International Journal of Mathematics And its Applications
Volume 4, Issue 2–C (2016), 13–22.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Bipolar Vague Boundary Space via Ring

Research Article

S.Cicily Flora[1]* and I.Arockiarani[1]

1 Department of Mathematics, Nirmala College for Women, Coimbatore, Tamilnadu, India.

Abstract: In this paper, we present the notions of bipolar vague rings and boundary space and characterize their properties.

Keywords: Bipolar vague rings, Bipolar vague structure ring space, Bipolar vague ring boundary.

1. Introduction

The concept of Fuzzy sets was introduced by Zadeh [10]. Fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundry is vague. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets etc. The concept of boundary of a fuzzy set was introduced by R. H Warren [9]. Bipolar-valued fuzzy sets, which are introduced by Lee [7, 8], are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0,1] to [-1,1]. Since then the theory of bipolar valued fuzzy sets has became a vigorous area of research in different disciplines such as algebraic structure, medical science, graph theory, decision making, machine theory and so on [1, 6]. The notion of vague set theory introduced by W. L. Gau and D.J. Buehrer [4], as a generalizations of Zadeh's fuzzy set. Ranjith Biswas [3] initiated the study of vague algebra by studying vague groups. The objective of this paper is to introduce the concept of bipolar vague rings and bipolar vague structure ring space.

2. Preliminaries

Definition 2.1 ([8])**.** *Let X be the universe of discourse. Then a bipolar valued fuzzy set, A on X is defined by positive membership function μ_A^+, that is $\mu_A^+ : X \to [0,1]$, and a negative membership function μ_A^-,that is $\mu_A^- : X \to [-1,0]$. For the sake of simplicity, we shall use the symbol $A = \{\langle x, \mu_A^+(x), \mu_A^-(x)\rangle : x \in X\}$.*

Definition 2.2 ([8])**.** *Let A and B be two bipolar valued fuzzy sets then their union, intersection and complement are defined as follows:*

(i). $\mu_{A\cup B}^+(x) = max\{\mu_A^+(x), \mu_B^+(x)\}$

(ii). $\mu_{A\cup B}^-(x) = min\{\mu_A^-(x), \mu_B^-(x)\}$

* *E-mail: cicily.sackriyas90@gmail.com*

(iii). $\mu^+_{A\cap B}(x) = min\{\mu^+_A(x), \mu^+_B(x)\}$

(iv). $\mu^-_{A\cap B}(x) = max\{\mu^-_A(x), \mu^-_B(x)\}$

(v). $\mu^+_{\bar{A}}(x) = 1 - \mu^+_A(x)$ *and* $\mu^-_{\bar{A}}(x) = -1 - \mu^-_A(x)$.

for all $x \in X$.

Definition 2.3 ([4]). *A Vague set A in the universe of discourse U is a pair* (t_A, f_A) *where* $t_A : U \to [0,1]$, $f_A : U \to [0,1]$, *are the mapping such that* $t_A(u) + f_A(u) \leq 1$ *for all* $u \in U$. *The function* t_A *and* f_A *are called true membership function and false membership function respectively, The interval* $[t_A, 1 - f_A]$ *is called the vague value of u in A, and it is denoted by* $v_A(u)$, *i.e,* $v_A(u) = [t_A(u), 1 - f_A(u)]$.

Definition 2.4 ([4]). *Let X be a nonempty set and the Vague set A and B in the form* $A = \{\langle x, t_A(x), 1 - f_A(x)\rangle : x \in X\}$,$B = \{\langle x, t_B(x), 1 - f_B(x)\rangle : x \in X\}$. *Then*

(i). $A \subseteq B$ *if and only if* $t_A(x) \leq t_B(x)$ *and* $1 - f_A(x) \leq 1 - f_B(x)$

(ii). $A \cup B = max\{t_A(x), t_B(x)\}$ *and* $max\{1 - f_A(x), 1 - f_B(x)\}$

(iii). $A \cap B = min\{t_A(x), t_B(x)\}$ *and* $min\{1 - f_A(x), 1 - f_B(x)\}$

(iv). $A^C = \{\langle x, f_A(x), 1 - t_A(x)\rangle : x \in X\}$.

for all $x \in X$.

