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Decompositions of $\star$-continuity and $\mathcal{A}^{\star}$-I_{ω}-continuity

Research Article

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Abstract: The aim of this paper is to introduce and study the notions of $\mathcal{A}^{\star}$-I_{ω}-sets, I_{ω}-$\mathcal{C}$-sets, η-I_{ω}-sets, $\mathcal{A}^{\star\star}$-I_{ω}-sets, $\eta^{\star}$-I_{ω}-sets, I_{ω}-$\mathcal{C}^{\star}$-sets, $\mathcal{C}^{\star\star}$-I_{ω}-sets and $\mathcal{C}^{\star}$-I_{ω}-sets in ideal topological spaces. Properties of such classes of sets are investigated. Moreover, decompositions of $\star$-continuous functions and decompositions of $\mathcal{A}^{\star}$-I_{ω}-continuous functions in ideal topological spaces are established.

MSC: 54A05, 54A10, 54C05, 54C08, 54C10.

Keywords: $\mathcal{A}^{\star}$-I_{ω}-set, I_{ω}-$\mathcal{C}$-set, $\mathcal{C}^{\star}$-I_{ω}-set, pre*-I_{ω}-open set, semi*-I_{ω}-open set, $\alpha^{\star}$-I_{ω}-open set, $I_{\omega^{\star}}$-submaximal space.

1. Introduction

In 1982, the class of ω-closed subsets of a space (X, τ) was defined to introduce ω-closed functions [7]. Several mathematicians have studied many weakened forms of continuous functions in topological spaces. Hdeib [8] introduced a new weakened form of continuous functions namely, ω-continuous functions. Noiri et al [11] introduced some new weakened forms of continuous functions, namely, pre-ω-continuous functions, α-ω-continuous functions, $\omega^{\star}$-continuous functions, etc.

In this paper, we introduce a new weakened form of continuous functions called $\mathcal{A}^{\star}$-I_{ω}-continuous functions and obtain decompositions of $\star$-continuous functions and $\mathcal{A}^{\star}$-I_{ω}-continuous functions.

2. Preliminaries

Throughout this paper, $\mathbb{R}$ (resp. $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Q}^{\star}$) denotes the set of all real numbers (resp. the set of all natural numbers, the set of all rational numbers, the set of all irrational numbers). By a space (X, τ), we always mean a topological space (X, τ) with no separation properties assumed. If $H \subset X$, $cl(H)$ and $int(H)$ will, respectively, denote the closure and interior of H in (X, τ).

Definition 2.1 ([6]). *A subset K of a space (X, τ) is said to be locally closed if $K = U \cap V$, where U is open and V is closed.*

Definition 2.2 ([4]). *A space (X, τ) is called submaximal if every dense subset is open.*

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Definition 2.3 ([19]). *Let H be a subset of a space (X, τ). A point p in X is called a condensation point of H if for each open set U containing p, $U \cap H$ is uncountable.*

Definition 2.4 ([7]). *A subset H of a space (X, τ) is called ω-closed if it contains all its condensation points. The complement of an ω-closed set is called ω-open.*

It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open sets, denoted by τ_ω, is a topology on X, which is finer than τ. The interior and closure operator in (X, τ_ω) are denoted by int_ω and cl_ω respectively.

Definition 2.5 ([13]). *A subset K of a space (X, τ) is said to be $\alpha^\star$-ω-open if $K \subset int(cl_\omega(int(K)))$.*

Definition 2.6 ([13]). *A subset K of a space (X, τ) is called*

(1). pre*-ω-closed if $cl(int_\omega(K)) \subset K$.

(2). pre*-ω-open if $K \subset int(cl_\omega(K))$.

The complement of a pre*-ω-open set is called pre*-ω-closed.

Definition 2.7 ([2]). *A subset K of a space (X, τ) is called ω-dense if $cl_\omega(K) = X$.*

Definition 2.8 ([12]). *A subset K of a space (X, τ) is called ω-codense if $X \backslash K$ is ω-dense.*

An ideal I [18] on a space (X, τ) is a non-empty collection of subsets of X which satisfies the following conditions.

(1). $H \in I$ and $G \subset H$ imply $G \in I$ and

(2). $H \in I$ and $G \in I$ imply $H \cup G \in I$.

Given a space (X, τ) with an ideal I on X if $\mathbb{P}(X)$ is the set of all subsets of X, a set operator $(.)^\star : \mathbb{P}(X) \to \mathbb{P}(X)$, called a local function of H with respect to τ and I is defined as follows: for $H \subset X$, $H^\star(I, \tau) = \{x \in X : U \cap H \notin I$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$ [10]. A Kuratowski closure operator $cl^\star(.)$ for a topology $\tau^\star(I, \tau)$, called the $\star$-topology, finer than τ is defined by $cl^\star(H) = H \cup H^\star(I, \tau)$ [17]. We will simply write $H^\star$ for $H^\star(I, \tau)$ and $\tau^\star$ for $\tau^\star(I, \tau)$. If I is an ideal on X, then (X, τ, I) is called an ideal topological space or an ideal space. $int^\star(H)$ will denote the interior of H in $(X, \tau^\star)$.

Definition 2.9 ([9]). *A subset H of an ideal topological space (X, τ, I) is said to be $\star$-closed if $H^\star \subset H$ or $cl^\star(H) = H$. The complement of an $\star$-closed set is called $\star$-open.*

Lemma 2.10 ([1]). *Let (X, τ, I) be an ideal topological space and H a subset of X. Then the following properties hold:*

(1). If O is open in (X, τ, I), then $O \cap cl^\star(H) \subset cl^\star(O \cap H)$.

(2). If $H \subset X_0 \subset X$, then $cl^\star_{X_0}(H) = cl^\star(H) \cap X_0$.

Proposition 2.11 ([1]). *Let (X, τ, I) be an ideal topological space and H a subset of X. If $I = \{\phi\}$ (resp. $\mathbb{P}(X)$, $\mathcal{N}$), then $H^\star = cl(H)$ (resp. ϕ, $cl(int(cl(H)))$) and $cl^\star(H) = cl(H)$ (resp. H, $H \cup cl(int(cl(H)))$) where $\mathcal{N}$ is the ideal of all nowhere dense sets of (X, τ).*

Remark 2.12 ([5]). *In any ideal topological space, every open set is $\star$-open but not conversely.*

Definition 2.13 ([15]). *A subset H of an ideal topological space (X, τ, I) is called*

(1). α-I_ω-open if $H \subset int_\omega(cl^\star(int_\omega(H)))$;

(2). pre-I_ω-open if $H \subset int_\omega(cl^\star(H))$;

(3). β-I_ω-open if $H \subset cl^\star(int_\omega(cl^\star(H)))$;

(4). b-I_ω-open if $H \subset int_\omega(cl^\star(H)) \cup cl^\star(int_\omega(H))$.

Definition 2.14 ([14]). *A subset K of an ideal topological space (X, τ, I) is said to be*

(1). semi-I_ω-closed if $int^\star(cl_\omega(K)) \subset K$.

(2). semi-I_ω-open if $K \subset cl^\star(int_\omega(K))$.

This complement of a semi-I_ω-open set is called semi-I_ω-closed.

Remark 2.15 ([14, 15]). *The diagram holds for any subset of an ideal topological space (X, τ, I):*

$$\begin{array}{ccccccc} \textbf{open} & \longrightarrow & \alpha\text{-}I_\omega\textbf{-open} & \longrightarrow & \textbf{pre-}I_\omega\textbf{-open} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \omega\textbf{-open} & \longrightarrow & \textbf{semi-}I_\omega\textbf{-open} & \longrightarrow & \textbf{b-}I_\omega\textbf{-open} & \longrightarrow & \beta\text{-}I_\omega\textbf{-open} \end{array}$$

In this diagram, none of the implications is reversible.

Proposition 2.16 ([14]). *A subset K of an ideal topological space (X, τ, I) is semi-I_ω-open if and only if $cl^\star(K) = cl^\star(int_\omega(K))$.*

Proposition 2.17 ([15]). *The intersection of a pre-I_ω-open set and an open set is pre-I_ω-open.*

Definition 2.18 ([14]). *A subset K of an ideal topological space (X, τ, I) is said to be semi*-I_ω-open if $K \subset cl_\omega(int^\star(K))$.*

3. On New Subsets of τ_ω in Ideal Spaces

Definition 3.1. *A subset K of an ideal topological space (X, τ, I) is called*

(1). pre*-I_ω-closed if $cl^\star(int_\omega(K)) \subset K$.

(2). pre*-I_ω-open if $K \subset int^\star(cl_\omega(K))$.

The complement of a pre*-I_ω-open set is called pre*-I_ω-closed.

Example 3.2. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \mathcal{N}$,*

(1). $K = \mathbb{Q}^\star$ is pre*-I_ω-open, since $int^\star(cl_\omega(K)) = int^\star(\mathbb{R}) = \mathbb{R} \supset \mathbb{Q}^\star = K$.

(2). $K = \mathbb{Q}$ is not pre*-I_ω-open, since $int^\star(cl_\omega(K)) = int^\star(\mathbb{Q}) = \mathbb{R} \backslash cl^\star(\mathbb{Q}^\star) = \mathbb{R} \backslash (\mathbb{Q}^\star \cup \mathbb{R}) = \mathbb{R} \backslash \mathbb{R} = \phi \not\supseteq \mathbb{Q} = K$.

