International Journal of Mathematics And its Applications
Volume 4, Issue 2–C (2016), 41–54.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Similarity Reductions of (2+1)-dimensional Equal Width Wave Equation

Research Article

G.Bharathi[1]* and R.Asokan[2]

1 Department of Mathematics, Sethupathy Govt. Arts College, Ramanathapuram, Tamil Nadu, India.
2 School of Mathematics, Madurai Kamaraj University, Madurai, Tamil Nadu, India.

Abstract: It has been shown to have solitary wave solutions and to govern a large number of important physical phenomena such as shallow water waves and plasma waves.

Keywords: Symmetry Group, Lie Algebra and Homotopy Perturbation Method (HPM).
© JS Publication.

1. Introduction

A simple model equation is the Korteweg-de Vries (KdV) equation [?]

$$v_t + 6vv_x + \delta v_{xxx} = 0 \ , \tag{1}$$

which describe the long waves in shallow water. Its modified version is,

$$u_t - 6u^2 u_x + u_{xxx} = 0 \tag{2}$$

and again there is Miura tramsformation [?]

$$v = u^2 + u_x \ , \tag{3}$$

between the KdV equation (1) and its modified version (2). In 2002, Liu and Yang [?] studied the bifurcation properties of generalized KdV equation (GKdVE)

$$u_t + au^n u_x + u_{xxx} = 0 \ , \quad a \ \in \mathbb{R} \ , \ n \ \in \ \mathbb{Z}^+ \ . \tag{4}$$

Gungor and Winternitz [?] transformed the Generalized Kadomtsev-Petviashvili Equation (GKPE) $(u_t + p(t)uu_x + q(t)u_{xxx})_x + \sigma(y,t)u_{yy} + a(y,t)u_y + b(y,t)u_{xy}$

$$+c(y,t)u_{xx} + e(y,t)u_x + f(y,t)u + h(y,t) = 0 \ , \tag{5}$$

* E-mail: bhabhu1977@gmail.com

to its canonical form and established conditions on the coefficient functions under which (5) has an infinite dimensional symmetry group having a Kac-Moody-Virasoro structure. In [?], they carried out the symmetry analysis of Variable Coefficient Kadomtsev Petviashvili Equation (VCKP) in the form,

$$(u_t + f(x, y, t)uu_x + g(x, y, t)u_{xxx})_x + h(x, y, t)u_y = 0.$$

The equal width (EW) wave equation was suggested by Morrison and Meiss [?] to be used as a model partial differential equation for the simulation of one-dimensional wave propagation in a nonlinear medium with a dispersion process. The EW wave equation is an alternative description of the nonlinear dispersive waves to the more usual Korteweg de Vries (KdV) equation. It has been shown to have solitary wave solutions and to govern a large number of important physical phenomena such as shallow water waves and plasma waves. In this chapter, we discuss the symmetry reductions of the (2+1)-dimensional Equal Width wave equation with damping term

$$u_t + u + uu_x - \mu(u_{xxt} + u_{yyt}) = 0\ , \quad \text{where } \mu \in \mathbb{R}\ . \tag{6}$$

Our intention is to show that equation (6) admits a three-dimensional symmetry group and determine the corresponding Lie algebra, classify the one- and two-dimensional subalgebras of the symmetry algebra of (6) in order to reduce (6) to (1+1)-dimensional PDEs and then to ODEs. We shall establish that the symmetry generators form a closed Lie algebra and this allowed us to use the recent method due to Ahmad, Bokhari, Kara and Zaman [?] to successively reduce (6) to (1+1)-dimensional PDEs and ODEs with the help of two-dimensional Abelian and non-Abelian solvable subalgebras. This chapter is organised as follows: First, we determine the symmetry group of (6) and write down the associated Lie algebra. secondly, we consider all one-dimensional subalgebras and obtain the corresponding reductions to (1+1)-dimensional PDEs. Next, we show that the generators form a closed Lie algebra and use this fact to reduce (6) successively to (1+1)- dimensional PDEs and ODEs. Finally, we summarises the conclusions of the present work.