Definition 2.5 ([3]). *A Vague set* $A = (t_A, f_A)$ *of a set U with* $t_A(x) = 0$ *and* $f_A(x) = 1$ *for all* $x \in U$ *is called the zero vague set.*

Definition 2.6 ([3]). *A Vague set* $A = (t_A, f_A)$ *of a set U with* $t_A(x) = 1$ *and* $f_A(x) = 0$ *for all* $x \in U$ *is called the unit vague set.*

3. Bipolar Vague Structure Ring Space

Definition 3.1. *Let X be the universe of discourse. A Bipolar-valued vague set A in X is an object having the form* $A = \{(x, [t^+_A(x), 1 - f^+_A(x)], [t^-_A(x), -1 - f^-_A(x)]) : x \in X\}$ *where* $[t^+_A, 1 - f^+_A] : X \to [0,1]$ *and* $[t^-_A, -1 - f^-_A] : X \to [-1, 0]$ *are mappings and* $t^+_A + f^+_A \leq 1, -1 \leq t^-_A + f^-_A$. *The positive membership degree* $[t^+_A(x), 1 - f^+_A(x)]$ *denotes the satisfaction region of an element x to the property corresponding to a bipolar-valued set A and the negative membership degree* $[t^-_A(x), -1 - f^-_A(x)]$ *denote the satisfication region of x to some implict counter property of A.*

Note 3.2. *For sake of simplicity, we shall use the symbol* $A = (X, v^+_A, v^-_A)$ *for the bipolar valued vague set* $A = \{(x, [t^+_A(x), 1 - f^+_A(x)], [t^-_A(x), -1 - f^-_A(x)]) : x \in X\}$, *and use the notion of bipolar vague set* $v^+_A = [t^+_A, 1 - f^+_A]$ *and* $v^-_A = [t^-_A, -1 - f^-_A]$.

Definition 3.3. *Let X be a nonempty set and the Vague set A and B in the form* $A = \{\langle x, v^+_A(x), v^-_A(x)\rangle : x \in X\}$,$B = \{\langle x, v^+_B(x), v^-_B(x)\rangle : x \in X\}$.

(i). $A \subseteq B$ *if and only if* $v^+_A(x) \leq v^+_B(x)$ *and* $v^-_A(x) \geq v^-_B(x)$

(ii). $A \cup B = max\{v^+_A(x), v^+_B(x)\}$ *and* $min\{v^-_A(x), v^-_B(x)\}$

(iii). $A \cap B = min\{v^+_A(x), v^+_B(x)\}$ *and* $max\{v^-_A(x), v^-_B(x)\}$

(iv). $\overline{A} = \{(x, [f_A^+(x), 1 - t_A^+(x)], [f_A^-(x), -1 - t_A^-(x)]) : x \in X\}$

for all $x \in X$.

Definition 3.4. *A Vague set* $A = (v_A^+, v_A^-)$ *of a set* U *with* $v_A^+ = 0$ *implies that* $t_A^+ = 0, 1 - f_A^+ = 0$ *and* $v_A^- = 0$ *implies that* $t_A^- = 0, -1 - f_A^- = 0$ *for all* $x \in U$ *is called zero bipolar vague set and it denoted by* 0.

Definition 3.5. *A Vague set* $A = (v_A^+, v_A^-)$ *of a set* U *with* $v_A^+ = 1$ *implies that* $t_A^+ = 1, 1 - f_A^+ = 1$ *and* $v_A^- = -1$ *implies that* $t_A^- = -1, -1 - f_A^- = -1$ *for all* $x \in U$ *is called unit bipolar vague set and it denoted by* 1.

Definition 3.6. *Let* R *be a ring. A bipolar vague set* $A = \left\{x, v_A^+, v_A^-\right\}$ *in* R *is called a bipolar vague ring on* R *if it satisfies the following conditions:*

(i). $v_A^+ (x + y) \geq min\left\{v_A^+ (x), v_A^+ (y)\right\}$

(ii). $v_A^- (x + y) \leq max\left\{v_A^- (x), v_A^- (y)\right\}$

(iii). $v_A^+ (xy) \geq min\left\{v_A^+ (x), v_A^+ (y)\right\}$

(iv). $v_A^- (xy) \leq max\left\{v_A^- (x), v_A^- (y)\right\}$

for all $x, y \in R$.

Definition 3.7. *Let* R *be a ring. A family* τ *of a bipolar vague rings in* R *is said to be bipolar vague structure ring on* R *if it satisfies the following axioms:*

(i). $0, 1 \in \tau$

(ii). If $G_1, G_2 \in \tau$ *then* $G_1 \cap G_2 \in \tau$

(iii). If $G_i \in \tau$ *for all* $i \in I$*, then* $\cup_{i \in I} G_i \in \tau$.

Then the ordered pair (R, τ) *is called a bipolar vague structure ring space. Every member of* τ *is called a bipolar vague open ring in* (R, τ)*. Then the complement* $\overline{A}$ *of a bipolar vague open ring* A *in* (R, τ) *is a bipolar vague closed ring in* (R, τ).