Proposition 3.3. *In an ideal topological space (X, τ, I),*

(1). Every $\star$-open set is pre*-I_ω-open.

(2). Every open set is pre*-I_ω-open.

Proof. (1) Let K be an $\star$-open set in X. Then $K = int^{\star}(K) \subset int^{\star}(cl_\omega(K))$. Thus K is pre*-$I_\omega$-open in X.
(2) Let K be an open set. Then K is $\star$-open. By (1), K is pre*-I_ω-open. □

Example 3.4. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \mathcal{N}$,*

(1). $K = \mathbb{Q}^{\star}$ is pre*-I_ω-open by (1) of Example 3.2. But $K = \mathbb{Q}^{\star}$ is not $\star$-open, since $int^{\star}(K) = \mathbb{R}\backslash cl^{\star}(\mathbb{Q}) = \mathbb{R}\backslash(\mathbb{Q}\cup\mathbb{R}) = \mathbb{R}\backslash\mathbb{R} = \phi \neq \mathbb{Q}^{\star} = K$.

(2). $K = \mathbb{Q}^{\star}$ is pre*-I_ω-open by (1) of Example 3.2. But K is not open, since $int(K) = \phi \neq \mathbb{Q}^{\star} = K$.

Proposition 3.5. *In an ideal topological space (X, τ, I), every pre*-ω-open set is pre*-I_ω-open.*

Proof. Let K be a pre*-ω-open set in X. Then $K \subset int(cl_\omega(K)) \subset int^{\star}(cl_\omega(K))$. Thus K is pre*-I_ω-open in X. □

Example 3.6. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \mathbb{P}(\mathbb{R})$, $K = \mathbb{Q}$ is pre*-I_ω-open, since $int^{\star}(cl_\omega(K)) = int^{\star}(\mathbb{Q}) = \mathbb{R}\backslash cl^{\star}(\mathbb{Q}^{\star}) = \mathbb{R}\backslash\mathbb{Q}^{\star} = \mathbb{Q} \supset \mathbb{Q} = K$. But $K = \mathbb{Q}$ is not pre*-ω-open, since $int(cl_\omega(K)) = int(\mathbb{Q}) = \phi \not\supseteq \mathbb{Q} = K$.*

Definition 3.7. *A subset K of an ideal topological space (X, τ, I) is called an I_ω-t-set if $int^{\star}(K) = int^{\star}(cl_\omega(K))$.*

Example 3.8.

(1). In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \mathbb{P}(\mathbb{R})$, $K = \mathbb{Q}$ is an I_ω-t-set, since $int^{\star}(K) = int^{\star}(cl_\omega(K)) = \mathbb{Q}$ by Example 3.6.

(2). In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}$, $K = \mathbb{Q}^{\star}$ is not an I_ω-t-set, since $int^{\star}(K) = \mathbb{R}\backslash cl^{\star}(\mathbb{Q}) = \mathbb{R}\backslash cl(\mathbb{Q}) = \mathbb{R}\backslash\mathbb{R} = \phi$; $int^{\star}(cl_\omega(K)) = int^{\star}(\mathbb{R}) = \mathbb{R}$ and $int^{\star}(K) \neq int^{\star}(cl_\omega(K))$.

Proposition 3.9. *In an ideal topological space (X, τ, I), a subset K of X is semi-I_ω-closed in X if and only if K is an I_ω-t-set in X.*

Proof. K is semi-I_ω-closed in $X \iff X\backslash K$ is semi-I_ω-open in $X \iff cl^{\star}(X\backslash K) = cl^{\star}(int_\omega(X\backslash K))$ by Proposition 2.16 $\iff X\backslash int^{\star}(K) = cl^{\star}(X\backslash cl_\omega(K)) = X\backslash int^{\star}(cl_\omega(K)) \iff int^{\star}(K) = int^{\star}(cl_\omega(K)) \iff K$ is an I_ω-t-set in X. □

Definition 3.10. *A subset K of an ideal topological space (X, τ, I) is called a $\mathcal{B}$-I_ω-set if $K = U \cap V$, where U is an $\star$-open set and V is an I_ω-t-set.*

Remark 3.11. *In an ideal topological space (X, τ, I),*

(1). Every $\star$-open set is a $\mathcal{B}$-I_ω-set.

(2). Every I_ω-t-set is a $\mathcal{B}$-I_ω-set.

Example 3.12.

(1). In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \mathbb{P}(\mathbb{R})$, $K = \mathbb{Q}$ is a $\mathcal{B}$-I_ω-set by (2) of Remark 3.11 since $K = \mathbb{Q}$ is an I_ω-t-set by (1) of Example 3.8.

(2). In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}$, $K = \mathbb{Q}^{\star}$ is not a $\mathcal{B}$-I_ω-set. If $K = U \cap V$ where U is $\star$-open and V is an I_ω-t-set, then $K \subset U$. But $\mathbb{R}$ is the only open (= $\star$-open) set containing K. Hence $U = \mathbb{R}$ and $K = \mathbb{R} \cap V = V$. This is a contradiction since $K = \mathbb{Q}^{\star}$ is not an I_ω-t-set by (2) of Example 3.8. This proves that $K = \mathbb{Q}^{\star}$ is not a $\mathcal{B}$-I_ω-set.

Remark 3.13. *The converses of (1) and (2) in Remark 3.11 are not true as seen from the following Example.*

Example 3.14.

(1). In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \mathcal{N}, K = \mathbb{Q}$ is an I_ω-t-set, for $int^\star(cl_\omega(K)) = int^\star(K) = \mathbb{R}\backslash cl^\star(\mathbb{Q}^\star) = \mathbb{R}\backslash(\mathbb{Q}^\star \cup \mathbb{R}) = \mathbb{R}\backslash\mathbb{R} = \phi$. Also $K = \mathbb{Q}$ is a $\mathcal{B}$-I_ω-set by (2) of Example 3.11. But $K = \mathbb{Q}$ is not $\star$-open, since $int^\star(K) = \phi \neq \mathbb{Q} = K$.

(2). In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$ and the ideal $I = \mathbb{P}(\mathbb{R})$, since $K = \mathbb{Q}^\star$ is open and hence $\star$-open, $K = \mathbb{Q}^\star$ is a $\mathcal{B}$-I_ω-set by (1) of Remark 3.11. But $K = \mathbb{Q}^\star$ is not an I_ω-t-set, since $int^\star(K) = \mathbb{R}\backslash cl^\star(\mathbb{Q}) = \mathbb{R}\backslash\mathbb{Q} = \mathbb{Q}^\star$; $int^\star(cl_\omega(K)) = int^\star(\mathbb{R}) = \mathbb{R}$ and $int^\star(K) \neq int^\star(cl_\omega(K))$.

Proposition 3.15. *For a subset K of an ideal topological space (X, τ, I), the following are equivalent:*

(1). K is $\star$-open.

(2). K is pre*-I_ω-open and a $\mathcal{B}$-I_ω-set.

Proof. (1) $\Rightarrow$ (2): (2) follows by Proposition 3.3(1) and Remark 3.11(1).
(2) $\Rightarrow$ (1): Given K is a $\mathcal{B}$-I_ω-set. So $K = U \cap V$ where U is $\star$-open and $int^\star(V) = int^\star(cl_\omega(V))$. Then $K \subset U = int^\star(U)$. Also K is pre*-I_ω-open implies $K \subset int^\star(cl_\omega(K)) \subset int^\star(cl_\omega(V)) = int^\star(V)$ by assumption. Thus $K \subset int^\star(U) \cap int^\star(V) = int^\star(U \cap V) = int^\star(K)$ and hence K is $\star$-open. □

Remark 3.16. *The following Examples show that the concepts of pre*-I_ω-openness and being a $\mathcal{B}$-I_ω-set are independent.*

Example 3.17. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}$, $K = \mathbb{Q}^\star$ is pre*-I_ω-open, since $int^\star(cl_\omega(K)) = int^\star(\mathbb{R}) = \mathbb{R} \supset \mathbb{Q}^\star = K$. But $K = \mathbb{Q}^\star$ is not a $\mathcal{B}$-I_ω-set by (2) of Example 3.12.*

Example 3.18. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \mathcal{N}$, $K = \mathbb{Q}$ is a $\mathcal{B}$-I_ω-set by (1) of Example 3.14. But $K = \mathbb{Q}$ is not pre*-I_ω-open by (2) of Example 3.2.*

Definition 3.19. *A subset K of an ideal topological space (X, τ, I) is called*

(1). $\alpha^\star$-I_ω-open if $K \subset int^\star(cl_\omega(int^\star(K)))$.

(2). $\alpha^\star$-I_ω-closed if $cl^\star(int_\omega(cl^\star(K))) \subset K$.

The complement of an $\alpha^\star$-I_ω-open set is called $\alpha^\star$-I_ω-closed.