2. The Symmetry Group & Lie Algebra of $u_t+u+uu_x-\mu(u_{xxt}+u_{yyt}) = 0$

If (6) is invariant under a one parameter Lie group of point transformations (Bluman and Kumei [3], Olver[?])

$$x^* = x + \epsilon\, \xi(x, y, t; u) + O(\epsilon^2)\ , \tag{7}$$

$$y^* = y + \epsilon\, \eta(x, y, t; u) + O(\epsilon^2)\ , \tag{8}$$

$$t^* = t + \epsilon\, \tau(x, y, t; u) + O(\epsilon^2)\ , \tag{9}$$

$$u^* = u + \epsilon\, \phi(x, y, t; u) + O(\epsilon^2)\ . \tag{10}$$

Then the third Prolongation $Pr^3(V)$ of the corresponding vector field

$$V = \xi(x, y, t; u)\frac{\partial}{\partial x} + \eta(x, y, t; u)\frac{\partial}{\partial y} + \tau(x, y, t; u)\frac{\partial}{\partial t} + \phi(x, y, t; u)\frac{\partial}{\partial u}, \tag{11}$$

satisfies

$$pr^3(V)\Omega(x, y, t; u)|_{\Omega(x,y,t;u=0} = 0. \tag{12}$$

The determining equations are obtained from (12) and solved for the infinitesimals ξ, η, τ and ϕ. They are as follows

$$\xi = k_1 , \tag{13}$$

$$\eta = k_3 , \tag{14}$$

$$\tau = k_2 , \tag{15}$$

$$\phi = 0 . \tag{16}$$

At this stage, we construct the symmetry generators corresponding to each of the constants involved. Totally there are three generators given by

$$V_1 = \partial x ,$$

$$V_2 = \partial t ,$$

$$V_3 = \partial y .$$

The symmetry generators found in Eq.(17) form a closed Lie Algebra whose commutation table is shown below.

$[V_i, V_j]$	V_1	V_2	V_3
V_1	0	0	0
V_2	0	0	0
V_3	0	0	0

Table 1. Commutator Table

The commutation relations of the Lie algebra, determined by V_1, V_2 and V_3 are shown in the above table. For this three-dimensional Lie algebra the commutator table for V_i is a $(3 \otimes 3)$ table whose $(i,j)^{th}$ entry expresses the Lie Bracket $[V_i, V_j]$ given by the above Lie algebra L. The table is skew-symmetric and the diagonal elements all vanish. The coefficient $C_{i,j,k}$ is the coefficient of V_i of the $(i,j)^{th}$ entry of the commutator table. The Lie algebra L is solvable. In the next section, we derive the reduction of (6) to PDEs with two independent variables and ODEs. These are three one-dimensional Lie subalgebras

$$L_{s,1} = \{V_1\} \ , \ L_{s,2} = \{V_2\} \ , \ L_{s,3} = \{V_3\} \ ,$$

and corresponding to each one-dimensional subalgebras we may reduce (6) to a PDE with two independent variables. Further reductions to ODEs are associated with two-dimensional subalgebras. It is evident from the commutator table that there are no two-dimensional solvable non-abelian subalgebras. And there are three two-dimensional Abelian subalgebras, namely,

$$L_{A,1} = \{V_1, V_2\} \ , \ L_{A,2} = \{V_1, V_3\} \ , \ L_{A,3} = \{V_2, V_3\} \ .$$

3. Homotopy Perturbation Method (HPM)

To describe the HPM, consider the following general nonlinear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{17}$$

under the boundary condition

$$B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \partial\Omega, \tag{18}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function and $\partial\Omega$ is a boundary of the domain Ω. The operator A can be divided into two parts L and N, where L is a linear operator while N is a nonlinear operator. Then Eq. (17) can be rewrittten as

$$L(u) + N(u) - f(r) = 0, \tag{19}$$

Using the homotopy technique, we construct a homotopy:

$$V(r,p) : \Omega \times [0,1] \to \mathbb{R}$$

which satisfies

$$H(V,p) = (1-p)[L(V) - L(u_0)] + p[A(V) - f(r)]$$

or

$$H(V,p) = L(V) - L(u_0) + pL(u_0) + p[N(V) - f(r)], \; r \in \Omega, \quad p \in [0,1], \tag{20}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is the initial approximation of Eq. (17) which satisfies the boundary conditions. Obviously, considering Eq. (20), we will have

$$\begin{aligned} H(V,0) &= L(V) - L(u_0) = 0, \\ H(V,1) &= A(V) - f(r) = 0, \end{aligned} \tag{21}$$

changing the process of p from zero to unity is just that $V(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called the deformation also $A(V) - f(r)$ and $L(u)$ are called as homotopy. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter [**?** **?** **?**] to obtain

$$V = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots = \sum_{n=0}^{\infty} p^n v_n. \tag{22}$$

$p \to 1$ results the approximate solution of eq (17) as

$$u = \lim_{p\to 1} V = v_0 + v_1 + v_2 + \cdots = \sum_{n=0}^{\infty} v_n. \tag{23}$$

A comparison of like powers of p gives the solutions of various orders. Series (23) is convergent for most of the cases. However, convergence rate depends on the nonlinear operator,$N(V)$. He [**?**] suggested the following opinions:

(1). The second derivative of $N(V)$ with respect to v must be small as the parameter p may be relatively large.