Example 3.8. *Let* $R = \{0, 1\}$ *be a set of integers of modulo 2 with two binary operations as follows:*

+	0	1
0	0	1
1	1	0

and

·	0	1
0	0	0
1	0	1

Then $(R, +, \cdot)$ *is a ring. Define bipolar vague rings* B *and* C *on* R *as follows:*

$t_B^+ (0) = 0.7 t_B^+ (1) = 0.61 - f_B^+ (0) = 0.8 \qquad 1 - f_B^+ (1) = 0.7$

$t_C^+ (0) = 0.5 t_C^+ (1) = 0.41 - f_C^+ (0) = 0.6 \qquad 1 - f_C^+ (1) = 0.5$

$t_B^- (0) = -0.6 t_B^- (1) = -0.51 - f_B^- (0) = -0.61 - f_B^- (1) = -0.6$

$t_C^- (0) = -0.3 t_C^- (1) = -0.21 - f_C^- (0) = -0.51 - f_C^- (1) = -0.5$

Then $\tau = \{0, 1, B, C\}$ *is a bipolar vague structure ring on* R*. Thus the pair* (R, τ) *is a bipolar vague structure ring space.*

Notation 3.9. *Let* (R, τ) *be a bipolar vague structure ring space. Then*

(i). $O(R)$ *denotes the family of bipolar vague open ring of a bipolar vague structure ring space* (R, τ).

(ii). $C(R)$*denotes the family of bipolar vague closed ring of a bipolar vague structure ring space* (R, τ).

Definition 3.10. *Let (R, τ) be a bipolar vague structure ring space. Let $A = \{x, v_A^+, v_A^-\}$ be a bipolar vague ring in R. Then*

(i). the bipolar vague interior of A is defined and denoted as $BV_R int(A) = \cup\{B = \{x, v_B^+, v_B^-\} \mid B \in O(R)$ and $B \subseteq A\}$.

(ii). the bipolar vague closure of A is defined and denoted $BV_R cl(A) = \cap\{B = \{x, v_B^+, v_B^-\} \mid B \in C(R)$ and $A \subseteq B\}$.

Remark 3.11. *Let (R, τ) be a bipolar vague structure ring space. Let $A = \{x, v_A^+, v_A^-\}$ be a bipolar vague ring in R. Then the following statements hold:*

(i). $BV_R cl(A) = A$ if and only if A is a bipolar vague closed ring.

(ii). $BV_R int(A) = A$ if and only if A is a bipolar vague open ring.

(iii). $BV_R int(A) \subseteq A \subseteq BV_R cl(A)$.

(iv). $BV_R int(1) = 1$ and $BV_R int(0) = 0$.

(v). $BV_R cl(0) = 0$ and $BV_R cl(1) = 1$.

(vi). $BV_R cl(\overline{A}) = \overline{BV_R int(A)}$ and $BV_R int(\overline{A}) = \overline{BV_R cl(A)}$.

Definition 3.12. *Let (R, τ) be a bipolar vague structure ring space. Let $A = \{x, v_A^+, v_A^-\}$ be any bipolar vague ring in R. Then the bipolar vague boundary of A is denoted and defined as $BV_R bd(A) = BV_R cl(A) \cap BV_R cl(\overline{A})$.*

Proposition 3.13. *Let (R, τ) be a bipolar vague structure ring space. Let A and B be any two bipolar vague rings. Then the following conditions hold:*

(i). $BV_R bd(A) = BV_R bd(\overline{A})$

(ii). If A is a bipolar vague closed ring, then $BV_R bd(A) \subseteq A$

(iii). If A is a bipolar vague open ring, then $BV_R bd(A) \subseteq \overline{A}$

(iv). Let $A \subseteq B$ and B be any bipolar vague closed ring (resp., A be any bipolar vague open ring). Then $BV_R bd(A) \subseteq B(resp., BV_R bd(A) \subseteq \overline{B})$

(v). $(\overline{BV_R bd(A)}) = BV_R int(A) \cup BV_R int(\overline{A})$

Proof.

(i).

$$\begin{aligned} BV_R bd(A) &= BV_R cl(A) \cap BV_R cl(\overline{A}) \\ &= BV_R cl(\overline{A}) \cap BV_R cl(A) \\ &= BV_R cl(\overline{A}) \cap BV_R cl(\overline{\overline{A}}) \\ &= BV_R bd(\overline{A}) \end{aligned}$$

(ii). $BV_R bd(A) = BV_R cl(A) \cap BV_R cl(\overline{A}) \subseteq BV_R cl(A) = A$. Hence, $BV_R bd(A) \subseteq A$.

(iii). Let A be any bipolar vague open ring. Then $\overline{A}$ is bipolar vague closed ring. By (ii),$BV_R bd(\overline{A}) \subseteq \overline{A}$ and by (i), $BV_R bd(A) \subseteq \overline{A}$.

