International Journal of Mathematics And its Applications
Volume 4, Issue 2–C (2016), 73–83.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

$g^{\#}$-closed Sets in Ideal Topological Spaces

Research Article

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Abstract: The notion of $g^{\#}$-closed sets is introduced in ideal topological spaces. Characterizations and properties of $\mathcal{I}_{g\#}$-closed sets and $\mathcal{I}_{g\#}$-open sets are given. A characterization of normal spaces is given in terms of $\mathcal{I}_{g\#}$-open sets. Also, it is established that an $\mathcal{I}_{g\#}$-closed subset of an $\mathcal{I}$-compact space is $\mathcal{I}$-compact.

MSC: 54A05, Secondary 54D15, 54D30.

Keywords: $g^{\#}$-closed set, $\mathcal{I}_{g\#}$-closed set, $\mathcal{I}$-compact space.

1. Introduction and Preliminaries

An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies

(1). $A \in \mathcal{I}$ and $B \subseteq A \Rightarrow B \in \mathcal{I}$ and

(2). $A \in \mathcal{I}$ and $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$ [15].

Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp$(X) is the set of all subsets of X, a set operator $(.)^\star$: $\wp(X)\rightarrow\wp(X)$, called a local function [15] of A with respect to τ and $\mathcal{I}$ is defined as follows: for A⊆X, $A^\star(\mathcal{I},\tau)=\{x\in X \mid U\cap A\notin\mathcal{I}$ for every $U\in\tau(x)\}$ where $\tau(x)=\{U\in\tau \mid x\in U\}$. We will make use of the basic facts about the local functions [[12], Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^\star(.)$ for a topology $\tau^\star(\mathcal{I},\tau)$, called the $\star$-topology, finer than τ is defined by $cl^\star(A)=A\cup A^\star(\mathcal{I},\tau)$ [29]. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathcal{I},\tau)$ and $\tau^\star$ for $\tau^\star(\mathcal{I},\tau)$.
If $\mathcal{I}$ is an ideal on X, then (X, τ, $\mathcal{I}$) is called an ideal topological space. $\mathcal{N}$ is the ideal of all nowhere dense subsets in (X, τ). A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is $\star$-closed [12] (resp. $\star$-dense in itself [10]) if $A^\star\subseteq A$ (resp. $A\subseteq A^\star$). A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is $\mathcal{I}_g$-closed [2] if $A^\star\subseteq U$ whenever A⊆U and U is open.
By a space, we always mean a topological space (X, τ) with no separation properties assumed. If A⊆X, cl(A) and int(A) will, respectively, denote the closure and interior of A in (X, τ) and $int^\star(A)$ will denote the interior of A in (X, $\tau^\star$).
A subset A of a space (X, τ) is an α-open [24] (resp. a semi-open [16], a preopen [19], a regular open [28]) set if A⊆int(cl(int(A))) (resp. A⊆cl(int(A)), A⊆int(cl(A)), A = int(cl(A))).

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The family of all α-open sets in (X, τ), denoted by τ^{α}, is a topology on X finer than τ. The closure of A in (X, τ^{α}) is denoted by cl_{α}(A).

Definition 1.1. *A subset A of a space (X, τ) is said to be*

(1). g-closed [17] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is open.

(2). g-open [17] if its complement is g-closed.

(3). αg-closed [18] if cl_{α}(A)$\subseteq$U whenever A$\subseteq$U and U is open.

(4). αg-open [18] if its complement is αg-closed.

(5). $g^{\#}$-closed [31] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is αg-open.

(6). $g^{\#}$-open [31] if its complement is $g^{\#}$-closed.

(7). $\hat{g}$-closed [30] or ω-closed [27] or $s^{\star}g$-closed [14, 20, 25] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is semi-open.

(8). $\hat{g}$-open [30] if its complement is $\hat{g}$-closed.

Definition 1.2. *An ideal $\mathcal{I}$ is said to be*

(1). codense [3] or τ-boundary [23] if $\tau \cap \mathcal{I}=\{\phi\}$,

(2). completely codense [3] if $PO(X) \cap \mathcal{I} = \{\phi\}$, where PO(X) is the family of all preopen sets in (X, τ).