(2). The norm of $L^{-1}\frac{\partial N}{\partial u}$ must be smaller than one so that the series converges.

Now, we implement the HPM method to Eq.(**??**). According to HPM, we construct the following simple homotopy

$$u_t + p[(-\mu_1 u_{x_1,x_1 t} - \mu_2 u_{x_2,x_2 t} - \cdots - \mu_n u_{x_n,x_n t}) - \nu u u_{x_1}] = 0. \tag{24}$$

With the initial approximation $u(x_1,\ldots,x_n,0) = u_0(x_1,x_2,\ldots,x_n)$, we suppose that the solution has the following form

$$u(x_1,\ldots,x_n,t) = u_0(x_1,\ldots,x_n,t) + pu_1(x_1,\ldots,x_n,t) + p^2 u_2(x_1,\ldots,x_n,t) + \ldots. \tag{25}$$

Insertion of eqn. (25) into eqn. (24) and equating the term with like powers of p, we get

$$p^0 : \frac{\partial u_0}{\partial t} = 0. \tag{26}$$

$$p^1 : \frac{\partial u_1}{\partial t} = \left(\mu_1 \frac{\partial^3 u_0}{\partial x_1^2 \partial t} + \mu_2 \frac{\partial^3 u_0}{\partial x_2^2 \partial t} + \cdots + \mu_n \frac{\partial^3 u_0}{\partial x_n^2 \partial t}\right) + \nu u_0 \frac{\partial u_0}{\partial x_1}. \tag{27}$$

$$p^2 : \frac{\partial u_2}{\partial t} = \left(\mu_1 \frac{\partial^3 u_0}{\partial x_1^2 \partial t} + \mu_2 \frac{\partial^3 u_0}{\partial x_2^2 \partial t} + \cdots + \mu_n \frac{\partial^3 u_0}{\partial x_n^2 \partial t}\right) + \nu \left(u_0 \frac{\partial u_1}{\partial x_1} + u_1 \frac{\partial u_0}{\partial x_1}\right), \tag{28}$$

and so on. We solve Eqs. (26)-(28), to get the values of u_0, u_1, u_2 etc. Thus as considering Eq. (25) and letting $p = 1$, we obtain the approximate analytic solution of Eq. (**??**) as

$$u(x_1, \ldots, x_n, t) = u_0(x_1, \ldots, x_n, t) + u_1(x_1, \ldots, x_n, t) + u_2(x_1, \ldots, x_n, t) + \ldots \tag{29}$$

4. Computational Illustrations of HPM

In this section, we describe the above method by the following examples to validate the efficiency of the HPM.

Example 4.1. *Consider the (n+1)-dimensional Equal Width wave equation and assuming the constants $\nu = \mu_i' s = 1$ as,*

$$u_t = u_{x_1 x_1 t} + u_{x_2 x_2 t} + \cdots + u_{x_n x_n t} + u u_{x_1}, \tag{30}$$

under the initial condition

$$u(x_1, x_2, \ldots, x_n, 0) = u_0(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n. \tag{31}$$

Applying the homotopy perturbation method to Equation (30), we have

$$u_t + p[(-u_{x_1,x_1 t} - u_{x_2,x_2 t} - \cdots - u_{x_n,x_n t}) - u u_{x_1}] = 0. \tag{32}$$

In the view of HPM, we use the homotopy parameter p to expand the solution

$$u(x_1, x_2, \ldots, x_n, t) = u_0 + p u_1 + p^2 u_2 + \ldots \tag{33}$$

The approximate solution can be obtained by taking $p = 1$ in Eq. (33) as

$$u(x_1, x_2, \ldots, x_n, t) = u_0 + u_1 + u_2 + \ldots \tag{34}$$

Now substituting from Eq. (32) into Equation (31) and equating the terms with identical powers of p, we obtain the series of linear equations, which can be easily solved. First few linear equations are given as