(iv). Since $A \subseteq B$ implies that $BV_Rcl(A) \subseteq BV_Rcl(B)$, we have $BV_Rbd(A) = BV_Rcl(A) \cap BV_Rcl(\overline{A}) \subseteq BV_Rcl(B) \cap BV_Rcl(\overline{A}) \subseteq BV_Rcl(B) = B$, since B is a bipolar vague closed ring.

(v). $(\overline{BV_Rbd(A)})$=$(\overline{BV_Rcl(A) \cap BV_Rcl(\overline{A}}$=$(\overline{BV_Rcl(A)}) \cup (\overline{BV_Rcl(\overline{A}}) = BV_Rint(\overline{A}) \cup BV_Rint(A)$.

□

The converse of (i) and (ii) of proposition is not true in general as seen in the following example.

Example 3.14. *In example 3.6 , Consider a bipolar vague ring G,* $t_G^+(0) = 0.6$, $t_G^+(1) = 0.5$, $1 - f_G^+(0) = 0.7$, $1 - f_G^+(1) = 0.6$, $t_G^-(0) = -0.6$, $t_G^-(1) = -0.5$, $1 - f_G^-(0) = -0.7$, $1 - f_G^-(1) = -0.6$ *clearly,* $BV_Rbd(G) \subseteq G$. *But G is not a bipolar vague closed ring.*

Example 3.15. *Let* $R = \{0, 1\}$ *be a set of integers of modulo 2 with two binary operations as follows:*

+	0	1
0	0	1
1	1	1

and

·	0	1
0	0	0
1	0	1

Then $(R, +, \cdot)$ *is a ring. Define bipolar vague rings B and C on R as follows:*

$t_A^+(0) = 0.6 t_A^+(1) = 0.5 \qquad 1 - f_A^+(0) = 0.81 - f_A^+(1) = 0.7$

$t_B^+(0) = 0.5 t_B^+(1) = 0.41 - f_B^+(0) = 0.5 \qquad 1 - f_B^+(1) = 0.4$

$t_A^-(0) = -0.5 t_A^-(1) = -0.41 - f_A^-(0) = -0.71 - f_A^-(1) = -0.6$

$t_B^-(0) = -0.3 t_B^-(1) = -0.21 - f_B^-(0) = -0.51 - f_B^-(1) = -0.5$

Then $\tau = \{0, 1, A, B\}$ *is a bipolar vague structure ring on R. Let*

$t_F^+(0) = 0.5 t_F^+(1) = 0.11 - f_F^+(0) = 0.4 \qquad 1 - f_F^+(1) = 0.3$

$t_F^-(0) = -0.3 t_F^-(1) = -0.21 - f_G^-(0) = -0.51 - f_F^-(1) = -0.4$

be a bipolar vague ring. Now $BV_Rbd(F) \subseteq \overline{F}$. *But F is not a bipolar vague open ring.*

Definition 3.16. *Let A and B be any two bipolar vague rings. Then* $A - B = A \cap \overline{B}$.

Proposition 3.17. *Let* (R, τ) *be a bipolar vague structure ring space. Let* $A = \{x, v_A^+, v_A^-\}$ *be any bipolar vague ring. Then the following conditions hold:*

(i). $BV_Rbd(A) = BV_Rcl(A) - BV_Rint(A)$

(ii). $BV_Rbd(BV_Rint(A)) \subseteq BV_Rbd(A)$

(iii) $BV_Rbd(BV_Rcl(A)) \supseteq BV_Rbd(A)$

(iv) $BV_Rint(A) \subseteq A - BV_Rbd(A)$

Proof.

(i). Since $(\overline{BV_Rcl(\overline{A})}) = BV_Rint(A)$. Therefore,

$$\begin{aligned} BV_Rbd(A) &= BV_Rcl(A) \cap BV_Rcl(\overline{A}) \\ &= BV_Rcl(A) - (\overline{BV_Rcl(\overline{A})}) \\ &= BV_Rcl(A) - BV_Rint(A). \end{aligned}$$

Thus $BV_Rbd(A) = BV_Rcl(A) - BV_Rint(A)$. Hence (i).

(ii).

$$\begin{aligned} BV_Rbd(BV_Rint(A)) &= BV_Rcl(BV_Rint(A)) - BV_Rint(BV_Rint(A)) \\ &= BV_Rcl(BV_Rint(A)) - BV_Rint(A) \\ &\subseteq BV_Rcl(A) - BV_Rint(A) \\ &= BV_Rbd(A). \end{aligned}$$

(iii).

$$\begin{aligned} BV_Rbd(BV_Rcl(A)) &= BV_Rcl(BV_Rcl(A)) - BV_Rint(BV_Rcl(A)) \\ &= BV_Rcl(A) - BV_Rint(BV_Rcl(A)) \\ &\supseteq BV_Rcl(A) - BV_Rint(A) \\ &= BV_Rbd(A). \end{aligned}$$