Example 3.20. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \mathbb{P}(\mathbb{R})$, $K = \mathbb{Q}$ is $\alpha^\star$-I_ω-open, since $int^\star(cl_\omega(int^\star(K))) = int^\star(cl_\omega(\mathbb{R}\backslash cl^\star(\mathbb{Q}^\star))) = int^\star(cl_\omega(\mathbb{R}\backslash\mathbb{Q}^\star)) = int^\star(cl_\omega(\mathbb{Q})) = int^\star(\mathbb{Q}) = \mathbb{Q} \supset \mathbb{Q} = K$.*

Example 3.21. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \mathcal{N}$,*

(1). $K = \mathbb{Q}$ is not $\alpha^\star$-I_ω-open, since $int^\star(cl_\omega(int^\star(K))) = int^\star(cl_\omega(\mathbb{R}\backslash cl^\star(\mathbb{Q}^\star))) = int^\star(cl_\omega(\mathbb{R}\backslash(\mathbb{Q}^\star \cup \mathbb{R}))) = int^\star(cl_\omega(\phi)) = \phi \not\supseteq \mathbb{Q} = K$.

(2). $K = \mathbb{Q}^\star$ is not $\alpha^\star$-I_ω-closed, since $cl^\star(int_\omega(cl^\star(K))) = cl^\star(int_\omega(\mathbb{R})) = \mathbb{R} \not\subseteq \mathbb{Q}^\star = K$.

Proposition 3.22. *In an ideal topological space (X, τ, I),*

(1). Every $\star$-open set is $\alpha^\star$-I_ω-open.

(2). Every open set is $\alpha^\star$-I_ω-open.

Proof. (1) Let K be an $\star$-open set in X. Then $K = int^\star(K) \subset cl_\omega(int^\star(K))$. It implies that $K = int^\star(K) \subset int^\star(cl_\omega(int^\star(K)))$. Hence K is $\alpha^\star$-I_ω-open in X.

(2) Let K be an open set X. Then K is $\star$-open. By (1), K is $\alpha^\star$-I_ω-open. □

Example 3.23.

(1). In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$ and the ideal $I = \{\phi\}$, $K = \mathbb{R}\backslash\{1\}$ is not $\star$-open, since $int^\star(K) = \mathbb{R}\backslash cl^\star(\{1\}) = \mathbb{R}\backslash cl(\{1\}) = \mathbb{R}\backslash\mathbb{Q} = \mathbb{Q}^\star \neq \mathbb{R}\backslash\{1\} = K$. But $K = \mathbb{R}\backslash\{1\}$ is $\alpha^\star$-I_ω-open, since $int^\star(cl_\omega(int^\star(K))) = int^\star(cl_\omega(\mathbb{Q}^\star)) = int^\star(\mathbb{R}) = \mathbb{R} \supset \mathbb{R}\backslash\{1\} = K$.

(2). In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \mathbb{P}(\mathbb{R})$, $K = \mathbb{Q}$ is $\alpha^\star$-I_ω-open by Example 3.20. But $K = \mathbb{Q}$ is not open, since $int(K) = \phi \neq \mathbb{Q} = K$.

Proposition 3.24. *In an ideal topological space (X, τ, I), every $\alpha^\star$-I_ω-open set is pre*-I_ω-open.*

Proof. Let K be an $\alpha^\star$-I_ω-open set in X. Then $K \subset int^\star(cl_\omega(int^\star(K))) \subset int^\star(cl_\omega(K))$. Thus K is pre*-I_ω-open in X. □

Example 3.25. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \mathcal{N}, K = \mathbb{Q}^\star$ is pre*-I_ω-open by (1) of Example 3.2. But K is not $\alpha^\star$-I_ω-open, since $int^\star(cl_\omega(int^\star(K))) = int^\star(cl_\omega(\mathbb{R}\backslash cl^\star(\mathbb{Q}))) = int^\star(cl_\omega(\mathbb{R}\backslash(\mathbb{Q}\cup\mathbb{R}))) = int^\star(cl_\omega(\mathbb{R}\backslash\mathbb{R})) = int^\star(cl_\omega(\phi)) = \phi \not\supseteq \mathbb{Q}^\star = K$.*

Proposition 3.26. *In an ideal topological space (X, τ, I), every $\alpha^\star$-ω-open set is $\alpha^\star$-I_ω-open.*

Proof. Let K be an $\alpha^\star$-ω-open set in X. Then $K \subset int(cl_\omega(int(K))) \subset int^\star(cl_\omega(int^\star(K)))$. Thus K is an $\alpha^\star$-I_ω-open set in X. □

Example 3.27. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \mathbb{P}(\mathbb{R})$, $K = \mathbb{Q}$ is $\alpha^\star$-I_ω-open by Example 3.20. But K is not $\alpha^\star$-ω-open, since $int(cl_\omega(int(K))) = int(cl_\omega(\phi)) = \phi \not\supseteq \mathbb{Q} = K$.*

Proposition 3.28. *Let K be a subset of an ideal topological space (X, τ, I).*

(1). If $K = cl^\star(int_\omega(K))$, then K is $\alpha^\star$-I_ω-closed in X.

(2). If $K = cl(int_\omega(K))$, then K is $\alpha^\star$-I_ω-closed in X.

Proof. (1) If $K = cl^\star(int_\omega(K))$, then we obtain that $cl^\star(int_\omega(cl^\star(K))) = cl^\star(int_\omega(cl^\star(int_\omega(K)))) = cl^\star(int_\omega(K)) = K$. Hence K is an $\alpha^\star$-I_ω-closed set in X.

(2) If $K = cl(int_\omega(K))$, then we obtain that $cl^\star(int_\omega(cl^\star(K))) = cl^\star(int_\omega(cl^\star(cl(int_\omega(K))))) \subset cl(int_\omega(cl(int_\omega(K)))) = cl(int_\omega(K)) = K$. Hence K is an $\alpha^\star$-I_ω-closed set in X. □

Definition 3.29. *A subset K of an ideal topological space (X, τ, I) is called pre-I_ω-regular if K is pre-I_ω-open and pre*-I_ω-closed.*

Example 3.30. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}$,*

(1). $K = \mathbb{Q}$ is pre-I_ω-regular, since $int_\omega(cl^\star(K)) = int_\omega(\mathbb{R}) = \mathbb{R} \supset \mathbb{Q} = K$; $cl^\star(int_\omega(K)) = cl^\star(\phi) = \phi \subset \mathbb{Q} = K$ and hence K is both pre-I_ω-open and pre*-I_ω-closed.

(2). $K = \mathbb{Q}^\star$ is not pre-I_ω-regular, since K is not pre*-I_ω-closed, for $cl^\star(int_\omega(K)) = cl^\star(\mathbb{Q}^\star) = cl(\mathbb{Q}^\star) = \mathbb{R} \not\subseteq \mathbb{Q}^\star = K$.

Remark 3.31. *In an ideal topological space (X, τ, I),*

(1). Every pre-I_ω-regular set is pre-I_ω-open.

(2). Every pre-I_ω-regular set is pre*-I_ω-closed.

The converses of (1) and (2) in Remark 3.31 are not true as seen from the following Examples.

Example 3.32. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}, K = \mathbb{Q}^\star$ is pre-I_ω-open, since $int_\omega(cl^\star(K)) = int_\omega(cl(K)) = int_\omega(\mathbb{R}) = \mathbb{R} \supset \mathbb{Q}^\star = K$. But K is not pre-I_ω-regular by (2) of Example 3.30.*

Example 3.33. *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{N}, \mathbb{Q}^\star, \mathbb{Q}^\star \cup \mathbb{N}\}$ and the ideal $I = \{\phi\}, K = \mathbb{Q}$ is pre*-I_ω-closed, since $cl^\star(int_\omega(K)) = cl^\star(\mathbb{N}) = cl(\mathbb{N}) = \mathbb{Q} = K \subset K$. But $K = \mathbb{Q}$ is not pre-I_ω-open, since $int_\omega(cl^\star(K)) = int_\omega(cl(K)) = int_\omega(K) = \mathbb{N} \not\supseteq \mathbb{Q} = K$. This implies $K = \mathbb{Q}$ is not pre-I_ω-regular.*

Proposition 3.34. *In an ideal topological space (X, τ, I), every $\alpha^\star$-I_ω-open set is semi*-I_ω-open.*

Proof. Let K be an $\alpha^\star$-I_ω-open set in X. Then $K \subset int^\star(cl_\omega(int^\star(K))) \subset cl_\omega(int^\star(K))$. Thus K is semi*-I_ω-open in X. □

Example 3.35. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}$, $K = [0, 1]$ is semi*-I_ω-open for $cl_\omega(int^\star([0, 1])) = cl_\omega(\mathbb{R}\backslash cl^\star(\mathbb{R}\backslash[0, 1])) = cl_\omega(\mathbb{R}\backslash cl(\mathbb{R}\backslash[0, 1])) = cl_\omega(\mathbb{R}\backslash cl((-\infty, 0) \cup (1, \infty))) = cl_\omega(\mathbb{R}\backslash((-\infty, 0] \cup [1, \infty))) = cl_\omega(\mathbb{R}\backslash\mathbb{R}\backslash(0, 1)) = cl_\omega((0, 1)) = [0, 1] \supset [0, 1]$. But $K = [0, 1]$ is not $\alpha^\star$-I_ω-open, since $int^\star(cl_\omega(int^\star([0, 1]))) = int^\star([0, 1]) = (0, 1) \not\supseteq [0, 1]$.*

Theorem 3.36. *Let K be a subset of an ideal topological space (X, τ, I). Then K is $\alpha^\star$-I_ω-open if and only if K is semi*-I_ω-open and pre*-I_ω-open.*