$$p^0 : \frac{\partial u_0}{\partial t} = 0. \tag{35}$$

$$p^1 : \frac{\partial u_1}{\partial t} = \left(\frac{\partial^3 u_0}{\partial x_1^2 \partial t} + \frac{\partial^3 u_0}{\partial x_2^2 \partial t} + \cdots + \frac{\partial^3 u_0}{\partial x_n^2 \partial t}\right) + u_0 \frac{\partial u_0}{\partial x_1}. \tag{36}$$

$$p^2 : \frac{\partial u_2}{\partial t} = \left(\frac{\partial^3 u_0}{\partial x_1^2 \partial t} + \frac{\partial^3 u_0}{\partial x_2^2 \partial t} + \cdots + \frac{\partial^3 u_0}{\partial x_n^2 \partial t}\right) + \left(u_0 \frac{\partial u_1}{\partial x_1} + u_1 \frac{\partial u_0}{\partial x_1}\right). \tag{37}$$

Using the initial condition (31), the solution of Equation (35) is given by

$$u(x_1, x_2, \ldots, x_n, 0) = u_0(x_1, x_2, \ldots, x_n) = (x_1 + x_2 + \cdots + x_n). \tag{38}$$

Then the solution of Equation (36) will be

$$u_1(x_1, \ldots, x_n, t) = \int_0^t \left(\left(\frac{\partial^3 u_0}{\partial x_1^2 \partial t} + \frac{\partial^3 u_0}{\partial x_2^2 \partial t} + \cdots + \frac{\partial^3 u_0}{\partial x_n^2 \partial t} \right) + u_0 \frac{\partial u_0}{\partial x_1} \right) dt. \tag{39}$$

$$u_1(x_1, x_2, \ldots, x_n, t) = (x_1 + x_2 + \cdots + x_n)t. \tag{40}$$

Also, we can find the solution of Equation (37) by using the following formula

$$u_2(x_1, \ldots, x_n, t) = \int_0^t \left(\frac{\partial^3 u_0}{\partial x_1^2 \partial t} + \frac{\partial^3 u_0}{\partial x_2^2 \partial t} + \cdots + \frac{\partial^3 u_0}{\partial x_n^2 \partial t} \right) + \left(u_0 \frac{\partial u_1}{\partial x_1} + u_1 \frac{\partial u_0}{\partial x_1} \right) dt. \tag{41}$$

$$u_2(x_1, x_2, \ldots, x_n, t) = (x_1 + x_2 + \cdots + x_n)t^2. \tag{42}$$

etc. Therefore, from Equation (38), the approximate solution of Equation (30) is given as

$$u(x_1, \ldots, x_n, t) = (x_1 + \cdots + x_n) + (x_1 + \cdots + x_n)t + (x_1 + \cdots + x_n)t^2 + \ldots \tag{43}$$

Hence the exact solution can be expressed as

$$u(x_1, x_2, \ldots, x_n, t) = \frac{(x_1 + x_2 + \cdots + x_n)}{(1 - t)}, \tag{44}$$

provided that $0 \leq t < 1$.

4.1. Illustration of HPM for (3+1)-dimensional Equal Width wave equation

Consider the (3+1)-dimensional Equal Width wave equation as,

$$u_t = u_{xxt} + u_{yyt} + u_{zzt} + uu_x, \tag{45}$$

under the initial condition

$$u(x, y, z, 0) = u_0(x, y, z) = x + y + z. \tag{46}$$

Applying the homotopy perturbation method to Equation (45), we have

$$u_t + p[(-u_{xxt} - u_{yyt} - u_{zzt}) - uu_x] = 0. \tag{47}$$

In the view of HPM, we use the homotopy parameter p to expand the solution

$$u(x, y, z, t) = u_0 + pu_1 + p^2 u_2 + \ldots \tag{48}$$

The approximate solution can be obtained by taking $p = 1$ in Equation (48) as

$$u(x, y, z, t) = u_0 + u_1 + u_2 + \ldots \tag{49}$$

Now, substituting Equation (48) into Equation (47) and equating the terms with identical powers of p, we obtain the series of linear equations. First few linear equations are given as

$$p^0 : \frac{\partial u_0}{\partial t} = 0. \tag{50}$$

$$p^1 : \frac{\partial u_1}{\partial t} = \left(\frac{\partial^3 u_0}{\partial x^2 \partial t} + \frac{\partial^3 u_0}{\partial y^2 \partial t} + \frac{\partial^3 u_0}{\partial z^2 \partial t}\right) + u_0 \frac{\partial u_0}{\partial x}. \tag{51}$$

$$p^2 : \frac{\partial u_2}{\partial t} = \left(\frac{\partial^3 u_0}{\partial x^2 \partial t} + \frac{\partial^3 u_0}{\partial y^2 \partial t} + \frac{\partial^3 u_0}{\partial z^2 \partial t}\right) + \left(u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x}\right). \tag{52}$$