(iv).

$$\begin{aligned} A - BV_Rbd(A) &= A \cap (\overline{BV_Rbd(A)}) \\ &= A \cap (\overline{BV_Rcl(A) \cap BV_Rcl(\overline{A})}) \\ &= A \cap (BV_Rint(\overline{A}) \cup BV_Rint(A)) \\ &= (A \cap BV_Rint(\overline{A})) \cup (A \cap BV_Rint(A)) \\ &= (A \cap BV_Rint(\overline{A})) \cup BV_Rint(A) \\ &\supseteq BV_Rint(A). \end{aligned}$$

□

Remark 3.18. *Let (X, τ) be a bipolar vague topological space. Let A and B be any two bipolar vague sets. Then $\cap_{i \in J} BVcl(A_i) \supseteq BVcl(\cap_{i \in J}(A_i))$, where J is an indexed set.*

Proposition 3.19. *Let (R, τ) be a bipolar vague structure ring space. Let A and B be any two bipolar vague rings. Then, $BV_Rbd(A \cup B) \subseteq BV_Rbd(A) \cup BV_Rbd(B)$.*

Proof.

$$\begin{aligned} BV_Rbd(A \cup B) &= BV_Rcl(A \cup B) \cap BV_Rcl(\overline{A \cup B}) \\ &\subseteq (BV_Rcl(A) \cup BV_Rcl(B)) \cap (BV_Rcl(\overline{A}) \cap BV_Rcl(\overline{B})) \\ &= [BV_Rcl(A) \cap (BV_Rcl(\overline{A}) \cap BV_Rcl(\overline{B}))] \cup \\ [BV_Rcl(B) \cap (BV_Rcl(\overline{A}) \cap BV_Rcl(\overline{B}))] &= (BV_Rbd(A) \cap BV_Rcl(\overline{B})) \cup (BV_Rbd(B) \cap BV_Rcl(\overline{A})) \\ &\subseteq BV_Rbd(A) \cup BV_Rbd(B). \end{aligned}$$

□

In above Theorem, the equality does not hold as shown in the following example.

Example 3.20.

+	0	1
0	0	1
1	1	0

and

$\cdot$	0	1
0	0	0
1	0	1

Then $(R, +, \cdot)$ is a ring. Define bipolar vague rings A, B, C and D on R as follows:

$$t_A^+(0) = 0.7 t_A^+(1) = 0.51 - f_A^+(0) = 0.71 - f_A^+(1) = 0.7$$

$$t_B^+(0) = 0.7 t_B^+(1) = 0.51 - f_B^+(0) = 0.81 - f_B^+(1) = 0.7$$

$$t_C^+(0) = 0.4 t_C^+(1) = 0.31 - f_C^+(0) = 0.61 - f_C^+(1) = 0.5$$

$$t_D^+(0) = 0.5 t_D^+(1) = 0.41 - f_D^+(0) = 0.71 - f_D^+(1) = 0.5$$

$$t_A^-(0) = -0.4 t_A^-(1) = -0.31 - f_A^-(0) = -0.51 - f_A^-(1) = -0.4$$

$$t_B^-(0) = -0.5 t_B^-(1) = -0.41 - f_B^-(0) = -0.61 - f_B^-(1) = -0.5$$

$$t_C^-(0) = -0.3 t_C^-(1) = -0.21 - f_C^-(0) = -0.41 - f_C^-(1) = -0.3$$

$$t_D^-(0) = -0.4 t_D^-(1) = -0.21 - f_D^-(0) = -0.51 - f_D^-(1) = -0.4$$

Then $\tau = \{0, 1, A, B, C, D\}$ is a bipolar vague structure ring on R. Let $t_G^+(0) = 0.5$; $t_G^+(1) = 0.4$; $1 - f_G^+(0) = 0.5$; $1 - f_G^+(1) = 0.4$; $t_G^-(0) = -0.6$; $t_G^-(1) = -0.4$; $1 - f_G^-(0) = -0.6$; $1 - f_G^-(1) = -0.4$ be the bipolar vague rings.It is easily verify that $BV_R bd(F \cup G) \neq BV_R bd(F) \cup BV_R bd(G)$.

