha}^{+}|) dP = 0 \tag{3}$$

uniformly in n .

Proof. Let $\{x_n\}$ is uniformly integrable. Then $E(\alpha(x_n)_{\alpha}^{-} V(x_n)_{\alpha}^{+}) \leq C < \infty$. Hence

$$\begin{aligned} P(\alpha|(x_n)_{\alpha}^{-} V(x_n)_{\alpha}^{+}|) \geq b & \\ & \leq \frac{1}{b} E(\alpha(|(x_n)_{\alpha}^{-} V(x_n)_{\alpha}^{+}|)) \\ & \leq \frac{C}{b} \rightarrow 0 \text{ as } b \rightarrow \infty \text{ uniformly in } n. \end{aligned}$$

Therefore from (1),

$$\sup_{n \geq 1} \int_{(|(x_n)_{\alpha}^{-} V(x_n)_{\alpha}^{+}|) \geq b} (\alpha|(x_n)_{\alpha}^{-} V(x_n)_{\alpha}^{+}|) dP \leq \epsilon.$$

If b is large enough for a given $\epsilon > 0$. So (1) and (2) implies (3)

Conversely for $\epsilon > 0$ let

$$\int_{(|(x_n)_\alpha^- V(x_n)_\alpha^+)| > b} (\alpha |(x_n)_\alpha^- V(x_n)_\alpha^+|) dP \leq \epsilon$$

for some large $b > 0$ and for every $n \geq 1$. Then

$$\begin{aligned} E(\alpha |(x_n)_\alpha^- V(x_n)_\alpha^+|) &= \int_{(|(x_n)_\alpha^- V(x_n)_\alpha^+)| < b} \alpha |(x_n)_\alpha^- V(x_n)_\alpha^+| dP \\ &\quad + \int_{(|(x_n)_\alpha^- V(x_n)_\alpha^+)| \geq b} (\alpha |(x_n)_\alpha^- V(x_n)_\alpha^+|) dP \\ &< b + \epsilon \text{ from (3)} \\ &< \infty \text{ for all } n \geq 1. \end{aligned}$$

Put $\delta = \frac{\epsilon}{b}$, then for $A \in \mathcal{A}$ with $P(A) < \delta$.

$$\begin{aligned} \int_A \alpha |(x_n)_\alpha^- V(x_n)_\alpha^+| dP &= \int_{A(|(x_n)_\alpha^- V(x_n)_\alpha^+)| < b} (\alpha |(x_n)_\alpha^- V(x_n)_\alpha^+|) dP \\ &= \int_{A(|(x_n)_\alpha^- - (x_n)_\alpha^+| \geq b)} \alpha |(x_n)_\alpha^- V(x_n)_\alpha^+| dP \\ &< bP(A) + \int_{(|(x_n)_\alpha^- V(x_n)_\alpha^+)| \geq b} \alpha |(x_n)_\alpha^- V(x_n)_\alpha^+| dP \\ &< b \frac{\epsilon}{b} + \epsilon \\ &= 2\epsilon \end{aligned}$$

uniformly in n . So (1) holds. □

Definition 3.3. A measurable fuzzy function f is said to be in $L_p(0 < p < \infty)$ if

$$\int |(f)_\alpha^- v(f)_\alpha^+|^p d\mu < \infty.$$

Lemma 3.4. Let $\{x_n\}$ be a sequence of a fuzzy random variables which converges in mean (L_1) to a fuzzy random variable X . Then $E(\alpha |(x_n)_\alpha^- v(x_n)_\alpha^+|) < \infty$.

Proof.

$$\begin{aligned} E(\alpha |(x_n)_\alpha^- v(x_n)_\alpha^+|) &\leq E(\alpha |(x_n)_\alpha^- v(x_n)_\alpha^+|) \\ &\quad + E(\alpha |(x_n)_\alpha^- - (x_n)_\alpha^+ - v(x_n)_\alpha^+ - (x_n)_\alpha^+|) \text{ (Triangle the quality)} \\ &< \infty. \end{aligned}$$

□

4. Theorem Mean Convergence Theorem

Theorem 4.1. Let $\{x_n\}$ be a sequence of fuzzy random variables (integrable) and $x_n \xrightarrow{P} X$. Then $\{x_n\}$ converges in mean if and only if $\{x_n\}$ is uniformly integrable.