Proof. Let K be an $\alpha^\star$-I_ω-open set in X. Then $K \subset int^\star(cl_\omega(int^\star(K)))$. It follows that $K \subset cl_\omega(int^\star(K))$ and $K \subset int^\star(cl_\omega(K))$. Thus, K is semi*-I_ω-open and pre*-I_ω-open.
Conversely, suppose that K is semi*-I_ω-open and pre*-I_ω-open in X. Then $K \subset cl_\omega(int^\star(K))$ and $K \subset int^\star(cl_\omega(K))$. It follows that $K \subset int^\star(cl_\omega(K)) \subset int^\star(cl_\omega(int^\star(K)))$ which implies that K is $\alpha^\star$-I_ω-open in X. □

Remark 3.37. *The following Examples show that the concepts of semi*-I_ω-openness and pre*-I_ω-openness are independent.*

Example 3.38. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}$, $K = [0, 1]$ is semi*-I_ω-open by Example 3.35. But $K = [0, 1]$ is not pre*-I_ω-open, since $int^\star(cl_\omega(K)) = int^\star([0, 1]) = (0, 1) \not\supseteq [0, 1] = K$.*

Example 3.39. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \mathcal{N}$, $K = \mathbb{Q}^\star$ is pre*-I_ω-open by (1) of Example 3.2. But $K = \mathbb{Q}^\star$ is not semi*-I_ω-open, since $cl_\omega(int^\star(K)) = cl_\omega(\mathbb{R}\backslash cl^\star(\mathbb{Q})) = cl_\omega(\mathbb{R}\backslash(\mathbb{Q} \cup \mathbb{R})) = cl_\omega(\mathbb{R}\backslash\mathbb{R}) = cl_\omega(\phi) = \phi \not\supseteq \mathbb{Q}^\star = K$.*

4. $\mathcal{A}^\star$-I_ω-sets, I_ω-$\mathcal{C}$-sets and $\mathcal{C}^\star$-I_ω-sets

Definition 4.1. *A subset K of an ideal topological space (X, τ, I) is called a $\mathcal{C}^\star$-I_ω-set if $K = U \cap V$, where U is an open set and V is a pre-I_ω-regular set.*

Remark 4.2. *In an ideal topological space (X, τ, I),*

(1). Every open set is a $\mathcal{C}^\star$-I_ω-set.

(2). Every pre-I_ω-regular set is a $\mathcal{C}^\star$-I_ω-set.

The converses of (1) and (2) in Remark 4.2 are not true as seen from the following Example.

Example 4.3. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}$,*

(1). $K = \mathbb{Q}$ is pre-I_ω-regular by (1) of Example 3.30 and hence is a $\mathcal{C}^\star$-I_ω-set by (2) of Remark 4.2. But $K = \mathbb{Q}$ is not open, since $int(K) = \phi \neq \mathbb{Q} = K$.

(2). $K = (0,1)$ is a $\mathcal{C}^\star$-I_ω-set by (1) of Remark 4.2, since K is open. But $K = (0,1)$ is not pre*-I_ω-closed, for $cl^\star(int_\omega(K)) = cl^\star((0,1)) = cl((0,1)) = [0,1] \nsubseteq (0,1) = K$ and hence not pre-I_ω-regular.

Example 4.4. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}$,*

(1). $K = \mathbb{Q}$ is pre-I_ω-regular by (1) of Example 3.30 and hence is a $\mathcal{C}^\star$-I_ω-set by (2) of Remark 4.2.

(2). $K = \mathbb{Q}^\star$ is not a $\mathcal{C}^\star$-I_ω-set. If $K = U \cap V$ where U is open and V is a pre-I_ω-regular set, then $K \subset U$. But $\mathbb{R}$ is the only open set containing K. Hence $U = \mathbb{R}$ and $K = \mathbb{R} \cap V = V$. This is a contradiction since $K = \mathbb{Q}^\star$ is not a pre-I_ω-regular set by (2) of Example 3.30. This proves that $K = \mathbb{Q}^\star$ is not a $\mathcal{C}^\star$-I_ω-set.

Theorem 4.5. *In an ideal topological space (X, τ, I), each $\mathcal{C}^\star$-I_ω-set is pre-I_ω-open.*

Proof. Let K be a $\mathcal{C}^\star$-I_ω-set in X. It follows that $K = L \cap M$, where L is an open set and M is a pre-I_ω-regular set in X. By Remark 3.31(1), M is pre-I_ω-open. Since M is pre-I_ω-open, by Proposition 2.17, $K = L \cap M$ is a pre-I_ω-open set in X. □

Example 4.6. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}$, $K = \mathbb{Q}^\star$ is pre-I_ω-open by Example 3.32. But $K = \mathbb{Q}^\star$ is not a $\mathcal{C}^\star$-I_ω-set by (2) of Example 4.4.*

Remark 4.7. *By Remark 4.2(2) and Theorem 4.5, the following diagram holds for any subset of an ideal topological space (X, τ, I).*

$$\text{pre-}I_\omega\text{-regular} \longrightarrow \mathcal{C}^\star\text{-}I_\omega\text{-set} \longrightarrow \text{pre-}I_\omega\text{-open}$$

Definition 4.8. *A subset K of an ideal topological space (X, τ, I) is called*

(1). an I_ω-$\mathcal{C}$-set if $K = U \cap V$, where U is an open set and V is pre*-I_ω-closed.

(2). a η-I_ω-set if $K = U \cap V$, where U is an open set and V is $\alpha^\star$-I_ω-closed.

(3). an $\mathcal{A}^\star$-I_ω-set if $K = U \cap V$, where U is an open set and $V = cl^\star(int_\omega(V))$.

Remark 4.9. *In an ideal topological space (X, τ, I),*

(1). Every pre*-I_ω-closed set is an I_ω-$\mathcal{C}$-set.

(2). Every $\alpha^\star$-I_ω-closed set is a η-I_ω-set.

(3). For a subset K of X if $K = cl^\star(int_\omega(K))$, then K is an $\mathcal{A}^\star$-I_ω-set.

Example 4.10. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}$,*

(1). $K = \mathbb{Q}$ is pre*-I_ω-closed, for $cl^\star(int_\omega(K)) = cl^\star(\phi) = \phi \subset \mathbb{Q} = K$ and hence is an I_ω-$\mathcal{C}$-set by (1) of Remark 4.9.

(2). $K = \mathbb{Q}^\star$ is not an I_ω-$\mathcal{C}$-set. If $K = U \cap V$ where U is open and V is pre*-I_ω-closed, then $K \subset U$. But $\mathbb{R}$ is the only open set containing K. Hence $U = \mathbb{R}$ and $K = \mathbb{R} \cap V = V$. This is a contradiction since $K = \mathbb{Q}^\star$ is not pre*-I_ω-closed by (2) of Example 3.30. This proves that $K = \mathbb{Q}^\star$ is not an I_ω-$\mathcal{C}$-set.

Example 4.11. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \mathcal{N}$,*

(1). For $K = [0,1]$, $cl^\star(int_\omega(cl^\star(K))) = cl^\star(int_\omega(K)) = cl^\star((0,1)) = [0,1] = K \subset K$. Thus K is $\alpha^\star$-I_ω-closed and hence is a η-I_ω-set by (2) of Remark 4.9.

(2). $K = \mathbb{Q}^\star$ is not a η-I_ω-set. If $K = U \cap V$ where U is open and V is $\alpha^\star$-I_ω-closed, then $K \subset U$. But $\mathbb{R}$ is the only open set containing K. Hence $U = \mathbb{R}$ and $K = \mathbb{R} \cap V = V$. This is a contradiction since $K = \mathbb{Q}^\star$ is not an $\alpha^\star$-I_ω-closed set by (2) of Example 3.21. This proves that $K = \mathbb{Q}^\star$ is not a η-I_ω-set.

(3). For $K = [0,1]$, $cl^\star(int_\omega(K)) = cl^\star((0,1)) = [0,1] = K$. Thus $K = [0,1]$ is an $\mathcal{A}^\star$-I_ω-set by (3) of Remark 4.9.

(4). $K = \mathbb{Q}$ is not an $\mathcal{A}^\star$-I_ω-set. If $K = U \cap V$ where U is open and $V = cl^\star(int_\omega(V))$, then $K \subset U$. But $\mathbb{R}$ is the only open set containing K. Hence $U = \mathbb{R}$ and $K = \mathbb{R} \cap V = V$. This is a contradiction since $cl^\star(int_\omega(K)) = cl^\star(\phi) = \phi \neq \mathbb{Q} = K$. This proves that $K = \mathbb{Q}$ is not an $\mathcal{A}^\star$-I_ω-set.

Remark 4.12. *In an ideal topological space (X, τ, I), every open set is an $\mathcal{A}^\star$-I_ω-set.*

Example 4.13. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \mathcal{N}$, $K = [0,1]$ is an $\mathcal{A}^\star$-I_ω-set by (3) of Example 4.11. But K is not open, since $int(K) = (0,1) \neq [0,1] = K$.*

Remark 4.14. *By Proposition 3.28(1), Proposition 3.24, Remark 3.31(2) and Definition 4.8, the following diagram holds for any subset of an ideal topological space (X, τ, I).*

$$\begin{array}{ccc} \mathcal{C}^\star\text{-}I_\omega\text{-}\textbf{set} & \longrightarrow & I_\omega\text{-}\mathcal{C}\text{-}\textbf{set} \\ & & \uparrow \\ \mathcal{A}^\star\text{-}I_\omega\text{-}\textbf{set} & \longrightarrow & \eta\text{-}I_\omega\text{-}\textbf{set} \end{array}$$

Remark 4.15. *The reverse implications in Remark 4.14 are not true as seen from the following Example.*

Example 4.16.