Then the solution of Equation (50) using the initial condition (46) is given by

$$u(x, y, z, 0) = u_0(x, y, z) = (x + y + z). \tag{53}$$

We derive the solution of Equation (51) in the following form

$$u_1(x, y, z, t) = \int_0^t \left(\frac{\partial^3 u_0}{\partial x^2 \partial t} + \frac{\partial^3 u_0}{\partial y^2 \partial t} + \frac{\partial^3 u_0}{\partial z^2 \partial t}\right) + u_0 \frac{\partial u_0}{\partial x}. \tag{54}$$

$$u_1(x, y, z, t) = (x + y + z)t. \tag{55}$$

Also, we can find the solution of Equation (52) by using the following formula

$$u_2(x, y, z, t) = \int_0^t \left(\frac{\partial^3 u_0}{\partial x^2 \partial t} + \frac{\partial^3 u_0}{\partial y^2 \partial t} + \frac{\partial^3 u_0}{\partial z^2 \partial t}\right) + \left(u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x}\right) dt. \tag{56}$$

$$u_2(x, y, z, t) = (x + y + z)t^2. \tag{57}$$

etc. Therefore, from Equation (49), the approximate solution of Equation (45) is given as

$$u(x, y, z, t) = (x + y + z) + (x + y + z)t + (x + y + z)t^2 + \dots \tag{58}$$

Thus, the exact solution can be expressed as

$$u(x, y, z, t) = \frac{(x + y + z)}{(1 - t)}, \tag{59}$$

provided that $0 \leq t < 1$.

4.2. Illustration of HPM for (2+1)-dimensional Equal Width wave equation

Consider the (2+1)-dimensional Equal Width wave equation as,

$$u_t = u_{xxt} + u_{yyt} + uu_x, \tag{60}$$

under the initial condition

$$u(x, y, 0) = u_0(x, y) = x + y. \tag{61}$$

Applying the homotopy perturbation method to Equation (60), we have

$$u_t + p[(-u_{xxt} - u_{yyt}) - uu_x] = 0. \tag{62}$$

In the view of HPM, we use the homotopy parameter p to expand the solution

$$u(x, y, t) = u_0 + pu_1 + p^2 u_2 + \dots \tag{63}$$

The approximate solution can be obtained by taking $p = 1$ in Equation (63) as

$$u(x, y, t) = u_0 + u_1 + u_2 + \dots \tag{64}$$

Now, substituting Equation (63) into Equation (62) and equating the terms with identical powers of p, we obtain the series of linear equations. First few linear equations are given as

$$p^0 : \frac{\partial u_0}{\partial t} = 0. \tag{65}$$

$$p^1 : \frac{\partial u_1}{\partial t} = \left(\frac{\partial^3 u_0}{\partial x^2 \partial t} + \frac{\partial^3 u_0}{\partial y^2 \partial t} \right) + u_0 \frac{\partial u_0}{\partial x}. \tag{66}$$

$$p^2 : \frac{\partial u_2}{\partial t} = \left(\frac{\partial^3 u_0}{\partial x^2 \partial t} + \frac{\partial^3 u_0}{\partial y^2 \partial t} \right) + \left(u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x} \right). \tag{67}$$

Then the solution of Equation (65) using the initial condition (46) is given by

$$u(x, y, 0) = u_0(x, y) = (x + y). \tag{68}$$

We derive the solution of Equation (66) in the following form

$$u_1(x, y, t) = \int_0^t \left(\frac{\partial^3 u_0}{\partial x^2 \partial t} + \frac{\partial^3 u_0}{\partial y^2 \partial t} \right) + u_0 \frac{\partial u_0}{\partial x}. \tag{69}$$

$$u_1(x, y, t) = (x + y)t. \tag{70}$$

Also, we can find the solution of Equation (67) by using the following formula

$$u_2(x, y, t) = \int_0^t \left(\frac{\partial^3 u_0}{\partial x^2 \partial t} + \frac{\partial^3 u_0}{\partial y^2 \partial t} \right) + \left(u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x} \right) dt. \tag{71}$$

$$u_2(x, y, t) = (x + y)t^2. \tag{72}$$

etc. Therefore, from Equation (64), the approximate solution of Equation (60) is given as

$$u(x, y, t) = (x + y) + (x + y)t + (x + y)t^2 + \dots \tag{73}$$

The exact solution can be expressed as

$$u(x, y, t) = \frac{(x + y)}{(1 - t)}, \tag{74}$$

provided that $0 \leq t < 1$.