Proposition 3.21. *Let (R, τ) be a bipolar vague structure ring space. Let A and B be any two bipolar vague rings. Then, $BV_R bd(A \cap B) \subseteq BV_R bd(A) \cup BV_R bd(B)$.*

Proof.

$$\begin{aligned}
BV_R bd(A \cap B) &= BV_R cl(A \cap B) \cap BV_R cl(\overline{A \cap B}) \\
&\subseteq (BV_R cl(A) \cap BV_R cl(B)) \cap (BV_R cl(\overline{A}) \cup BV_R cl(\overline{B})) \\
&= [(BV_R cl(A) \cap (BV_R cl(B)) \cap BV_R cl(\overline{A})] \cup [(BV_R cl(A) \cap BV_R cl(B)) \cap BV_R cl(\overline{B})] \\
&= (BV_R bd(A) \cap BV_R cl(B)) \cup (BV_R bd(B) \cap BV_R cl(A)) \\
&\subseteq BV_R bd(A) \cup BV_R bd(B).
\end{aligned}$$

□

In general, the equality of the above theorem does not hold as seen from the following.

Example 3.22. *In example choose bipolar vague rings $t_F^+(0) = 0.6$; $t_F^+(1) = 0.5$; $1 - f_F^+(0) = 0.7$; $1 - f_F^+(1) = 0.6$; $t_F^-(0) = -0.4$; $t_F^-(1) = -0.3$; $1 - f_F^-(0) = -0.6$; $1 - f_F^-(1) = -0.7$; $t_G^+(0) = 0.2$; $t_G^+(1) = 0.1$; $1 - f_G^+(0) = 0.3$; $1 - f_G^+(1) = 0.4$; $t_G^-(0) = -0.3$; $t_G^-(1) = -0.2$; $1 - f_G^-(0) = -0.5$; $1 - f_G^-(1) = -0.5$ gives that $BV_R bd(F \cap G) \neq BV_R bd(F) \cup BV_R bd(G)$.*

Proposition 3.23. *Let (R, τ)be a bipolar vague structure ring space. Let A be any bipolar vague rings. Then the following conditions hold:*

(i). $BV_R bd(BV_R bd(A)) \subseteq BV_R bd(A)$

(ii). $BV_Rbd(BV_Rbd(BV_Rbd(A))) \subseteq BV_Rbd(BV_Rbd(A))$.

Proof.

(i).

$$\begin{aligned} BV_Rbd(BV_Rbd(A)) &= BV_Rcl(BV_Rbd(A)) \cap BV_Rcl(\overline{BV_Rbd(A)}) \\ &\subseteq BV_Rcl(BV_Rbd(A)) \\ &= BV_Rbd(A). \end{aligned}$$

(ii).

$$\begin{aligned} BV_Rbd(BV_Rbd(BV_Rbd(A))) &= BV_Rcl(BV_Rbd(BV_R(bd(A))) \cap BV_Rcl(\overline{BV_Rbd(BV_Rbd(A)})) \\ &= BV_Rbd(BV_Rbd(A)) \cap (BV_Rcl(\overline{BV_Rbd(BV_Rbd(A)})) \\ &\subseteq BV_Rbd(BV_Rbd(A)). \end{aligned}$$

□

Definition 3.24. *Let $A = (x, v_A^+(x), v_A^-(x))$ be a bipolar vague ring of X and $B = (x, v_B^+(x), v_B^-(x))$ be a bipolar vague ring of Y, then the bipolar vague ring $A \times B = (v_A^+ \times v_B^+, v_A^- \times v_B^-)$ of $X \times Y$ is defined as $(v_A^+ \times v_B^+)(x, y) = min\ \{v_A^+(x), v_B^+(y)\}$ for every $(x, y) \in X \times Y$ and $(v_A^- \times v_B^-)(x, y) = max\ \{v_A^-(x), v_B^-(y)\}$ for every $(x, y) \in X \times Y$.*

Definition 3.25. *Let (R_1, τ) and (R_2, σ) be any two bipolar vague structure spaces. The bipolar vague ring product space of (R_1, τ) and (R_2, σ) is the cartesian product $(R_1, \tau) \times (R_2, \sigma)$ of sets (R_1, τ) and (R_2, σ) together with the bipolar vague structure ring $R_1 \times R_2$ generated by the family, $\{p_1^{-1}(A), p_2^{-1}(B) \mid A \in R_1, B \in R_2$, where p_1 and p_2 are projections of $(R_1, \tau) \times (R_2, \sigma)$ onto (R_1, τ) and $(R_2, \sigma)\}$.*

Proposition 3.26. *Let A, B, C and D be bipolar vague rings in X, Then $(A \cap B) \times (C \cap D) = (A \times D) \cap (B \times D)$.*

Proof.

$$\begin{aligned} ((v_A^+ \cap v_B^+) \times (v_C^+ \cap v_D^+))(x, y) &= \min((v_A^+ \cap v_B^+)(x), (v_C^+ \cap v_D^+)(y)) \\ &= \min(\min(v_A^+(x), v_B^+(x)), \min(v_C^+(y), v_D^+(y))) \\ &= \min(\min(v_A^+(x), v_D^+(y)), \min(v_B^+(x), v_C^+(y))) \\ &= \min((v_A^+ \times v_D^+)(x, y), (v_B^+ \times v_C^+)(x, y)) \\ &= ((v_A^+ \times v_D^+) \cap (v_B^+ \times v_C^+))(x, y) \end{aligned}$$