Lemma 1.3. *Every completely codense ideal is codense but not conversely [3].*

The following Lemmas, Result, Definitions, Remarks and Theorem will be useful in the sequel.

Lemma 1.4 ([12])**.** *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A, B subsets of X. Then the following properties hold:*

(1). $A\subseteq B \Rightarrow A^{\star}\subseteq B^{\star}$,

(2). $A^{\star}=cl(A^{\star})\subseteq cl(A)$,

(3). $(A^{\star})^{\star}\subseteq A^{\star}$,

(4). $(A\cup B)^{\star}=A^{\star}\cup B^{\star}$,

(5). $(A\cap B)^{\star}\subseteq A^{\star}\cap B^{\star}$.

Lemma 1.5. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A$\subseteq$X. If $A\subseteq A^{\star}$, then $A^{\star}=cl(A^{\star})=cl(A)=cl^{\star}(A)$ [[26], Theorem 5].*

Lemma 1.6. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. Then $\mathcal{I}$ is codense if and only if $G\subseteq G^{\star}$ for every semi-open set G in X [[26], Theorem 3].*

Lemma 1.7. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. If $\mathcal{I}$ is completely codense, then $\tau^{\star}\subseteq\tau^{\alpha}$ [[26], Theorem 6].*

Result 1.8. *For a subset of a topological space, the following properties hold:*

(1). Every closed set is $g^{\#}$-closed but not conversely [31].

(2). Every $g^{\#}$-closed set is g-closed but not conversely [31].

(3). Every closed set is $\hat{g}$-closed but not conversely [30].

(4). Every $\hat{g}$-closed set is g-closed but not conversely [30].

Definition 1.9. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be a $T_{\mathcal{I}}$-space [2] if every $\mathcal{I}_g$-closed subset of X is a $\star$-closed set.*

Lemma 1.10. *If $(X, \tau, \mathcal{I})$ is a $T_{\mathcal{I}}$-space and A is an $\mathcal{I}_g$-closed set, then A is a $\star$-closed set [[21], Corollary 2.2].*

Lemma 1.11. *Every g-closed set is $\mathcal{I}_g$-closed but not conversely [[2], Theorem 2.1].*

Definition 1.12. *A subset G of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be*

(1). $\mathcal{I}_{rg}$-closed [22] if $G^\star \subseteq H$ whenever $G \subseteq H$ and H is regular open.

(2). pre$^\star_{\mathcal{I}}$-open [4] if $G \subseteq int^\star(cl(G))$.

(3). pre$^\star_{\mathcal{I}}$-closed [4] if $X \backslash G$ is pre$^\star_{\mathcal{I}}$-open.

(4). $\mathcal{I}$-R closed [1] if $G = cl^\star(int(G))$.

Remark 1.13 ([5]). *In any ideal topological space, every $\mathcal{I}$-R closed set is $\star$-closed but not conversely.*

Definition 1.14 ([5]). *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset G of X is said to be a weakly $\mathcal{I}_{rg}$-closed set if $(int(G))^\star \subseteq H$ whenever $G \subseteq H$ and H is a regular open set in X.*

Remark 1.15 ([5]). *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. The following diagram holds for a subset $G \subseteq X$:*

$$\begin{array}{ccccc} \mathcal{I}_g\text{-closed} & \longrightarrow & \mathcal{I}_{rg}\text{-closed} & \longrightarrow & \text{weakly } \mathcal{I}_{rg}\text{-closed} \\ \uparrow & & & & \uparrow \\ \star\text{-closed} & \longleftarrow & \mathcal{I}\text{-R-closed} & & \text{pre}^\star_{\mathcal{I}}\text{-closed} \end{array}$$

These implications are not reversible.

Definition 1.16 ([6, 7]). *A subset K of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be*

(1). semi*-$\mathcal{I}$-open if $K \subseteq cl(int^\star(K))$,

(2). semi*-$\mathcal{I}$-closed if its complement is semi*-$\mathcal{I}$-open.