Proof. First note that the limiting fuzzy random variable in our convergence is unique. If possible let $x_n \rightarrow x^1$ in L1, then $x_n \xrightarrow{P} x^1$ also. But $x_n \xrightarrow{P} x$ and hence $X = X^1$ also. Now $E(\alpha | (x_n)_\alpha^- - (x_\alpha)^- v(x_n)_\alpha^+ - (x_\alpha)^+ |) \rightarrow 0$ as $n \rightarrow \infty$ and $E(\alpha | (x_n)_\alpha^- v(x_\alpha)^+ |) < \infty$. We need to prove that $x_n \xrightarrow{L_i} x$ implies $\{x_n\}$ is uniformly integrable. Now $\sup_n E(\alpha | (x_n)_\alpha^- - (x_\alpha)^- v(x_n)_\alpha^+ - (x_\alpha)^+ |) + E(\alpha | (x_\alpha)^- v(x_\alpha)^+ |) < \infty$. Let $\epsilon > 0$. For $A \in \mathcal{A}$, $P(A) < \delta$ and $n \geq 0$,

$$\begin{aligned} \int_A \alpha | (x_n)_\alpha^- v(x_n)_\alpha^+ | dP &\leq \int_A \alpha (| (x_n)_\alpha^- - (x_\alpha)^- v(x_n)_\alpha^+ - (x_\alpha)^+ |) dP \\ &\quad + \int_A \alpha | (x_\alpha)^- v(x_\alpha)^+ | dP \quad (\text{Triangle inequality}) \\ &\leq \int_A \alpha | (x_\alpha)^- v(x_\alpha)^+ | dP \\ &\leq 2\epsilon = \epsilon^* \end{aligned}$$

So we have in other words $\int_A \alpha | (x_n)_\alpha^- v(x_n)_\alpha^+ | dP < \epsilon^*$ (> 0) if $n \geq n_0$ and $P(A) < \delta$ since $E(\alpha | (x_n)_\alpha^- v(x_n)_\alpha^+ |) < \infty$, $\int_A \alpha | (x_n)_\alpha^- v(x_n)_\alpha^+ | dP < \epsilon_n$ for a fixed $n \geq 1$ and $P(A) < \delta$.

Conversely let $\{x_n\}$ is uniformly integrable. Then $E(\alpha | (x_n)_\alpha^- v(x_n)_\alpha^+ |) < C < \infty$. Since $x_n \xrightarrow{P} x$ there exists a subsequence $\{x_{nk}\}$ such that $|x_{nk}| \rightarrow |\alpha|$ a.s. By Fatous Lemma

$$\begin{aligned} E(\alpha | (x_n)_\alpha^- v(x_n)_\alpha^+ |) &= E(\liminf_{k \rightarrow \infty} \alpha | (x_{nk})_\alpha^- v(x_{nk})_\alpha^+ |) \\ &= E(\alpha \liminf_{k \rightarrow \infty} | (x_{nk})_\alpha^- v(x_{nk})_\alpha^+ |) \\ &= \liminf_{k \rightarrow \infty} E(\alpha | (x_{nk})_\alpha^- v(x_{nk})_\alpha^+ |) \\ &\leq C < \infty \end{aligned}$$

□

Corollary 4.2 (Lebesgue Dominated Convergence Theorem). *Let $\{x_n\}$ be a sequence of fuzzy random variables $x_n \xrightarrow{P} x$ and $E(\sup_{n \geq 1} \alpha | (x_n)_\alpha^- v(x_n)_\alpha^+ |) < \infty$. Then $E(\alpha | (x_n)_\alpha^- - (x_\alpha)^- v(x_n)_\alpha^+ - (x_\alpha)^+ |) \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. For $\epsilon > 0$ choose $\delta > 0$, $A \in \mathcal{A}$ and $P(A) < \delta$. Which will imply $\int_A \alpha | (x_\alpha)^- v(x_\alpha)^+ | dP < \epsilon$ and $\sup_m \int_A \alpha | (x_n)_\alpha^- v(x_n)_\alpha^+ | < \epsilon < \delta$ for $n > N$. Hence

