International Journal of Mathematics And its Applications
Volume 4, Issue 2–C (2016), 23–28.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

On Cen-EP Matrices

Research Article

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Abstract: The concept of Cen-EP matrix is introduced and relations between Cen-EP and EP matrices are also discussed.

MSC: Primary 15A09 ; Secondary 15A18.

Keywords: EP matrix, Cen-EP matrix, Centrohermitian.

1. Introduction

Let $C_{n\times n}$ be the space of $n\times n$ complex matrices of order n. Let C_n be the space of complex n-tuples. For $A\in C_{n\times n}$, let $A^T, \bar{A}, A^*, A^\dagger, R(A), N(A)$ and $\rho(A)$ denote the transpose, conjugate, conjugate transpose, Moore-Penrose inverse, range space, null space and rank of A respectively. We denote a solution X of the equation $AXA=A$ by A^-. Let J be the unit perdiagonal matrix that has 1's on the secondary diagonal and 0's elsewhere. That is

$$J=\begin{pmatrix} 0 & 0 & . & . & . & 0 & 1\\ 0 & 0 & . & . & . & 1 & 0\\ . & . & . & . & . & . & .\\ . & . & . & . & . & . & .\\ . & . & . & . & . & . & .\\ 1 & 0 & . & . & . & 0 & 0 \end{pmatrix}$$

A matrix $A=(a_{ij})\in C_{n\times n}$ is centrohermitian if $a_{ij}=\bar{a}_{n-i+1,n-j+1}$, for $i,j=1$ to n or

$$A=J\bar{A}J \tag{1}$$

A theory for centrohermitian matrices is developed in [3]. In this paper, we introduce the concept of Cen-EP matrices as a generalization of centrohermitian and EP matrices and extend many of the basic results on centrohermitian [3] and EP matrices [1,5,7,9]. A matrix $A\in C_{n\times n}$ is EP if

$$N(A)=N(A^*) \tag{2}$$

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2. Cen-EP Matrices

In this section, we present equivalent characterizations of a Cen-EP matrix. As an application, it is shown that the class of all Cen-EP matrices having the same range spaces forms a group under multiplication. For $x = (x_1, x_2, ..., x_n)^T \in C_n$, let us define the function $j(x) = Jx$ and J satisfies the following properties :

$$J = J^T = J^{-1} = J^* \tag{3}$$

$$(JA)^\dagger = A^\dagger J \text{ and } (AJ)^\dagger = JA^\dagger \text{ for } A \in C_{n\times n} \ (by\ [2]) \tag{4}$$

Definition 2.1. *A matrix $A \in C_{n\times n}$ is said to Cen-EP if it satisfies the condition $Ax = 0 \Leftrightarrow \bar{A}j(x) = 0$ or equivalently, $N(A) = N(J\bar{A}J) = N(\bar{A}J)$. If A is nonsingular, then A is Cen-EP.*

Remark 2.2. *Every centrohermitian matrix is Cen-EP matrix.But the converse need not be true.*

Example 2.3. *Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Then, $J\bar{A}J = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \neq A$. Therefore, A is not centrohermitian. Since A is a nonsingular matrix, A is a Cen-EP matrix.*

Theorem 2.4. *For $A \in C_{n\times n}$, the following are equivalent :*

(1). A is Cen-EP.

(2). JA is EP.

(3). AJ is EP.

(4). $A^\dagger$ is Cen-EP.

(5). $N(A) = N(A^\dagger J)$.

(6). $N(\bar{A}) = N(AJ)$.

(7). $R(A^T) = R(JA^*)$.

(8). $R(A^*) = R(JA)$.

(9). $AA^\dagger J = JA^\dagger A$.

(10). $JAA^\dagger = A^\dagger AJ$.

(11). $A = JA^*JH$ for a non singular $n \times n$ matrix H.

(12). $A = HJA^*J$ for a non singular $n \times n$ matrix H.

(13). $A^* = HJAJ$ for a non singular $n \times n$ matrix H.

(14). $A^* = JAJH$ for a non singular $n \times n$ matrix H.

(15). $C_n = R(A) \oplus N(AJ)$.

(16). $C_n = R(JA) \oplus N(A)$.

Proof.

(1) ⇔ (2) ⇔ (3):

$$\begin{aligned} A \text{ is Cen-EP} &\Leftrightarrow N(A) = N(J\bar{A}J) && \text{(by (2.1))} \\ &\Leftrightarrow N(JA) = N(\bar{A}J) && \text{(by [3])} \\ &\Leftrightarrow N(JA) = N(JA)^* \\ &\Leftrightarrow JA \text{ is EP} && \text{(by [2])} \\ &\Leftrightarrow J(JA)J^* \text{ is EP} && \text{(by [1])} \\ &\Leftrightarrow AJ \text{ is EP} && \text{(by [3])} \end{aligned}$$