Definition 1.17 ([6]). *The semi*-$\mathcal{I}$-closure of a subset K of an ideal topological space $(X, \tau, \mathcal{I})$, denoted by $s^\star_{\mathcal{I}}cl(K)$, is defined by the intersection of all semi*-$\mathcal{I}$-closed sets of X containing K.*

Theorem 1.18 ([6]). *For a subset K of an ideal topological space $(X, \tau, \mathcal{I})$, $s^\star_{\mathcal{I}}cl(K) = K \cup int(cl^\star(K))$.*

Definition 1.19 ([8]). *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $K \subseteq X$. K is called*

(1). generalized semi*-$\mathcal{I}$-closed ($gs^\star_{\mathcal{I}}$-closed) in $(X, \tau, \mathcal{I})$ if $s^\star_{\mathcal{I}}cl(K) \subseteq O$ whenever $K \subseteq O$ and O is an open set in $(X, \tau, \mathcal{I})$.

(2). generalized semi*-$\mathcal{I}$-open ($gs^\star_{\mathcal{I}}$-open) in $(X, \tau, \mathcal{I})$ if $X \backslash K$ is a $gs^\star_{\mathcal{I}}$-closed set in $(X, \tau, \mathcal{I})$.

2. $\mathcal{I}_{g^{\#}}$-closed Sets

Definition 2.1. *A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is said to be*

(1). $\mathcal{I}_{g^{\#}}$-closed if $A^{\star}\subseteq U$ whenever $A\subseteq U$ and U is αg-open.

(2). $\mathcal{I}_{g^{\#}}$-open if its complement is $\mathcal{I}_{g^{\#}}$-closed.

Theorem 2.2. *If (X, τ, $\mathcal{I}$) is any ideal topological space, then every $\mathcal{I}_{g^{\#}}$-closed set is $\mathcal{I}_g$-closed but not conversely.*

Proof. It follows from the fact that every open set is αg-open. □

Example 2.3. *Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{c\}, \{a, b\}\}$ and $\mathcal{I} = \{\phi, \{a, b\}\}$. It is clear that $\{b\}$ is $\mathcal{I}_g$-closed but not $\mathcal{I}_{g^{\#}}$-closed.*

Theorem 2.4. *If (X, τ, $\mathcal{I}$) is any ideal topological space and $A\subseteq X$, then the following are equivalent.*

(1). A is $\mathcal{I}_{g^{\#}}$-closed.

(2). $cl^{\star}(A)\subseteq U$ whenever $A\subseteq U$ and U is αg-open in X.

Proof. (1) ⇒ (2) Let A ⊆ U where U is αg-open in X. Since A is $\mathcal{I}_{g^{\#}}$-closed, $A^{\star} \subseteq U$ and so $cl^{\star}(A) = A \cup A^{\star} \subseteq U$.
(2) ⇒ (1) It follows from the fact that $A^{\star} \subseteq cl^{\star}(A) \subseteq U$. □

Theorem 2.5. *Every $\star$-closed set is $\mathcal{I}_{g^{\#}}$-closed but not conversely.*

Proof. Let A be a $\star$-closed set. To prove A is $\mathcal{I}_{g^{\#}}$-closed, let U be any αg-open set such that A ⊆ U. Since A is $\star$-closed, $A^{\star} \subseteq A \subseteq U$. Thus A is $\mathcal{I}_{g^{\#}}$-closed. □

Example 2.6. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ and $\mathcal{I} = \{\phi\}$. It is clear that $\{a, c, d\}$ is $\mathcal{I}_{g^{\#}}$-closed but not $\star$-closed.*

Theorem 2.7. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. For every $A\in\mathcal{I}$, A is $\mathcal{I}_{g^{\#}}$-closed.*

Proof. Let $A \in \mathcal{I}$ and let A ⊆ U where U is αg-open. Since $A \in \mathcal{I}$, $A^{\star} = \phi \subseteq U$. Thus A is $\mathcal{I}_{g^{\#}}$-closed. □

Theorem 2.8. *If (X, τ, $\mathcal{I}$) is an ideal topological space, then $A^{\star}$ is always $\mathcal{I}_{g^{\#}}$-closed for every subset A of X.*

Proof. Let $A^{\star}\subseteq U$ where U is αg-open. Since $(A^{\star})^{\star}\subseteq A^{\star}$ [12], we have $(A^{\star})^{\star} \subseteq U$. Hence $A^{\star}$ is $\mathcal{I}_{g^{\#}}$-closed. □