(1). In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$ and the ideal $I = \{\phi\}$, $K = \mathbb{Q}$ is pre*-I_ω-closed by Proposition 3.3(2) for $K = \mathbb{Q}$ is closed and hence is an I_ω-$\mathcal{C}$-set by (1) of Remark 4.9. But K is not a $\mathcal{C}^\star$-I_ω-set. If $K = U \cap V$ where U is open and V is pre-I_ω-regular, then $K \subset U$. But $\mathbb{R}$ is the only open set containing K. Hence $U = \mathbb{R}$ and $K = \mathbb{R} \cap V = V$. This is a contradiction since $K = \mathbb{Q}$ is not pre-I_ω-regular, being not pre-I_ω-open for $int_\omega(cl^\star(K = \mathbb{Q})) = int_\omega(cl(\mathbb{Q})) = int_\omega(\mathbb{Q}) = \phi \not\supseteq \mathbb{Q} = K$. This proves that $K = \mathbb{Q}$ is not a $\mathcal{C}^\star$-I_ω-set.

(2). In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}$, $K = \mathbb{Q}$ is an I_ω-$\mathcal{C}$-set by (1) of Example 4.10. But $K = \mathbb{Q}$ is not a η-I_ω-set. If $K = U \cap V$ where U is open and V is $\alpha^\star$-I_ω-closed, then $K \subset U$. But $\mathbb{R}$ is the only open set containing K. Hence $U = \mathbb{R}$ and $K = \mathbb{R} \cap V = V$. This is a contradiction since $K = \mathbb{Q}$ is not $\alpha^\star$-I_ω-closed for $cl^\star(int_\omega(cl^\star(K))) = cl^\star(int_\omega(\mathbb{R})) = \mathbb{R} \not\subseteq \mathbb{Q} = K$. This proves that $K = \mathbb{Q}$ is not a η-I_ω-set.

(3). In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$ and the ideal $I = \{\phi\}$, $K = \mathbb{N}$ is $\alpha^\star$-I_ω-closed for $cl^\star(int_\omega(cl^\star(K))) = cl^\star(int_\omega(\mathbb{Q})) = cl^\star(\phi) = \phi \subset \mathbb{N} = K$ and hence $K = \mathbb{N}$ is a η-I_ω-set by (2) of Remark 4.9. But $K = \mathbb{N}$ is not an $\mathcal{A}^\star$-I_ω-set. If $K = U \cap V$ where U is open and $V = cl^\star(int_\omega(V))$, then $K \subset U$. But $\mathbb{R}$ is the only open set containing K. Hence $U = \mathbb{R}$ and $K = \mathbb{R} \cap V = V$. This is a contradiction since $cl^\star(int_\omega(K)) = cl^\star(\phi) = \phi \neq \mathbb{N} = K$. This proves that $K = \mathbb{N}$ is not an $\mathcal{A}^\star$-I_ω-set.

Theorem 4.17. *For a subset K of an ideal topological space (X, τ, I), the following are equivalent:*

(1). K is an I_ω-$\mathcal{C}$-set and a semi-I_ω-open set in X.

(2). $K = L \cap cl^\star(int_\omega(K))$ for an open set L.

Proof. (1) $\Rightarrow$ (2) : Since K is an I_ω-$\mathcal{C}$-set, $K = L \cap M$, where L is an open set and M is a pre*-I_ω-closed set in X. We have $K \subset M$ and $cl^\star(int_\omega(K)) \subset cl^\star(int_\omega(M)) \subset M$ since M is pre*-I_ω-closed in X. Since K is semi-I_ω-open in X, we have $K \subset cl^\star(int_\omega(K))$. It follows that $K = K \cap cl^\star(int_\omega(K)) = L \cap M \cap cl^\star(int_\omega(K)) = L \cap cl^\star(int_\omega(K))$.

(2) $\Rightarrow$ (1): Let $K = L \cap cl^\star(int_\omega(K))$ for an open set L. Then $K \subset cl^\star(int_\omega(K))$ and thus K is semi-I_ω-open in X. Since $cl^\star(int_\omega(K))$ is a $\star$-closed set, by Proposition 3.3(1), it is a pre*-I_ω-closed set in X. Hence, K is an I_ω-$\mathcal{C}$-set in X. □

Theorem 4.18. *For a subset K of an ideal topological space (X, τ, I), the following are equivalent:*

(1). K is an $\mathcal{A}^\star$-I_ω-set.

(2). K is semi-I_ω-open and a η-I_ω-set.

(3). K is semi-I_ω-open and an I_ω-$\mathcal{C}$-set.

Proof. (1) $\Rightarrow$ (2): Suppose that K is an $\mathcal{A}^\star$-I_ω-set in X. It follows that $K = L \cap M$, where L is an open set and $M = cl^\star(int_\omega(M))$. This implies that $K = L \cap M = L \cap cl^\star(int_\omega(M)) \subset cl^\star(L \cap int_\omega(M))$ (by Lemma 2.10) $= cl^\star(int(L) \cap int_\omega(M)) \subset cl^\star(int_\omega(L) \cap int_\omega(M)) = cl^\star(int_\omega(L \cap M)) = cl^\star(int_\omega(K))$. Thus $K \subset cl^\star(int_\omega(K))$ and hence K is a semi-I_ω-open set. Moreover, by Remark 4.14, K is a η-I_ω-set in X.

(2) $\Rightarrow$ (3): It follows from the fact that every η-I_ω-set is an I_ω-$\mathcal{C}$-set in X by Remark 4.14.

(3) $\Rightarrow$ (1): Suppose K is semi-I_ω-open and an I_ω-$\mathcal{C}$-set in X. By Theorem 4.17, $K = L \cap cl^\star(int_\omega(K))$ for an open set L. We have $cl^\star(int_\omega(cl^\star(int_\omega(K)))) = cl^\star(int_\omega(K))$. It follows that K is an $\mathcal{A}^\star$-I_ω-set in X. □

Remark 4.19. *The following Example shows that*

(1). The concepts of semi-I_ω-openness and being a η-I_ω-set are independent.

(2). The concepts of semi-I_ω-openness and being an I_ω-$\mathcal{C}$-set are independent.

Example 4.20.

(1). In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \mathcal{N}$, $K = \mathbb{Q}^\star$ is semi-I_ω-open, since $cl^\star(int_\omega(K)) = cl^\star(\mathbb{Q}^\star) = \mathbb{R} \supset \mathbb{Q}^\star = K$. But $K = \mathbb{Q}^\star$ is not a η-I_ω-set by (2) of Example 4.11.

(2). In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$ and the ideal $I = \{\phi\}$, $K = \mathbb{N}$ is a η-I_ω-set by (3) of Example 4.16. But $K = \mathbb{N}$ is not semi-I_ω-open, since $cl^\star(int_\omega(K)) = cl^\star(\phi) = \phi \not\supseteq \mathbb{N} = K$.

(3). In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}, K = \mathbb{Q}$ is an I_ω-$\mathcal{C}$-set by (1) of Example 4.10. But $K = \mathbb{Q}$ is not semi-I_ω-open, since $cl^\star(int_\omega(K)) = cl^\star(\phi) = \phi \not\supseteq \mathbb{Q} = K$.

(4). In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}, K = \mathbb{Q}^\star$ is semi-I_ω-open, since $cl^\star(int_\omega(K)) = cl^\star(\mathbb{Q}^\star) = cl(\mathbb{Q}^\star) = \mathbb{R} \supset \mathbb{Q}^\star = K$. But $K = \mathbb{Q}^\star$ is not an I_ω-$\mathcal{C}$-set by (2) of Example 4.10.

5. $A^{\star\star}$-I_ω-sets, I_ω-$\mathcal{C}^\star$-sets and $\mathcal{C}^{\star\star}$-I_ω-sets

Definition 5.1. *A subset K of an ideal topological space (X, τ, I) is called a $\mathcal{C}^{\star\star}$-I_ω-set if $K = U \cap V$, where U is an $\star$-open set and V is a pre-I_ω-regular set.*

Remark 5.2. *In an ideal topological space (X, τ, I),*

(1). Every $\star$-open set is a $\mathcal{C}^{\star\star}$-I_ω-set.

(2). Every pre-I_ω-regular set is a $\mathcal{C}^{\star\star}$-I_ω-set.

The converses of (1) and (2) in Remark 5.2 are not true as seen from the following Example.

Example 5.3. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}$,*

(1). $K = \mathbb{Q}$ is pre-I_ω-regular by (1) of Example 3.30 and hence is a $\mathcal{C}^{\star\star}$-I_ω-set by (2) of Remark 5.2. But $K = \mathbb{Q}$ is not $\star$-open, since $int^\star(K) = \mathbb{R}\backslash cl^\star(\mathbb{Q}^\star) = \mathbb{R}\backslash cl(\mathbb{Q}^\star) = \mathbb{R}\backslash\mathbb{R} = \phi \neq \mathbb{Q} = K$.

(2). $K = (0, 1)$ is a $\mathcal{C}^{\star\star}$-I_ω-set by (1) of Remark 5.2, since K is open and hence $\star$-open. But $K = (0, 1)$ is not pre*-I_ω-closed, for $cl^\star(int_\omega(K)) = cl^\star((0, 1)) = cl((0, 1)) = [0, 1] \nsubseteq (0, 1) = K$ and hence not pre-I_ω-regular.