(2) ⇔ (4):

$$\begin{aligned} JA \text{ is EP} &\Leftrightarrow (JA)^\dagger \text{ is EP} && \text{(by [2])} \\ &\Leftrightarrow A^\dagger J \text{ is EP} && \text{(by [4])} \\ &\Leftrightarrow A^\dagger \text{ is EP} && \text{(by equivalence of (1) and (3) applied to } A^\dagger) \end{aligned}$$

(1) ⇔ (5):

$$\begin{aligned} A \text{ is Cen-EP} &\Leftrightarrow N(A) = N(\bar{A}J) && \text{(by (2.1))} \\ &\Leftrightarrow N(A) = N(JA)^* \\ &\Leftrightarrow N(A) = N(JA)^\dagger \\ &\Leftrightarrow N(A) = N(A^\dagger J) \end{aligned}$$

(1) ⇔ (6) ⇔ (7): Now, we shall prove the equivalence of (1), (6) and (7) using $\rho(A) = \rho(\bar{A}) = \rho(\bar{A}J) = \rho(AJ)$ in the following way :

$$\begin{aligned} A \text{ is Cen-EP} &\Leftrightarrow N(A) = N(\bar{A}J) \\ &\Leftrightarrow N(A) \subseteq N(\bar{A}J) \\ &\Leftrightarrow \bar{A}J = \bar{A}JA^- A && \text{(by [2])} \\ &\Leftrightarrow \bar{A} = \bar{A}JA^- AJ \\ &\Leftrightarrow \bar{A} = \bar{A}J^{-1}A^- AJ \\ &\Leftrightarrow \bar{A} = \bar{A}(AJ)^- AJ && \text{(by [4])} \\ &\Leftrightarrow N(AJ) \subseteq N(\bar{A}) && \text{(by [2])} \\ &\Leftrightarrow N(\bar{A}) = N(AJ) \\ &\Leftrightarrow R(\bar{A})^* = R(AJ)^* && \text{(by [8])} \\ &\Leftrightarrow R(A^T) = R(JA^*) \end{aligned}$$

(1) ⇔ (8):

$$\begin{aligned} A \text{ is Cen-EP} &\Leftrightarrow N(A) = N(\bar{A}J) \\ &\Leftrightarrow N(A) = N(JA)^* \\ &\Leftrightarrow R(A^*) = R(JA) && \text{(by [8])} \end{aligned}$$

(3) ⇔ (9):

$$\begin{aligned} AJ \text{ is EP} &\Leftrightarrow (AJ)(AJ)^\dagger = (AJ)^\dagger(AJ) && \text{(by [2])} \\ &\Leftrightarrow (AJ)(JA^\dagger) = (JA^\dagger)(AJ) && \text{(by (4))} \\ &\Leftrightarrow AA^\dagger = JA^\dagger AJ \\ &\Leftrightarrow AA^\dagger J = JA^\dagger A \end{aligned}$$

(9) ⇔ (10): By (3), $J^2 = I$, this equivalence follows by pre and post multiplying $AA^\dagger J = JA^\dagger A$ by J.

(2) ⇔ (11) : JA is EP ⇔ $(JA)^* = (JA)H_1$ for a non singular $n \times n$ matrix H_1 [2].

$$\begin{aligned} &\Leftrightarrow A^*J = JAH_1 \\ &\Leftrightarrow JA^*J = AH_1 \\ &\Leftrightarrow A = JA^*JH, \quad \text{where } H = H_1^{-1} \text{ is a non singular } n \times n \text{ matrix.} \end{aligned}$$

(3) ⇔ (12): AJ is EP ⇔ $(AJ)^* = H_1(AJ)$ for a non singular $n \times n$ matrix H_1 [2].

$$\begin{aligned} &\Leftrightarrow JA^* = H_1AJ \\ &\Leftrightarrow JA^*J = H_1A \\ &\Leftrightarrow A = HJA^*J, \quad \text{where } H = H_1^{-1} \text{ is a non singular } n \times n \text{ matrix.} \end{aligned}$$

The equivalences (11) ⇔ (13) and (12) ⇔ (14) follow immediately by taking conjugate transpose and using $J = J^*$.

(13) ⇔ (16):

$$\begin{aligned} A^* &\Leftrightarrow \text{HJAJ for a non singular } n \times n \text{ matrix } H \\ &\Leftrightarrow A^*A = H(JA)(JA) \\ &\Leftrightarrow A^*A = H(JA)^2 \\ &\Leftrightarrow \rho(A^*A) = \rho(H(JA)^2) \\ &\Leftrightarrow \rho(A^*A) = \rho((JA)^2) \end{aligned}$$

Over the complex field, A^*A and A have the same rank. Therefore,

$$\begin{aligned} \rho((JA)^2) = \rho(A^*A) = \rho(A) = \rho(JA) &\Leftrightarrow R(JA) \cap N(JA) = \{0\} \\ &\Leftrightarrow R(JA) \cap N(A) = \{0\} \\ &\Leftrightarrow C_n = R(JA) \oplus N(A) \end{aligned}$$

(14) ⇔ (15): This can by proved along the same lines and using $\rho(AA^*) = \rho(A)$. Hence the proof is omitted. □

Remark 2.5. *It is well known that a complex normal matrix is EP. However, a normal matrix need not be Cen-EP.*

Example 2.6.