4.3. Illustration of HPM for (1+1)-dimensional Equal Width wave equation

Consider the (1+1)-dimensional Equal Width wave equation as,

$$u_t = u_{xxt} + uu_x, \tag{75}$$

under the initial condition

$$u(x,0) = u_0(x) = 2x. \tag{76}$$

Applying the homotopy perturbation method to Equation (75), we have

$$u_t + p[-u_{xxt} - uu_x] = 0. \tag{77}$$

In the view of HPM, we use the homotopy parameter p to expand the solution

$$u(x,t) = u_0 + pu_1 + p^2 u_2 + \dots \tag{78}$$

The approximate solution can be obtained by taking $p = 1$ in Eq. (78) as

$$u(x,t) = u_0 + u_1 + u_2 + \dots \tag{79}$$

Now, substituting Equation (78) into Eq.(77) and equating the terms with identical powers of p, we obtain the series of linear equations. First few linear equations are given as

$$p^0 : \frac{\partial u_0}{\partial t} = 0. \tag{80}$$

$$p^1 : \frac{\partial u_1}{\partial t} = \left(\frac{\partial^3 u_0}{\partial x^2 \partial t}\right) + u_0 \frac{\partial u_0}{\partial x}. \tag{81}$$

$$p^2 : \frac{\partial u_2}{\partial t} = \left(\frac{\partial^3 u_0}{\partial x^2 \partial t}\right) + \left(u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x}\right). \tag{82}$$

Then the solution of Equation (80) using the initial condition (46) is given by

$$u(x,0) = u_0(x) = (2x). \tag{83}$$

We derive the solution of Equation (81) in the following form

$$u_1(x,t) = \int_0^t \left(\frac{\partial^3 u_0}{\partial x^2 \partial t}\right) + u_0 \frac{\partial u_0}{\partial x}. \tag{84}$$

$$u_1(x,t) = -4(x)t. \tag{85}$$

Also, we can find the solution of Equation (82) by using the following formula

$$u_2(x,t) = \int_0^t \left(\frac{\partial^3 u_0}{\partial x^2 \partial t}\right) + \left(u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x}\right) dt. \tag{86}$$

$$u_2(x,t) = 8(x)t^2. \tag{87}$$

etc. Therefore, from Equation (79), the approximate solution of Equation (75) is given as

$$u(x,t) = 2x - 4xt + 8xt^2 + \dots \tag{88}$$

Hence, the exact solution can be expressed as

$$u(x,t) = \frac{(2x)}{(1+2t)}. \tag{89}$$

5. Adomian Decomposition Method (ADM)

Consider the following linear operator and their inverse operators:

$$L_t = \frac{\partial}{\partial t};\ L_{x_i,x_i,t} = \frac{\partial^3}{\partial x_i^2 \partial t},\ i = 1, 2, \ldots, n.$$

$$L_t^{-1} = \int_0^t (.)dt,\ L_{x_i,x_i,t} = \int_0^{x_i}\int_0^t (.)d\tau d\tau d\gamma,\ i = 1, 2, \ldots, n.$$

Using the above notations, Equation (**??**) becomes

$$L_t(u) = \left(\sum_{i=1}^{n} \alpha_i L_{x_i,x_i,t}(u)\right) + \nu u \frac{\partial u}{\partial x_i}, \tag{90}$$

operating the inverse operators L_t^{-1} to Equation (90) and using the initial condition gives

$$u(x_1, \ldots, x_n, t) = u_0(x_1, \ldots, x_n, t) + L_t^{-1}\left(\sum_{i=1}^{n} \alpha_i L_{x_i,x_i,t}(u)\right) + \nu L_t^{-1} u \frac{\partial u}{\partial x_1}. \tag{91}$$

The decomposition method consists of representing the solution $u(x_1, x_2, \ldots, x_n, t)$ by the decomposition series

$$u(x_1, x_2, \ldots, x_n, t) = \sum_{q=0}^{\infty} u_q(x_1, x_2, \ldots, x_n, t). \tag{92}$$