Example 5.4. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}$,*

(1). $K = \mathbb{Q}$ is pre-I_ω-regular by (1) of Example 3.30 and hence is a $\mathcal{C}^{\star\star}$-I_ω-set by (2) of Remark 5.2.

(2). $K = \mathbb{Q}^\star$ is not a $\mathcal{C}^{\star\star}$-I_ω-set. If $K = U \cap V$ where U is $\star$-open and V is a pre-I_ω-regular set, then $K \subset U$. But $\mathbb{R}$ is the only open ($= \star$-open) set containing K. Hence $U = \mathbb{R}$ and $K = \mathbb{R} \cap V = V$. This is a contradiction since $K = \mathbb{Q}^\star$ is not a pre-I_ω-regular set by (2) of Example 3.30. This proves that $K = \mathbb{Q}^\star$ is not a $\mathcal{C}^{\star\star}$-I_ω-set.

Definition 5.5. *A subset K of an ideal topological space (X, τ, I) is called*

(1). an I_ω-$\mathcal{C}^\star$-set if $K = U \cap V$, where U is an $\star$-open set and V is pre*-I_ω-closed.

(2). a $\eta^\star$-I_ω-set if $K = U \cap V$, where U is an $\star$-open set and V is $\alpha^\star$-I_ω-closed.

(3). an $\mathcal{A}^{\star\star}$-I_ω-set if $K = U \cap V$, where U is an $\star$-open set and $V = cl(int_\omega(V))$.

Remark 5.6. *In an ideal topological space (X, τ, I),*

(1). Every pre*-I_ω-closed set is an I_ω-$\mathcal{C}^\star$-set.

(2). Every $\alpha^\star$-I_ω-closed set is a $\eta^\star$-I_ω-set.

(3). For a subset K of X if $K = cl(int_\omega(K))$, then K is an $\mathcal{A}^{\star\star}$-I_ω-set.

Example 5.7. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}$,*

(1). $K = \mathbb{Q}$ is pre*-I_ω-closed, for $cl^\star(int_\omega(K)) = cl^\star(\phi) = \phi \subset \mathbb{Q} = K$ and hence is an I_ω-$\mathcal{C}^\star$-set by (1) of Remark 5.6.

(2). $K = \mathbb{Q}^\star$ is not an I_ω-$\mathcal{C}^\star$-set. If $K = U \cap V$ where U is $\star$-open and V is pre*-I_ω-closed, then $K \subset U$. But $\mathbb{R}$ is the only open ($= \star$-open) set containing K. Hence $U = \mathbb{R}$ and $K = \mathbb{R} \cap V = V$. This is a contradiction since $K = \mathbb{Q}^\star$ is not pre*-I_ω-closed by (2) of Example 3.30. This proves that $K = \mathbb{Q}^\star$ is not an I_ω-$\mathcal{C}^\star$-set.

Example 5.8. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \mathcal{N}$,* *(1). Since $K=[0,1]$ is $\alpha^\star$-I_ω-closed by (1) of Example 4.11, $K=[0,1]$ is a $\eta^\star$-I_ω-set by (2) of Remark 5.6.*

(2). For $K=[0,1]$, $cl(int_\omega(K))=cl((0,1))=[0,1]=K$. Then $K=[0,1]$ is an $\mathcal{A}^{\star\star}$-I_ω-set by (3) of Remark 5.6.

Example 5.9. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I=\{\phi\}$,*

(1). $K=\mathbb{Q}$ is not a $\eta^\star$-I_ω-set. If $K=U\cap V$, where U is $\star$-open and V is $\alpha^\star$-I_ω-closed, then $K\subset U$. But $\mathbb{R}$ is the only open ($=\star$-open) set containing K. Hence $U=\mathbb{R}$ and $K=\mathbb{R}\cap V=V$. This is a contradiction since $K=\mathbb{Q}$ is not $\alpha^\star$-I_ω-closed for $cl^\star(int_\omega(cl^\star(K=\mathbb{Q})))=cl^\star(int_\omega(cl(\mathbb{Q})))=cl^\star(int_\omega(\mathbb{R}))=\mathbb{R}\nsubseteq\mathbb{Q}=K$. This proves that $K=\mathbb{Q}$ is not a $\eta^\star$-I_ω-set.

(2). $K=\mathbb{Q}$ is not an $\mathcal{A}^{\star\star}$-I_ω-set. If $K=U\cap V$, where U is $\star$-open and $V=cl(int_\omega(V))$, then $K\subset U$. But $\mathbb{R}$ is the only open ($=\star$-open) set containing K. Hence $U=\mathbb{R}$ and $K=\mathbb{R}\cap V=V$. This is a contradiction since $cl(int_\omega(K=\mathbb{Q}))=cl(\phi)=\phi\neq\mathbb{Q}=K$. This proves that $K=\mathbb{Q}$ is not an $\mathcal{A}^{\star\star}$-I_ω-set.

Remark 5.10. *By Proposition 3.28(2), Proposition 3.24 , Remark 3.31(2) and Definition 5.5, the following diagram holds for any subset of an ideal topological space (X,τ,I).*

$$\begin{array}{ccc} \mathcal{C}^{\star\star}\text{-}I_\omega\text{-set} & \longrightarrow & I_\omega\text{-}\mathcal{C}^{\star}\text{-set} \\ & & \uparrow \\ \mathcal{A}^{\star\star}\text{-}I_\omega\text{-set} & \longrightarrow & \eta^{\star}\text{-}I_\omega\text{-set} \end{array}$$

Remark 5.11. *The reverse implications in Remark 5.10 are not true as seen from the following Example.*

Example 5.12. *(1). In $\mathbb{R}$ with the topology $\tau=\{\phi,\mathbb{R},\mathbb{Q}^\star\}$ and the ideal $I=\{\phi\}, K=\mathbb{Q}$ is pre*-I_ω-closed by Proposition 3.3(2) for K is closed and hence is an I_ω-$\mathcal{C}^\star$-set by (1) of Remark 5.6. But $K=\mathbb{Q}$ is not a $\mathcal{C}^{\star\star}$-I_ω-set. If $K=U\cap V$ where U is $\star$-open and V is pre-I_ω-regular, then $K\subset U$. But $\mathbb{R}$ is the only open ($=\star$-open) set containing K. Hence $U=\mathbb{R}$ and $K=\mathbb{R}\cap V=V$. This is a contradiction since $K=\mathbb{Q}$ is not pre-I_ω-regular, being not pre-I_ω-open for $int_\omega(cl^\star(K=\mathbb{Q}))=int_\omega(cl(\mathbb{Q}))=int_\omega(\mathbb{Q})=\phi\nsupseteq\mathbb{Q}=K$. This proves that $K=\mathbb{Q}$ is not a $\mathcal{C}^{\star\star}$-I_ω-set.*

(2). In $\mathbb{R}$ with usual topology τ_u and the ideal $I=\{\phi\}, K=\mathbb{Q}$ is an I_ω-$\mathcal{C}^\star$-set by (1) of Example 5.7. But $K=\mathbb{Q}$ is not a $\eta^\star$-I_ω-set. If $K=U\cap V$ where U is $\star$-open and V is $\alpha^\star$-I_ω-closed, then $K\subset U$. But $\mathbb{R}$ is the only open ($=\star$-open) set containing K. Hence $U=\mathbb{R}$ and $K=\mathbb{R}\cap V=V$. This is a contradiction since $K=\mathbb{Q}$ is not $\alpha^\star$-I_ω-closed for $cl^\star(int_\omega(cl^\star(K)))=cl^\star(int_\omega(\mathbb{R}))=\mathbb{R}\nsubseteq\mathbb{Q}=K$. This proves that $K=\mathbb{Q}$ is not a $\eta^\star$-I_ω-set.

(3). In $\mathbb{R}$ with the topology $\tau=\{\phi,\mathbb{R},\mathbb{Q}^\star\}$ and the ideal $I=\{\phi\}, K=\mathbb{N}$ is $\alpha^\star$-I_ω-closed for $cl^\star(int_\omega(cl^\star(K)))=cl^\star(int_\omega(\mathbb{Q}))=cl^\star(\phi)=\phi\subset\mathbb{N}=K$ and hence $K=\mathbb{N}$ is a $\eta^\star$-I_ω-set by (2) of Remark 5.6. But $K=\mathbb{N}$ is not an $\mathcal{A}^{\star\star}$-I_ω-set. If $K=U\cap V$ where U is $\star$-open and $V=cl(int_\omega(V))$, then $K\subset U$. But $\mathbb{R}$ is the only open ($=\star$-open) set containing K. Hence $U=\mathbb{R}$ and $K=\mathbb{R}\cap V=V$. This is a contradiction since $cl(int_\omega(K))=cl(\phi)=\phi\neq\mathbb{N}=K$. This proves that $K=\mathbb{N}$ is not an $\mathcal{A}^{\star\star}$-I_ω-set.