(1). $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ *is EP as well as Cen-EP.*

(2). $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *is Cen-EP but not EP.*

(3). $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *is hermitian, normal and EP but not Cen-EP and hence not centrohermitian. This motivates the following result.*

Theorem 2.7. *Let $A \in C_{n\times n}$. Then any two of the following conditions imply the other one:*

(1). A *is EP.*

(2). A *is Cen-EP.*

(3). $R(A) = R(JA)$.

Proof. First, we prove that whenever (1) holds, then (2) and (3) are equivalent. Suppose (1) holds. Then A is EP $\Leftrightarrow R(A) = R(A^*)$. Now, by theorem 2.4, A is Cen-EP $\Leftrightarrow R(A^*) = R(JA)$. Therefore, A is Cen-EP $\Leftrightarrow R(A) = R(JA)$. This completes the proof of [(1) and (2)]$\Rightarrow$ (3) and [(1) and (3)]$\Rightarrow$(2).

Now, let us prove [(2) and (3)]$\Rightarrow$(1). Since A is Cen-EP, then $R(A^*) = R(JA)$. By using (3), we have $R(A) = R(A^*)$. Therefore, A is EP. Thus (1) holds. □

Corollary 2.8. *If $A \in C_{n\times n}$ is normal and AA^* is Cen-EP, then A is Cen-EP.*

Proof. Since A is normal, [A is EP and AA^* is Cen-EP]$\Leftrightarrow R(AA^*) = R(JAA^*) \Rightarrow R(A) = R(JA)$. From Theorem 2.7, A is Cen-EP. □

Corollary 2.9. *Let $E = E^* = E^2 \in C_{n\times n}$ be a hermitian idempotent that commutes with J. Then $H_j(E) = \{A; A$ is Cen-EP and $R(A) = R(E)\}$ forms a maximal subgroup of $C_{n\times n}$ containing E as identity.*

Proof. Since $EJ = JE$, by (iii) and (iv), we have $E = JEJ$ and $EE^\dagger = E^2 = E = (JE)(EJ) = (JE)(JE)^\dagger$; hence $R(E) = R(JE)$. Since E is hermitian, it is automatically EP and by theorem 2.7, E is Cen-EP. Thus, $E \in H_j(E)$. For $A \in H_j(E)$, A is Cen-EP and $R(A) = R(E) = R(JE) \Rightarrow AA^\dagger = EE^\dagger = E$ and $AA^\dagger = E = (JE)(JE)^\dagger = JEE^\dagger J^\dagger = JAA^\dagger J^\dagger = (JA)(JA)^\dagger$. Therefore, $R(A) = R(JA)$. Hence by Theorem 2.7, A is EP. Thus, $H_j(E) = \{A; A$ is EP and $R(A) = R(E)\}$. By [6], $H_j(E)$ forms a maximal subgroup of $C_{n\times n}$ containing E as identity. □

Remark 2.10. *For $A \in C_{n\times n}$, there exist unique centrohermitian matrices P and Q such that $A = P + iQ$, where $P = \frac{1}{2}(A + J\bar{A}J)$ and $Q = \frac{1}{2i}(A - J\bar{A}J)$ [4]. In the following theorem, an equivalent condition for a matrix A to be Cen-EP is obtained in terms of P, the centrohermitian part of A.*

Theorem 2.11. *For $A \in C_{n\times n}$, A is Cen-EP $\Leftrightarrow N(A) \subset N(P)$, where P is the centrohermitian part of A.*

Proof. If A is Cen-EP, then $N(A) = N(J\bar{A}J) = N(\bar{A}J)$. Therefore, for $x \in N(A)$, both $Ax = 0$ and $J\bar{A}Jx = 0$ which implies that $Px = \frac{1}{2}(A + J\bar{A}J)x = 0$. Thus, $N(A) \subseteq N(P)$. Conversely, let $N(A) \subseteq N(P)$. Then $Ax = 0$ implies $Px = 0$ and hence $Qx = 0$. Therefore, $N(A) \subseteq N(Q)$. Thus, $N(A) \subseteq N(P) \cap N(Q)$. Since both P and Q are centrohermitian, $P = J\bar{P}J$ and $Q = J\bar{Q}J$. Hence, $N(P) = N(J\bar{P}J) = N(\bar{P}J)$ and $N(Q) = N(J\bar{Q}J) = N(\bar{Q}J)$. Now $N(A) \subseteq N(P) \cap N(Q) = N(\bar{P}J) \cap N(\bar{Q}J) \subseteq N((\bar{P} - i\bar{Q})J)$. Therefore, $N(A) \subseteq N(\bar{A}J)$ and $\rho(A) = \rho(\bar{A}J)$. Hence, $N(A) = N(\bar{A}J)$. Therefore, A is Cen-EP. Hence the theorem. □

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