Similarly,

$$\begin{aligned} ((v_A^- \cap v_B^-) \times (v_C^- \cap v_D^-))(x, y) &= \max((v_A^- \cap v_B^-)(x), (v_C^- \cap v_D^-)(y)) \\ &= \max(\max(v_A^-(x), v_B^-(x)), \max(v_C^-(y), v_D^-(y))) \\ &= \max(\max(v_A^-(x), v_D^-(y)), \max(v_B^-(x), v_C^-(y))) \\ &= \max((v_A^- \times v_D^-)(x, y), (v_B^- \times v_C^-)(x, y)) \\ &= ((v_A^- \times v_D^-) \cap (v_B^- \times v_C^-))(x, y). \end{aligned}$$

Hence, $(A \cap B) \times (C \cap D) = (A \times D) \cap (B \times D)$. □

Proposition 3.27. *If A is a bipolar vague ring in X and B is a bipolar vague ring in Y, then*

(i). $(v_A^+ \times 1) \cap (1 \times v_B^+) = v_A^+ \times v_B^+$ *and* $(v_A^- \times -1) \cap (-1 \times v_B^-) = v_A^- \times v_B^-$.

Proof. Let $A = (x, v_A^+(x), v_A^-(x))$, and $B = (y, v_B^+(y), v_B^-(y))$. Since $v_A^+ \times 1 = (x, min(t_A^+, 1), min(1 - f_A^+, 1)) = (x, t_A^+, 1 - f_A^+) = v_A^+$ and $1 \times v_B^+ = (y, min(1, t_B^+), min(1, 1 - f_B^+)) = (y, t_B^+, 1 - f_B^+) = v_B^+$, we have $(v_A^+ \times 1) \cap (1 \times v_B^+) = v_A^+(x) \cap v_B^+(y) = ((x, y), min(t_A^+(x), t_B^+(y)), min(1 - f_A^+(x), 1 - f_B^+(y))) = v_A^+ \times v_B^+$. Similarly,
$v_A^- \times -1 = (x, max(t_A^-, -1), max(-1 - f_A^-, -1)) = (x, t_A^-, -1 - f_A^-) = v_A^-$ and $-1 \times v_B^- = (y, max(-1, t_B^-), max(-1, -1 - f_B^-)) = (y, t_B^-, -1 - f_B^+) = v_B^-$, we have $(v_A^- \times -1) \cap (-1 \times v_B^-) = v_A^-(x) \cap v_B^-(y) = ((x, y), max(t_A^-(x), t_B^-(y)), max(-1 - f_A^-(x), -1 - f_B^-(y))) = v_A^- \times v_B^-$. Hence the proof. □

Proposition 3.28. *If $A = (x, v_A^+(x), v_A^-(x))$ is a bipolar vague in R_1 structure space and $B = (x, v_B^+(x), v_B^-(x))$ is a bipolar vague in R_2 structure space, then*

(i). $BV_{R_1}cl(A) \times BV_{R_2}cl(B) \supseteq BV_Rcl(A \times B)$.

(ii). $BV_{R_1}int(A) \times BV_{R_2}int(B) \subseteq BV_Rint(A \times B)$.

Proof.

(i). Since $A \subseteq BV_{R_1}cl(A)$ and $B \subseteq BV_{R_2}cl(B)$, $A \times B \subseteq BV_{R_1}cl(A) \times BV_{R_2}cl(B)$ Now, $BV_Rcl(A \times B) \subseteq BV_Rcl(BV_{R_1}cl(A) \times BV_{R_2}cl(B))$ such that $BV_Rcl(A \times B) \subseteq BV_{R_1}cl(A) \times BV_{R_2}cl(B)$.

(ii). Follows from (i).

□

Proposition 3.29. *Let $(X, R_i)(i = 1, \ldots, n)$ be a family of bipolar vague product related structure spaces. If each A_i is a bipolar vague ring in X, then*

$$BV_Rbd\prod_{i=1}^{n}(A_i) = [BV_Rbd(A_1) \times \cdots \times BV_Rcl(A_n)] \cup [BV_Rcl(A_1) \times \cdots \times BV_Rcl(A_n)] \cup \cdots \cup [BV_Rcl(A_1) \times \cdots \times BV_Rbd(A_n)].$$