Remark 5.13. *In an ideal topological space (X,τ,I), every $\star$-open set is an $\mathcal{A}^{\star\star}$-I_ω-set.*

Example 5.14. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I=\mathcal{N}$, $K=[0,1]$ is an $\mathcal{A}^{\star\star}$-I_ω-set by (2) of Example 5.8. But $K=[0,1]$ is not $\star$-open, since $int^\star([0,1])=\mathbb{R}\backslash cl^\star(\mathbb{R}\backslash[0,1])=\mathbb{R}\backslash((\mathbb{R}\backslash[0,1])\cup(cl(int(cl(\mathbb{R}\backslash[0,1])))))=\mathbb{R}\backslash((\mathbb{R}\backslash[0,1])\cup(cl(int(\mathbb{R}\backslash(0,1)))))=\mathbb{R}\backslash((\mathbb{R}\backslash[0,1])\cup(cl(\mathbb{R}\backslash[0,1])))=\mathbb{R}\backslash((\mathbb{R}\backslash[0,1])\cup(\mathbb{R}\backslash(0,1)))=\mathbb{R}\backslash\mathbb{R}\backslash(0,1)=(0,1)\neq[0,1]=K$.*

Proposition 5.15. *For a subset K of an ideal topological space (X,τ,I), the following are equivalent:*

(1). K is $\star$-open.

(2). K is $\alpha^\star$-I_ω-open and an $\mathcal{A}^{\star\star}$-I_ω-set.

(3). K is pre*-I_ω-open and an $\mathcal{A}^{\star\star}$-I_ω-set.

Proof. (1) $\Rightarrow$ (2): (2) follows by Proposition 3.22(1) and Remark 5.13.
(2) $\Rightarrow$ (3): (3) follows by Proposition 3.24.
(3) $\Rightarrow$ (1): Suppose K is pre*-I_ω-open and an $\mathcal{A}^{\star\star}$-I_ω-set. Since K is an $\mathcal{A}^{\star\star}$-I_ω-set, we have $K = L \cap M$, where L is an $\star$-open set and $M = cl(int_\omega(M))$. It follows that $int^\star(cl_\omega(M)) \subset cl_\omega(M) \subset cl(M) = cl(cl(int_\omega(M))) = cl(int_\omega(M)) = M$. This implies that M is semi-I_ω-closed. By Proposition 3.9, M is an I_ω-t-set in X. By Definition 3.10, K is a $\mathcal{B}$-I_ω-set. Since K is pre*-I_ω-open and a $\mathcal{B}$-I_ω-set, K is $\star$-open by Proposition 3.15. □

Remark 5.16. *The following Example shows that*

(1). The concepts of $\alpha^\star$-I_ω-openness and being an $\mathcal{A}^{\star\star}$-I_ω-set are independent.

(2). The concepts of pre*-I_ω-openness and being an $\mathcal{A}^{\star\star}$-I_ω-set are independent.

Example 5.17.

(1). In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \mathcal{N}$, $K = [0,1]$ is an $\mathcal{A}^{\star\star}$-I_ω-set by Example 5.14. But $K = [0,1]$ is not $\alpha^\star$-I_ω-open, since $int^\star(cl_\omega(int^\star([0,1]))) = int^\star(cl_\omega((0,1))) = int^\star([0,1]) = (0,1) \not\supseteq [0,1] = K$.

(2). In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$ and the ideal $I = \{\phi\}$, $K = \mathbb{R}\backslash\{1\}$ is $\alpha^\star$-I_ω-open by (1) of Example 3.23. But K is not an $\mathcal{A}^{\star\star}$-I_ω-set. If $K = U \cap V$ where U is $\star$-open and $V = cl(int_\omega(V))$, then $K \subset U$. But $\mathbb{R}$ is the only open (= $\star$-open) set containing K. Hence $U = \mathbb{R}$ and $K = \mathbb{R} \cap V = V$. This is a contradiction since $cl(int_\omega(K)) = cl(K) = \mathbb{R} \neq \mathbb{R}\backslash\{1\} = K$. This proves that $K = \mathbb{R}\backslash\{1\}$ is not an $\mathcal{A}^{\star\star}$-I_ω-set.

(3). In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}$, $K = \mathbb{Q}^\star$ is pre*-I_ω-open by Example 3.17. But K is not an $\mathcal{A}^{\star\star}$-I_ω-set. If $K = U \cap V$ where U is $\star$-open and $V = cl(int_\omega(V))$, then $K \subset U$. But $\mathbb{R}$ is the only open (= $\star$-open) set containing K. Hence $U = \mathbb{R}$ and $K = \mathbb{R} \cap V = V$. This is a contradiction since $cl(int_\omega(K)) = cl(K) = \mathbb{R} \neq \mathbb{Q}^\star = K$. This proves that $K = \mathbb{Q}^\star$ is not an $\mathcal{A}^{\star\star}$-I_ω-set.

(4). In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \mathcal{N}$, $K = [0,1]$ is an $\mathcal{A}^{\star\star}$-I_ω-set by Example 5.14. But K is not pre*-I_ω-open, since $int^\star(cl_\omega(K)) = int^\star([0,1]) = (0,1) \not\supseteq [0,1] = K$.

Theorem 5.18. *For a subset K of an ideal topological space (X, τ, I), the following are equivalent:*

(1). K is an I_ω-$\mathcal{C}^\star$-set and a semi-I_ω-open set in X.

(2). $K = L \cap cl^\star(int_\omega(K))$ for an $\star$-open set L.

Proof. (1) $\Rightarrow$ (2) : Since K is an I_ω-$\mathcal{C}^\star$-set, $K = L \cap M$, where L is an $\star$-open set and M is a pre*-I_ω-closed set in X. We have $K \subset M$ and $cl^\star(int_\omega(K)) \subset cl^\star(int_\omega(M)) \subset M$ since M is pre*-I_ω-closed in X. Since K is semi-I_ω-open in X, we have $K \subset cl^\star(int_\omega(K))$. It follows that $K = K \cap cl^\star(int_\omega(K)) = L \cap M \cap cl^\star(int_\omega(K)) = L \cap cl^\star(int_\omega(K))$.
(2) $\Rightarrow$ (1): Let $K = L \cap cl^\star(int_\omega(K))$ for an $\star$-open set L. Then $K \subset cl^\star(int_\omega(K))$ and thus K is semi-I_ω-open in X. Since $cl^\star(int_\omega(K))$ is a $\star$-closed set, by Proposition 3.3(1), it is a pre*-I_ω-closed set in X. Hence, K is an I_ω-$\mathcal{C}^\star$-set in X. □

6. $I_{\omega^{\star}}$-submaximal Spaces

Definition 6.1. *A subset K of an ideal topological space (X, τ, I) is called locally $I_{\omega^{\star}}$-closed if $K = U \cap V$, where U is $\star$-open and V is ω-closed.*

Remark 6.2. *In an ideal topological space (X, τ, I),*

(1). Every $\star$-open set is locally $I_{\omega^{\star}}$-closed.

(2). Every ω-closed set is locally $I_{\omega^{\star}}$-closed.

Example 6.3. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}$,*

(1). $K = \mathbb{Q}$ is ω-closed and hence is locally $I_{\omega^{\star}}$-closed by (2) of Remark 6.2.

(2). $K = \mathbb{Q}^{\star}$ is not locally $I_{\omega^{\star}}$-closed. If $K = U \cap V$, where U is $\star$-open and V is ω-closed, then $K \subset U$. But $\mathbb{R}$ is the only open ($= \star$-open) set containing K. Hence $U = \mathbb{R}$ and $K = \mathbb{R} \cap V = V$. This is a contradiction since K is not ω-closed ((i.e) $cl_{\omega}(K) = \mathbb{R} \neq \mathbb{Q}^{\star} = K$).

Remark 6.4. *The converses of (1) and (2) in Remark 6.2 are not true as seen from the following Example.*

Example 6.5. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}$,*

(1). $K = \mathbb{Q}$ is locally $I_{\omega^{\star}}$-closed by (1) of Example 6.3. But K is not $\star$-open, since $int^{\star}(K) = \mathbb{R} \backslash cl^{\star}(\mathbb{Q}^{\star}) = \mathbb{R} \backslash cl(\mathbb{Q}^{\star}) = \mathbb{R} \backslash \mathbb{R} = \phi \neq \mathbb{Q} = K$.

(2). $K = (0, 1)$ is locally $I_{\omega^{\star}}$-closed by (1) of Remark 6.2, since K is open and hence $\star$-open. But K is not ω-closed, since $cl_{\omega}(K) = [0, 1] \neq (0, 1) = K$.

Proposition 6.6. *For a subset K of an ideal topological space (X, τ, I), the following are equivalent:*

(1). K is $\star$-open.

(2). K is pre*-I_{ω}-open and locally $I_{\omega^{\star}}$-closed.

Proof. (1) $\Rightarrow$ (2): (2) follows by Proposition 3.3(1) and Remark 6.2(1).
(2) $\Rightarrow$ (1): Given K is locally $I_{\omega^{\star}}$-closed. So $K = U \cap V$ where U is $\star$-open and $V = cl_{\omega}(V)$. Then $K \subset U = int^{\star}(U)$. Also K is pre*-I_{ω}-open implies $K \subset int^{\star}(cl_{\omega}(K)) \subset int^{\star}(cl_{\omega}(V)) = int^{\star}(V)$ by assumption. Thus $K \subset int^{\star}(U) \cap int^{\star}(V) = int^{\star}(U \cap V) = int^{\star}(K)$ and hence K is $\star$-open. □

Remark 6.7. *The following Example shows that the concepts of pre*-I_{ω}-openness and locally $I_{\omega^{\star}}$-closedness are independent.*

Example 6.8. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}$,*

(1). $K = \mathbb{Q}^{\star}$ is pre*-I_{ω}-open by Example 3.17. But K is not locally $I_{\omega^{\star}}$-closed by (2) of Example 6.3.