Theorem 2.9. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. Then every $\mathcal{I}_{g^{\#}}$-closed, αg-open set is $\star$-closed.*

Proof. Let A be $\mathcal{I}_{g^{\#}}$-closed and αg-open. We have A ⊆ A where A is αg-open. Since A is $\mathcal{I}_{g^{\#}}$-closed, $A^{\star} \subseteq A$. Thus A is $\star$-closed. □

Corollary 2.10. *If (X, τ, $\mathcal{I}$) is a $T_{\mathcal{I}}$-space and A is an $\mathcal{I}_{g^{\#}}$-closed set, then A is a $\star$-closed set.*

Proof. By assumption A is $\mathcal{I}_{g^{\#}}$-closed in (X, τ, $\mathcal{I}$) and so by Theorem 2.2, A is $\mathcal{I}_g$-closed. Since (X, τ, $\mathcal{I}$) is a $T_{\mathcal{I}}$-space, by Definition 1.9, A is $\star$-closed. □

Corollary 2.11. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A be an $\mathcal{I}_{g^{\#}}$-closed set. Consider the following statements.*

(1). A is a $\star$-closed set,

(2). $cl^\star(A) - A$ is an αg-closed set,

(3). $A^\star - A$ is an αg-closed set.

Then (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) hold.

Proof. (1) $\Rightarrow$ (2) By (1) A is $\star$-closed. Hence $A^\star \subseteq A$ and $cl^\star(A) - A = (A \cup A^\star) - A = \phi$ which is an αg-closed set. (2) $\Rightarrow$ (3) $cl^\star(A) - A = A^\star \cup A - A = (A^\star \cup A) \cap A^c = (A^\star \cap A^c) \cup (A \cap A^c) = (A^\star \cap A^c) \cup \phi = A^\star - A$ which is an αg-closed set by (2). □

Theorem 2.12. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then every $g^\#$-closed set is an $\mathcal{I}_{g\#}$-closed set but not conversely.*

Proof. Let A be a $g^\#$-closed set. Let U be any αg-open set such that $A \subseteq U$. Since A is $g^\#$-closed, $cl(A) \subseteq U$. So, by Lemma 1.4, $A^\star \subseteq cl(A) \subseteq U$ and thus A is $\mathcal{I}_{g\#}$-closed. □

Example 2.13. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{d\}, \{a, d\}\}$ and $\mathcal{I} = \{\phi, \{d\}\}$. It is clear that $\{d\}$ is $\mathcal{I}_{g\#}$-closed but not $g^\#$-closed.*

Theorem 2.14. *If $(X, \tau, \mathcal{I})$ is an ideal topological space and A is a $\star$-dense in itself, $\mathcal{I}_{g\#}$-closed subset of X, then A is $g^\#$-closed.*

Proof. Let $A \subseteq U$ where U is αg-open. Since A is $\mathcal{I}_{g\#}$-closed, $A^\star \subseteq U$. As A is $\star$-dense in itself, by Lemma 1.5, $cl(A) = A^\star$. Thus $cl(A) \subseteq U$ and hence A is $g^\#$-closed. □

Corollary 2.15. *If $(X, \tau, \mathcal{I})$ is any ideal topological space where $\mathcal{I}=\{\phi\}$, then A is $\mathcal{I}_{g\#}$-closed if and only if A is $g^\#$-closed.*

Proof. In $(X, \tau, \mathcal{I})$, if $\mathcal{I} = \{\phi\}$ then $A^\star = cl(A)$ for the subset A. A is $\mathcal{I}_{g\#}$-closed $\Leftrightarrow$ $A^\star \subseteq U$ whenever $A \subseteq U$ and U is αg-open $\Leftrightarrow$ $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is αg-open $\Leftrightarrow$ A is $g^\#$-closed. □

Corollary 2.16. *In an ideal topological space $(X, \tau, \mathcal{I})$ where $\mathcal{I}$ is codense, if A is a semi-open and $\mathcal{I}_{g\#}$-closed subset of X, then A is $g^\#$-closed.*

Proof. By Lemma 1.6, A is $\star$-dense in itself. By Theorem 2.14, A is $g^\#$-closed. □