The nonlinear term $u\frac{\partial u}{\partial x_1}$ is represented by a series of the so called Adomian polynomials, given by

$$u\frac{\partial u}{\partial x_i} = \sum_{q=0}^{\infty} A_q(x_1, x_2, \ldots, x_n, t). \tag{93}$$

The component $u_q(x_1, x_2, \ldots, x_n, t)$ of the solution $u(x_1, x_2, \ldots, x_n, t)$ is determined in a recursive manner. Replacing the decomposition series (92) and (93) for u into Equation (91) gives

$$\begin{aligned}\sum_{q=0}^{\infty} u_q(x_1, x_2, \ldots, x_n, t) &= u_0(x_1, \ldots, x_n, t) + L_t^{-1}\left(\sum_{i=1}^{n} \alpha_i L_{x_i,x_i,t}(u)\right) \\ &+ \nu L_t^{-1} \sum_{q=0}^{\infty} A_q(x_1, x_2, \ldots, x_n, t).\end{aligned} \tag{94}$$

According to ADM the zero-th component $u_0(x_1, \ldots, x_n, t)$ is identified from the initial or boundary conditions and from the source terms. The remaining components of $u(x_1, \ldots, x_n, t)$ are determined in a recursion manner as follows

$$u_0(x_1, \ldots, x_n, t) = u_0(x_1, \ldots, x_n), \tag{95}$$

$$u_k(x_1, \ldots, x_n, t) = L_t^{-1}\left(\sum_{i=1}^{n} \alpha_i L_{x_i,x_i,t}(u)\right) + \nu L_t^{-1}(A_k) + \gamma L^{-1}(u_k),\ k \geq 0, \tag{96}$$

where the Adomian polynomials for the nonlinear term $u\frac{\partial u}{\partial x_1}$ are derived from the following recursive formulation,

$$A_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left(\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right)_{\lambda=0},\ k = 0, 1, 2, \ldots. \tag{97}$$

First few Adomian polynomials are given as

$$A_0 = u_0 \frac{\partial u_0}{\partial x_1}, \; A_1 = u_0 \frac{\partial u_1}{\partial x_1} + u_1 \frac{\partial u_0}{\partial x_1}, \\ A_2 = u_2 \frac{\partial u_0}{\partial x_1} + u_1 \frac{\partial u_1}{\partial x_1} + u_0 \frac{\partial u_2}{\partial x_1}, \tag{98}$$

using Equation (96) for the adomian polynomials A_k, we get

$$u_0(x_1, \ldots, x_n, t) = u_0(x_1, \ldots, x_n), \tag{99}$$

$$u_1(x_1, \ldots, x_n, t) = L_t^{-1}\left(\sum_{i=1}^{n} \alpha_i L_{x_i,x_i,t}(u_0)\right) + \nu L_t^{-1}(A_0) + \gamma L^{-1}(u_0), \tag{100}$$

$$u_2(x_1, \ldots, x_n, t) = L_t^{-1}\left(\sum_{i=1}^{n} \alpha_i L_{x_i,x_i,t}(u_1)\right) + \nu L_t^{-1}(A_1) + \gamma L^{-1}(u_1), \tag{101}$$

and so on. Then the q-th term, u_q can be determined from

$$u_q = \sum_{0}^{q-1} u_k(x_1, \ldots, x_n, t). \tag{102}$$

Knowing the components of u, the analytical solution follows immediately.

6. Computational Illustrations of ADM for (n+1)-dimensional Equal Width Wave Equation

Using Equations (97) and (98), first few components of the decomposition series are given by

$$u_0(x_1, \ldots, x_n, t) = (x_1 + \cdots + x_n), \tag{103}$$

$$u_1(x_1, \ldots, x_n, t) = (x_1 + \cdots + x_n)t, \tag{104}$$

$$u_2(x_1, \ldots, x_n, t) = (x_1 + \cdots + x_n)t^2, \tag{105}$$

$$u_3(x_1, \ldots, x_n, t) = (x_1 + \cdots + x_n)t^3, \tag{106}$$

Then by the decomposition series, we get the solution

$$\begin{aligned} u(x_1, \ldots, x_n, t) &= \sum_{k=0}^{\infty} u_k(x_1, \ldots, x_n, t), \\ &= u_0(x_1, \ldots, x_n, t) + u_1(x_1, \ldots, x_n, t) + u_2(x_1, \ldots, x_n, t) + \ldots, \\ &= (x_1 + \cdots + x_n)(1 + t + t^2 + \ldots) \end{aligned} \tag{107}$$