$$\begin{aligned} E(\alpha | (x_n)_\alpha^- - (x_\alpha)^- v(x_n)_\alpha^+ - (x_\alpha)^+ |) &= \int_{(|(x_n)_\alpha^- - (x_\alpha)^- v(x_n)_\alpha^+ - (x_\alpha)^+ |) \leq \epsilon} \alpha | (x_n)_\alpha^- - (x_\alpha)^- v(x_n)_\alpha^+ - (x_\alpha)^+ | dP \\ &= \int_{(|(x_n)_\alpha^- - (x_\alpha)^- v(x_n)_\alpha^+ - (x_\alpha)^+ |) > \epsilon} \alpha | (x_n)_\alpha^- - (x_\alpha)^- v(x_n)_\alpha^+ - (x_\alpha)^+ | dP \\ &\leq \epsilon + \int_{(|(x_n)_\alpha^- - (x_\alpha)^- v(x_n)_\alpha^+ - (x_\alpha)^+ |) > \epsilon} \alpha | (x_n)_\alpha^- v(x_n)_\alpha^+ | dP \\ &\quad + \int_{(|(x_n)_\alpha^- - (x_\alpha)^- v(x_n)_\alpha^+ - (x_\alpha)^+ |) > \epsilon} | (x_n)_\alpha^- v(x_n)_\alpha^+ | dP \\ &\quad + \int_{(|(x_n)_\alpha^- - (x_\alpha)^- v(x_n)_\alpha^+ - (x_\alpha)^+ |) > \epsilon} \alpha | (x_\alpha)^- v(x_\alpha)^+ | dP \\ &< \epsilon + \epsilon + \epsilon \\ &\geq 3\epsilon \text{ if } n > N \text{ at } x_n \xrightarrow{L_i} x. \end{aligned}$$

□

Theorem 4.3 (Mean Ergodic Theorem). *Let x_1, x_2, \dots , are independent identically distributed fuzzy random variables with $E(\alpha|(x_1)_\alpha^- v(x_1)_\alpha^+)|) = C < \infty$. Then $E\left(\frac{S_n}{n} - C\right) \rightarrow 0$ as $n \rightarrow \infty$. That is $\frac{S_n}{n} \xrightarrow{L_i} C$.*

Proof. By Kolmogorov's strong law of large numbers for independent identically distributed fuzzy random variables $\frac{S_n}{n} \rightarrow C$ a.s. Since $C = E(\alpha|(x_1)_\alpha^- v(x_1)_\alpha^+)|) < \infty$, we need to prove that $\frac{S_n}{n}$ is uniformly integrable

$$\begin{aligned} E\left(\alpha\left|\frac{(S_n)_\alpha^-}{n} v\frac{(S_n)_\alpha^+}{n}\right|\right) &\leq \sum_1^n E\left(\alpha\left|\frac{(x_1)_\alpha^-}{n} v\frac{(x_1)_\alpha^+}{n}\right|\right) \\ &= E(\alpha|(x_1)_\alpha^- v(x_1)_\alpha^+|) \\ &< \infty \end{aligned} \tag{4}$$

and hence the first condition of uniform integrability is satisfied. For $\epsilon > 0$ choose N large enough such that $\int_{A(|(x_n)_\alpha^- v(x_n)_\alpha^+| > N)} \alpha|(x_n)_\alpha^- v(x_n)_\alpha^+| dP < \epsilon$ which implies $\int_{A(|(x_n)_\alpha^- v(x_n)_\alpha^+| < N)} \alpha|(x_n)_\alpha^- v(x_n)_\alpha^+| dP < \epsilon$. Then for $A \in \mathcal{A}$ and $P(A) < \delta$ put $\delta = \frac{\epsilon}{N}$.

$$\begin{aligned} \int_A \alpha|(x_n)_\alpha^- v(x_n)_\alpha^+| dP &= \int_{A(|(x_n)_\alpha^- v(x_n)_\alpha^+| > N)} \alpha|(x_n)_\alpha^- v(x_n)_\alpha^+| dP \\ &\leq \epsilon + NP(A) \\ &< \epsilon + N\frac{\epsilon}{N} \\ &= 2\epsilon \text{ for every } n \geq 1. \end{aligned}$$