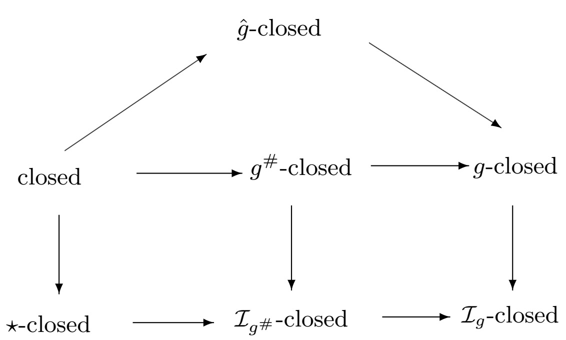
Example 2.17. *In Example 2.3, $\{a\}$ is g-closed but not $\mathcal{I}_{g\#}$-closed.*

Example 2.18. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{c\}, \{b, c\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $\mathcal{I} = \{\phi, \{c\}\}$. It is clear that $\{c\}$ is $\mathcal{I}_{g\#}$-closed but not g-closed.*

Remark 2.19. *We see that from Examples 2.17 and 2.18, g-closedness and $\mathcal{I}_{g\#}$-closedness are independent.*

Example 2.20. *In Example 2.3, $\{a\}$ is $\hat{g}$-closed but not $g^\#$-closed.*

Remark 2.21. *We have the following implications for the subsets stated above.*



Theorem 2.22. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. If $A \subseteq B \subseteq A^\star$, then $A^\star = B^\star$ and B is $\star$-dense in itself.*

Proof. Since A⊆B, then $A^\star \subseteq B^\star$ and since $B \subseteq A^\star$, then $B^\star \subseteq (A^\star)^\star \subseteq A^\star$. Therefore $A^\star = B^\star$ and $B \subseteq A^\star \subseteq B^\star$. Hence proved. □

Theorem 2.23. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then every subset of X is $\mathcal{I}_{g\#}$-closed if and only if every αg-open set is $\star$-closed.*

Proof. Suppose every subset of X is $\mathcal{I}_{g\#}$-closed. Let U be αg-open in X. Then U ⊆ U ⊆ X and U is $\mathcal{I}_{g\#}$-closed by assumption. It implies $U^\star \subseteq U$. Hence U is $\star$-closed.

Conversely, let A ⊆ X and U be αg-open such that A ⊆ U. Since U is $\star$-closed by assumption, we have $A^\star \subseteq U^\star \subseteq U$. Thus A is $\mathcal{I}_{g\#}$-closed. □

Theorem 2.24. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then A is $\mathcal{I}_{g\#}$-open if and only if $F \subseteq int^\star(A)$ whenever F is αg-closed and $F \subseteq A$.*

Proof. Suppose A is $\mathcal{I}_{g\#}$-open. If F is αg-closed and F⊆A, then X−A⊆X−F and so $cl^\star$(X−A)⊆X−F by Theorem 2.4(2). Therefore F⊆X−$cl^\star$(X−A)=$int^\star$(A). Hence F⊆$int^\star$(A).

Conversely, suppose the condition holds. Let U be an αg-open set such that X−A⊆U. Then X−U⊆A and so X−U⊆$int^\star$(A). Therefore $cl^\star$(X−A)⊆U. By Theorem 2.4(2), X−A is $\mathcal{I}_{g\#}$-closed. Hence A is $\mathcal{I}_{g\#}$-open. □

The following Theorem gives a characterization of normal spaces in terms of $\mathcal{I}_{g\#}$-open sets.

Theorem 2.25. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space where $\mathcal{I}$ is completely codense. Then the following are equivalent.*

(1). X is normal,

(2). For any disjoint closed sets A and B, there exist disjoint $\mathcal{I}_{g\#}$-open sets U and V such that A⊆U and B⊆V,

(3). For any closed set A and open set V containing A, there exists an $\mathcal{I}_{g\#}$-open set U such that A⊆U⊆$cl^\star$(U)⊆V.

Proof. (1)⇒(2) The proof follows from the fact that every open set is $\mathcal{I}_{g\#}$-open.