Proof. It suffices to prove this for $n = 2$. Consider

$$\begin{aligned}
BV_Rbd(A_1 \times A_2) &= BV_Rcl(A_1 \times A_2) - BV_Rint(A_1 \times A_2) \\
&= (BV_Rcl(A_1) \times BV_Rcl(A_2)) - (BV_Rint(A_1) \times BV_Rint(A_2)) \\
&= (BV_Rcl(A_1) \times BV_Rcl(A_2) - (BV_Rint(A_1) \cap BV_Rcl(A_1)) \times (BV_Rint(A_2) \cap BV_Rcl(A_2)) \\
&= (BV_Rcl(A_1) \times BV_Rcl(A_2)) - (BV_Rint(A_1) \times BV_Rcl(A_2)) \cap (BV_Rcl(A_1) \times BV_Rint(A_2)) \\
&= [(BV_Rcl(A_1) \times BV_Rcl(A_2)) - (BV_Rint(A_1) \times BV_Rcl(A_2))] \cup [(BV_Rcl(A_1) \times BV_Rcl(A_2)) \\
&\quad - (BV_Rcl(A_1) \times BV_Rint(A_2))] \\
&= [(BV_Rcl(A_1) - BV_Rint(A_1)) \times BV_Rcl(A_2)] \cup [BV_Rcl(A_1) \times (BV_Rcl(A_2) - BV_Rint(A_2))] \\
&= ((BV_Rbd(A_1) \times (BV_Rcl(A_2)) \cup (BV_Rcl(A_1) \times (BV_Rbd(A_2)).
\end{aligned}$$

□

Definition 3.30. *Let (R_1, τ) and (R_2, σ)be any two bipolar vague structure ring spaces. A function $f : (R_1, \tau) \to (R_2, \sigma)$ is said to be bipolar vague ring continuous if and only if for each bipolar vague open ring B in (R_2, σ) the inverse image $f^{-1}(B)$ is a bipolar vague open ring in (R_1, τ).*

Proposition 3.31. *Let (R_1,τ) and (R_2,σ) be any two bipolar vague structure ring spaces. Let $f:(R_1,\tau)\to(R_2,\sigma)$ be a bipolar vague ring continuous function. Then, $BV_Rbd(f^{-1}(A))\subseteq f^{-1}(BV_Rbd(A))$.*

Proof. Let f be a bipolar vague ring continuous function. Let A be any bipolar vague ring in (R_2,σ). Then, $BV_Rcl(A)$ is a bipolar vague closed ring in (R_2,σ), which implies that $f^{-1}(BV_Rcl(A))$ is a bipolar vague closed ring in (R_1,τ). Therefore,

$$\begin{aligned}BV_Rbd(f^{-1}(A)) &= BV_Rcl(f^{-1}(A))\cap BV_Rcl(\overline{f^{-1}(A)})\\ &\subseteq BV_Rcl(f^{-1}(BV_Rcl(A)))\cap BV_Rcl(f^{-1}(BV_Rcl(\overline{A})))\\ &= f^{-1}(BV_Rcl(A))\cap f^{-1}(BV_Rcl(\overline{A}))\\ &= f^{-1}(BV_Rcl(A)\cap BV_Rcl(\overline{A}))\\ &= f^{-1}(BV_Rbd(A))\end{aligned}$$

Therefore, $BV_Rbd(f^{-1}(A))\subseteq f^{-1}(BV_Rbd(A))$. □

References

[1] M.Akram, *Bipolar fuzzy graphs*, Information Science, 181(24)(2011), 5548-5564.

[2] K.K.Azad, *On fuzzy semi continuity, fuzzy almost continuity and weekly continuity*, Journal of Mathematical Analysis and Applications, 82(1981), 14-32.

[3] R.Biswas, *Vague Groups*, International Journal of Computational Cognition, 4(2)(2006), 20-23.

[4] W.L.Gau and D.J.Buehrer, *Vague sets*, IEEE Transcations on Systems, Man and Cybernatics, 23(2)(1993), 610-614.

[5] H.Gureay and D.Coker, *On Fuzzy continuity in intuitionistic fuzzy topological spaces*, Journal of Fuzzy Math, 5(2)(1997), 365-378.

[6] Y.B.Jun and S.Z.Song, *Subalgebras and closed ideals of BCK-algebras based on bipolar valued fuzzy sets*, Sci. Math. Jpn., 68(2)(2008), 287-297.

[7] K.M.Lee, *Comparision of interval- valued fuzzy sets, intuitionistic fuzzy sets and bipolar- valued fuzzy sets*, J. Fuzzy Logic Intelligent, 14(2)(2004), 125-129.

[8] K.M.Lee, *Bipolar-valued fuzzy sets and their operations*, Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand, (2000), 307-312.

[9] W.H.Warren, *Boundry of a fuzzy set*, Indiana Mathematics J., 226(1977), 191-197.

[10] L.A.Zadeh, *Fuzzy sets*, Information and Control, 8(1965), 338-353.