(2). $K = \mathbb{Q}$ is locally $I_{\omega^{\star}}$-closed by (1) of Example 6.3. But K is not pre*-I_{ω}-open, since $int^{\star}(cl_{\omega}(K)) = int^{\star}(K) = \mathbb{R} \backslash cl^{\star}(\mathbb{Q}^{\star}) = \mathbb{R} \backslash cl(\mathbb{Q}^{\star}) = \mathbb{R} \backslash \mathbb{R} = \phi \not\supseteq \mathbb{Q} = K$.

Proposition 6.9. *Every locally closed set is locally $I_{\omega^{\star}}$-closed.*

Proof. It follows from the facts that every closed set is ω-closed and every open set is $\star$-open. □

The converse of Proposition 6.9 is not true as seen from the following Example.

Example 6.10. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \{\phi\}$, $K = \mathbb{Q}$ is locally $I_{\omega^\star}$-closed by (1) of Example 6.3. But $K = \mathbb{Q}$ is not locally closed. If $K = U \cap V$ where U is open and V is closed, then $K \subset U$. But $\mathbb{R}$ is the only open set containing K. Hence $U = \mathbb{R}$ and $K = \mathbb{R} \cap V = V$. This is a contradiction since K is not closed. ((i.e) $cl(K = \mathbb{Q}) = \mathbb{R} \neq \mathbb{Q} = K$).*

Lemma 6.11. *In an ideal topological space (X, τ, I), if K is pre*-I_ω-open, then $K = L \cap M$ for some $L \in \tau^\star$ and an ω-dense subset M.*

Proof. If K is pre*-I_ω-open, then $K \subset int^\star(cl_\omega(K)) = L \in \tau^\star$. Also, $L = int^\star(cl_\omega(K)) \subset cl_\omega(K)$ and $(X - cl_\omega(K)) \subset (X - L)$. Let $M = X - (L - K) = (X - L) \cup K$. Then M is ω-dense, since $X = cl_\omega(K) \cup (X - cl_\omega(K)) \subset cl_\omega(K) \cup (X - L) \subset cl_\omega(K) \cup cl_\omega(X - L) = cl_\omega((X - L) \cup K) = cl_\omega(M)$. Also, $L \cap M = L \cap ((X - L) \cup K) = (L \cap (X - L)) \cup (L \cap K) = \phi \cup (L \cap K) = L \cap K = K$. □

Definition 6.12. *An ideal topological space (X, τ, I) is called $I_{\omega^\star}$-submaximal if every ω-dense subset of X is $\star$-open.*

Proposition 6.13. *Every submaximal space is $I_{\omega^\star}$-submaximal.*

Proof. Let K be ω-dense in X. Then $X = cl_\omega(K) \subset cl(K)$ and $X = cl(K)$. Thus K is dense in X. Since X is submaximal, K is open and hence $\star$-open in X. Hence, X is $I_{\omega^\star}$-submaximal. □

Example 6.14. *Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X, \{c\}, \{b, c\}\}$. Set $K = \{a, c\}$. Then $cl(K) = X$ and $K \notin \tau$. Hence X is not submaximal, since the dense set K is not open. But it is $I_{\omega^\star}$-submaximal, since the only ω-dense set is X which is $\star$-open.*

Theorem 6.15. *For an ideal topological space (X, τ, I), the following are equivalent.*

(1). X is $I_{\omega^\star}$-submaximal,

(2). Every ω-codense subset K of X is $\star$-closed.

Proof. (1) ⇒ (2): Let K be a ω-codense subset of X. Since $X \backslash K$ is ω-dense, $X \backslash K$ is $\star$-open. Thus K is $\star$-closed.
(2) ⇒ (1): Let K be a ω-dense subset of X. Since $X \backslash K$ is ω-codense, $X \backslash K$ is $\star$-closed. Thus K is $\star$-open and hence X is $I_{\omega^\star}$-submaximal. □

Lemma 6.16. *In an $I_{\omega^\star}$-submaximal space (X, τ, I), $\tau^\star = P^\star_I\omega O(X)$ where $P^\star_I\omega O(X)$ is the family of pre*-I_ω-open sets of X.*

Proof. Clearly $\tau^\star \subset P^\star_I\omega O(X)$ by Proposition 3.3(1). Now if $K \in P^\star_I\omega O(X)$, then by Lemma 6.11, $K = L \cap M$ for some $L \in \tau^\star$ and M is ω-dense in X. Since (X, τ, I) is $I_{\omega^\star}$-submaximal, $M \in \tau^\star$ and hence $K \in \tau^\star$. Therefore $P^\star_I\omega O(X) \subset \tau^\star$ and $\tau^\star = P^\star_I\omega O(X)$. □

Proposition 6.17. *In an ideal topological space (X, τ, I), every $\star$-open set is semi*-I_ω-open.*

Proof. Let K be an $\star$-open set in X. Then $K = int^\star(K) \subset cl_\omega(int^\star(K))$. Thus K is a semi*-I_ω-open set in X. □

Example 6.18. *In $\mathbb{R}$ with usual topology τ_u and the ideal $I = \mathcal{N}$, $K = [0, 1]$ is semi*-I_ω-open by using Example 5.14 but not $\star$-open.*

Theorem 6.19. *In an ideal topological space (X, τ, I), the following are equivalent:*

(1). X is $I_{\omega^{\star}}$-submaximal.

(2). Every pre*-I_{ω}-open set is $\star$-open.

(3). Every pre*-I_{ω}-open set is semi*-I_{ω}-open and every $\alpha^{\star}$-I_{ω}-open set is $\star$-open.

Proof. (1) $\Rightarrow$ (2): (2) follows by Lemma 6.16.
(2) $\Rightarrow$ (3): Let K be any pre*-I_{ω}-open set. By assumption K is $\star$-open and hence is semi*-I_{ω}-open by Proposition 6.17. Let $K \subset X$ be an $\alpha^{\star}$-I_{ω}-open set. Then K is pre*-I_{ω}-open by Proposition 3.24 and hence is $\star$-open by assumption.
(3) $\Rightarrow$ (1): Let K be a ω-dense subset of X. Since $cl_{\omega}(K) = X$, $int^{\star}(cl_{\omega}(K)) = X \supset K$ and so K is pre*-I_{ω}-open. By (3), K is semi*-I_{ω}-open. Since K is both semi*-I_{ω}-open and pre*-I_{ω}-open, by Theorem 3.36, K is $\alpha^{\star}$-I_{ω}-open. By (3), K is $\star$-open in X. Hence X is $I_{\omega^{\star}}$-submaximal. □

7. Decompositions of $\star$-continuity and $\mathcal{A}^{\star}$-I_{ω}-continuity

Definition 7.1. *A function $f : X \to Y$ is called semi-I_{ω}-continuous [16] (resp. $\star$-continuous [3]) if $f^{-1}(V)$ is semi-I_{ω}-open (resp. $\star$-open) in X for each open set V in Y.*

Definition 7.2. *A function $f : X \to Y$ is called pre*-I_{ω}-continuous (resp. $\alpha^{\star}$-I_{ω}-continuous, $\mathcal{B}$-I_{ω}-continuous, $\mathcal{A}^{\star}$-I_{ω}-continuous, I_{ω}-$\mathcal{C}$-continuous, η-I_{ω}-continuous, contra locally $I_{\omega^{\star}}$-continuous, $\mathcal{A}^{\star\star}$-I_{ω}-continuous if $f^{-1}(V)$ is pre*-I_{ω}-open (resp. $\alpha^{\star}$-I_{ω}-open, a $\mathcal{B}$-I_{ω}-set, an $\mathcal{A}^{\star}$-I_{ω}-set, an I_{ω}-$\mathcal{C}$-set, a η-I_{ω}-set, locally $I_{\omega^{\star}}$-closed, an $\mathcal{A}^{\star\star}$-I_{ω}-set in X for each open set V in Y.*

Theorem 7.3. *For a function $f : X \to Y$, then the following are equivalent:*

(1). f is $\star$-continuous.

(2). f is pre*-I_{ω}-continuous and $\mathcal{B}$-I_{ω}-continuous.

(3). f is $\alpha^{\star}$-I_{ω}-continuous and $\mathcal{A}^{\star\star}$-I_{ω}-continuous.

(4). f is pre*-I_{ω}-continuous and $\mathcal{A}^{\star\star}$-I_{ω}-continuous.

(5). f is pre*-I_{ω}-continuous and contra locally $I_{\omega^{\star}}$-continuous.

Proof. (1) $\Leftrightarrow$ (2): This is obvious by Proposition 3.15. (1) $\Rightarrow$ (3); (3) $\Rightarrow$ (4); (4) $\Rightarrow$ (1): This is obvious by Proposition 5.15. (1) $\Leftrightarrow$ (5): This is obvious by Proposition 6.6. □

Theorem 7.4. *For a function $f : X \to Y$, then the following are equivalent:*

(1). f is $\mathcal{A}^{\star}$-I_{ω}-continuous.

(2). f is semi-I_{ω}-continuous and η-I_{ω}-continuous.

(3). f is semi-I_{ω}-continuous and I_{ω}-$\mathcal{C}$-continuous.

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