Hence

$$\begin{aligned} \int_A \left|\frac{(S_n)_\alpha^-}{n} v\frac{(S_n)_\alpha^+}{n}\right| dP &= \int_A \frac{[|(x_1)_\alpha^- v(x_1)_\alpha^+| + \dots + |(x_n)_\alpha^- v(x_n)_\alpha^+|]}{n} dP \\ &= \int_A |(x_n)_\alpha^- v(x_n)_\alpha^+| dP \\ &\leq 2\epsilon \text{ for every } n \geq 1 \end{aligned} \tag{5}$$

at $\frac{S_n}{n}$ is uniformly integrable. □

Lemma 4.4. *For every real x , $F_n(x) \rightarrow F(x)$ a.s. and $F_n(x-) \rightarrow F(x-)$ a.s. as $n \rightarrow \infty$.*

Proof. Now $F(x-)_\alpha = P(\alpha|(x)_\alpha^- v(x)_\alpha^+| < x)$ and $F(x) = P(\alpha|(x)_\alpha^- v(x)_\alpha^+| \leq x)$. Also $F_n(x) = \frac{1}{n} \sum_{k=1}^n I_{[x_k \leq x]}$ and $F_n(x-) = \frac{1}{n} \sum_{k=1}^n I_{[x_k < x]}$. Now $Y_k = I_{[x_k \leq x]}$ are independent identically distributed random variables and $E(Y_k) = F(x)$. Then $Y_k \sim B(1, p), p = F(x)$. □

So by Kolmogorov's strong law of large numbers for independent identically distributed fuzzy random variables the result follows:

Theorem 4.5 (Fuzzy Distribution Functions $F_n(x)$). $P(\sup_{-\infty < x < \infty} |F_n(x) - F(x)| \rightarrow 0) = 1$.

Proof. Let r be any positive integer ≥ 2 . For $K = 1, 2, \dots, r - 1$ define

$$x_{r,k} = \min\{x : (F(x))_\alpha^- v(F(x))_\alpha^+ \geq \frac{K}{r}\}$$

the $\frac{k}{r}$ population quantile. Then $-\infty = x_{r,0} < x_{r,1} < x_{r,2} \cdots < x_{r,r} = \infty$. We need to consider only those intervals $[x_{r,k}, x_{r,k+1}]$ which are non-empty. If $x \in [x_{r,k}, x_{r,k+1}]$ then

$$\begin{aligned} (F_n(x))_{\alpha}^{-} - (F(x))_{\alpha}^{-} V((F_n(x))_{\alpha}^{+} - (F(x))_{\alpha}^{+}) &\leq F_n(x_{r,k+1})_{\alpha}^{-} - F(x_{r,k})_{\alpha}^{-} V F_n(x_{r,k+1})_{\alpha}^{+} - F(x_{r,k})_{\alpha}^{+} \\ &= F_n(x_{r,k+1})_{\alpha}^{-} - F(x_{r,k})_{\alpha}^{-} V F_n(x_{r,k+1})_{\alpha}^{+} - F(x_{r,k+1})_{\alpha}^{+} \\ &\quad + F(x_{r,k+1})_{\alpha}^{-} - F(x_{r,k})_{\alpha}^{-} V F(x_{r,k+1})_{\alpha}^{+} - F(x_{r,k})_{\alpha}^{+} \end{aligned}$$

Note that

$$\begin{aligned} F(x_{r,k+1})_{\alpha}^{-} - F(x_{r,k})_{\alpha}^{-} V F(x_{r,k+1})_{\alpha}^{+} - F(x_{r,k})_{\alpha}^{+} &\leq \frac{k+1}{r} - \frac{k}{r} = \frac{1}{r} \\ F(x_{r,1})_{\alpha}^{-} V F(x_{r,1})_{\alpha}^{+} &\leq \frac{1}{r} \end{aligned}$$

and

$$\begin{aligned} (F(x_{r,k})_{\alpha}^{-} V F(x_{r,k})_{\alpha}^{+}) &\geq 1 - \frac{1}{r} \\ &\leq (F_n(x_{r,k+1})_{\alpha}^{-} - F(x_{r,k})_{\alpha}^{-} V F_n(x_{r,k+1})_{\alpha}^{+} - F(x_{r,k+1})_{\alpha}^{+}) + \frac{1}{r} \end{aligned} \quad (6)$$