Hence, the exact solution can be expressed as

$$u(x_1, \ldots, x_n, t) = \frac{(x_1 + \cdots + x_n)}{(1 - t)}. \tag{108}$$

6.1. Adomian Decomposition Method for (3+1)-dimensional Equal Width wave equation

Using Equations (97) and (98), first few components of the decomposition series are given by

$$u_0(x, y, z, t) = (x + y + z), \tag{109}$$

$$u_1(x, y, z, t) = (x + y + z)t, \tag{110}$$

$$u_2(x, y, z, t) = (x + y + z)t^2, \tag{111}$$

$$u_3(x, y, z, t) = (x + y + z)t^3, \tag{112}$$

and so on. By the decomposition series, we get the solution

$$\begin{aligned} u(x, y, z, t) &= \sum_{k=0}^{\infty} u_k(x, y, z, t), \\ &= u_0(x, y, z, t) + u_1(x, y, z, t) + u_2(x, y, z, t) + \ldots, \\ &= (x + y + z)(1 + t + t^2 + t^3 + \ldots) \end{aligned} \tag{113}$$

Therefore the exact solution can be expressed as

$$u(x, y, z, t) = \frac{(x + y + z)}{(1 - t)}, \tag{114}$$

provided that $0 \leq t < 1$.

6.2. Adomian Decomposition Method for (2+1)-dimensional Equal Width wave equation

Using Equations (97) and (98), first few components of the decomposition series are given by

$$u_0(x, y, t) = (x + y), \tag{115}$$

$$u_1(x, y, t) = (x + y)t, \tag{116}$$

$$u_2(x, y, t) = (x + y)t^2, \tag{117}$$

$$u_3(x, y, t) = (x + y)t^3, \tag{118}$$

and so on. By the decomposition series, we get the solution

$$\begin{aligned} u(x, y, t) &= \sum_{k=0}^{\infty} u_k(x, y, t), \\ &= u_0(x, y, t) + u_1(x, y, t) + u_2(x, y, t) + \ldots, \\ &= (x + y)(1 + t + t^2 + t^3 + \ldots) \end{aligned} \tag{119}$$

Therefore the exact solution can be expressed as

$$u(x, y, t) = \frac{(x + y)}{(1 - t)}, \tag{120}$$

provided that $0 \leq t < 1$.

6.3. Adomian Decomposition Method for (1+1)-dimensional Equal Width wave equation

Using Equations (97) and (98), first few components of the decomposition series are given by

$$u_0(x,t) = (2x), \tag{121}$$

$$u_1(x,t) = (-4x)t, \tag{122}$$

$$u_2(x,t) = (8x)t^2, \tag{123}$$

$$u_3(x,t) = (-16x)t^3, \tag{124}$$

and so on. By the decomposition series, we get the solution

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} u_k(x,t), \\ &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \dots, \\ &= 2x - 4xt + 8xt^2 - 16xt^3 \end{aligned} \tag{125}$$

Finally, the exact solution can be expressed as

$$u(x,t) = \frac{(2x)}{(1+2t)}. \tag{126}$$

References

[1] G.Bluman and S.Kumei, *Symmetries and Differential Equations*, Springer-Verlag, New York, (1989).

[2] P.J.Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, (1986).

[3] A.Ahmad, Ashfaque H.Bokhari, A.H.Kara and F.D.Zaman, *Symmetry Classifications and Reductions of Some Classes of (2+1)-Nonlinear Heat Equation*, J. Math. Anal. Appl., 339(2008), 175-181.

[4] Z.Liu and C.Yang, *The application of bifurcation method to a higher-order KdV equation*, J. Math. Anal. Appl., 275(2002), 1-12.

[5] R.M.Miura, *Korteweg-de Vries equations and generalizations. A remarkable explicit nonlinear transformation*, I.Math. Phys., 9(1968), 1202-1204.

[6] D.J.Korteweg and G.de Vries, *On the Chans of Form of Long Waves Advancing in a Rectangular canal, and On a New type of Long Stationary Waves*, Philosophical Magazine, 39(1985), 422-443.

[7] F.Gungor and P Winternitz, *Generalized Kadomtsev Petviashvili equation with an infinitesimal dimensional symmetry algebra*, J. Math. Anal., 276(2002), 314-328.

[8] F.Gungor and P.Winternitz, *Equivalence Classes and Symmetries of the Variable Coefficient Kadomtsev Petviashvili Equation*, Nonlinear Dynamics, 35(2004), 381-396.