for almost all w , $1 \leq k \leq r-1$ and

$$\begin{aligned} (F_n(x))_{\alpha}^{-} - (F(x))_{\alpha}^{-} V((F_n(x))_{\alpha}^{+} - (F(x))_{\alpha}^{+}) &\geq F_n(x_{r,k})_{\alpha}^{-} - F(x_{r,k+1})_{\alpha}^{-} V F_n(x_{r,k})_{\alpha}^{+} - F(x_{r,k+1})_{\alpha}^{+} \\ &= F_n(x_{r,k})_{\alpha}^{-} - F(x_{r,k})_{\alpha}^{-} V F_n(x_{r,k})_{\alpha}^{+} - F(x_{r,k})_{\alpha}^{+} \\ &\quad - (F(x_{r,k+1})_{\alpha}^{-} - F(x_{r,k})_{\alpha}^{-} V F(x_{r,k+1})_{\alpha}^{+} - F(x_{r,k+1})_{\alpha}^{+}) \\ &\geq (F_n(x_{r,k})_{\alpha}^{-} - F(x_{r,k})_{\alpha}^{-} V F_n(x_{r,k})_{\alpha}^{+} - F(x_{r,k})_{\alpha}^{+}) - \frac{1}{r} \end{aligned} \quad (7)$$

for almost all w , $1 \leq k \leq r-1$, Lemma says that for each x there is a Set A with $P(A_x) = 0$ such that

$$\begin{aligned} \lim (F_n(x, w)_{\alpha}^{-} - V(F_n(x, w))_{\alpha}^{+}) &= (F(x))_{\alpha}^{-} V(F(x))_{\alpha}^{+} \text{ for } x \in A_x \\ \lim (F_n(x-w)_{\alpha}^{-} - V(F_n(x, w))_{\alpha}^{+}) &= (F(x-))_{\alpha}^{-} V(F(x-))_{\alpha}^{+} \text{ for } x \in A_x \end{aligned}$$

except on a set B_x with $P(B_x) = 0$. Similar arguments held for $x < x_{r,1}$ and $x < x_{r,r-1}$. From (7) and (8) for almost all real x .

$$\begin{aligned} 0 &\leq (F_n(x))_{\alpha}^{-} - (F(x))_{\alpha}^{-} V(F_n(x))_{\alpha}^{+} - (F(x))_{\alpha}^{+} j \\ &\leq \max_{\substack{1 \leq k \leq r \\ 1 \leq j \leq r}} \{ |(F_n(x_{r,k})_{\alpha}^{-} - F(x_{r,k})_{\alpha}^{-} V F_n(x_{r,k})_{\alpha}^{+} - F(x_{r,k})_{\alpha}^{+}) j, |(F_n(x_{r,j})_{\alpha}^{-} - F(x_{r,j})_{\alpha}^{-} V F_n(x_{r,j})_{\alpha}^{+} - F(x_{r,j})_{\alpha}^{+}) + \frac{1}{r} j \} \end{aligned}$$

Let

$$\begin{aligned} D_{r,n}(w) &= \max_{\substack{1 \leq k \leq r \\ 1 \leq j \leq r}} [|(F_n(x_{r,k})_{\alpha}^{-} - F(x_{r,k})_{\alpha}^{-} V F_n(x_{r,k})_{\alpha}^{+} - F(x_{r,k})_{\alpha}^{+}) j, |(F_n(x_{r,j})_{\alpha}^{-} - F(x_{r,j})_{\alpha}^{-} V F_n(x_{r,j})_{\alpha}^{+} - F(x_{r,j})_{\alpha}^{+}) + j \end{aligned}$$

and

$$D_n = (D_{r,n}(w))_{\alpha}^{-} v(D_{r,n}(w))_{\alpha}^{+} = \sup_x j(F_n(x))_{\alpha}^{-} - (F(x))_{\alpha}^{-} V(F_n(x))_{\alpha}^{+} - (F(x))_{\alpha}^{+} j$$

Hence $D_n(w) = (D_{r,n}(w))_{\alpha}^{-} v(D_{r,n}(w))_{\alpha}^{+} + r^{-1}$. If w lies outside the union A of all the countably many $A_{x,r,k}$ and $B_{x,r,k}$ then by lemma

$$\lim_{n \rightarrow \infty} (D_{r,n}(w))_{\alpha}^{-} v(D_{r,n}(w))_{\alpha}^{+} = 0 \quad \text{for } w \in A \text{ and } P(A) = 0$$

Therefore $\lim_{n \rightarrow \infty} \sup D_n \leq \frac{1}{r}$ a.s. Now by the arbitraryness and countability of the values of r the result follows by taking $\lim_{r \rightarrow \infty}$. □

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