(2)⇒(3) Suppose A is closed and V is an open set containing A. Since A and X−V are disjoint closed sets, there exist disjoint $\mathcal{I}_{g\#}$-open sets U and W such that A⊆U and X−V⊆W. Since X−V is αg-closed and W is $\mathcal{I}_{g\#}$-open, X−V⊆$int^\star$(W). Then X−$int^\star$(W)⊆V. Again U∩W=ϕ which implies that U∩$int^\star$(W)=ϕ and so U⊆X−$int^\star$(W). Then $cl^\star$(U)⊆X−$int^\star$(W)⊆V and thus U is the required $\mathcal{I}_{g\#}$-open sets with A⊆U⊆$cl^\star$(U)⊆V.

(3)⇒(1) Let A and B be two disjoint closed subsets of X. Then A is a closed set and X − B an open set containing A. By hypothesis, there exists an $\mathcal{I}_{g\#}$-open set U such that A⊆U⊆$cl^\star$(U)⊆X−B. Since U is $\mathcal{I}_{g\#}$-open and A is αg-closed we have, by Theorem 2.24 , A⊆$int^\star$(U). Since $\mathcal{I}$ is completely codense, by Lemma 1.7, $\tau^\star \subseteq \tau^\alpha$ and so $int^\star$(U) and X−$cl^\star$(U)$\in \tau^\alpha$. Hence A⊆$int^\star$(U)⊆ int(cl(int($int^\star$(U))))=G and B⊆X−$cl^\star$(U)⊆ int(cl(int(X−$cl^\star$(U)))) = H. G and H are the required disjoint open sets containing A and B respectively, which proves (1). □

Definition 2.26. *A subset A of a topological space (X, τ) is said to be an $\alpha g^{\#}$-closed set if $cl_\alpha(A) \subseteq U$ whenever $A \subseteq U$ and U is αg-open. The complement of an $\alpha g^{\#}$-closed set is said to be an $\alpha g^{\#}$-open set.*

If $\mathcal{I}=\mathcal{N}$, it is not difficult to see that $\mathcal{I}_{g\#}$-closed sets coincide with $\alpha g^{\#}$-closed sets and so we have the following Corollary.

Corollary 2.27. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space where $\mathcal{I}=\mathcal{N}$. Then the following are equivalent.*

(1). X is normal,

(2). For any disjoint closed sets A and B, there exist disjoint $\alpha g^{\#}$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$,

(3). For any closed set A and open set V containing A, there exists an $\alpha g^{\#}$-open set U such that $A \subseteq U \subseteq cl_{\alpha}(U) \subseteq V$.

Definition 2.28. *A subset A of an ideal topological space is said to be $\mathcal{I}$-compact [9] or compact modulo $\mathcal{I}$ [23] if for every open cover $\{U_{\alpha} \mid \alpha \in \Delta\}$ of A, there exists a finite subset Δ_0 of Δ such that $A - \cup\{U_{\alpha} \mid \alpha \in \Delta_0\} \in \mathcal{I}$. The space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-compact if X is $\mathcal{I}$-compact as a subset.*

Theorem 2.29. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If A is an $\mathcal{I}_g$-closed subset of X, then A is $\mathcal{I}$-compact [[21], Theorem 2.17].*

Corollary 2.30. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If A is an $\mathcal{I}_{g^{\#}}$-closed subset of X, then A is $\mathcal{I}$-compact.*

Proof. The proof follows from the fact that every $\mathcal{I}_{g^{\#}}$-closed is $\mathcal{I}_g$-closed. □

Remark 2.31. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. By Remark 1.15, Definition 1.19, Definition 2.1 and Theorem 2.2, the following diagram holds for a subset $G \subseteq X$:*

$$\begin{array}{ccccc} gs^{\star}_{\mathcal{I}}\text{-closed} & & & & \text{weakly } \mathcal{I}_{rg}\text{-closed} \\ \uparrow & & & & \uparrow \\ \mathcal{I}_{g^{\#}}\text{-closed} & \longrightarrow & \mathcal{I}_g\text{-closed} & \longrightarrow & \mathcal{I}_{rg}\text{-closed} \end{array}$$

These implications are not reversible.

Example 2.32. *In Example 2.3, $\{b\}$ is $gs^{\star}_{\mathcal{I}}$-closed set but not $\mathcal{I}_{g^{\#}}$-closed.*

Definition 2.33. *A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be a $s^{\star}C_{\mathcal{I}}$-set if $A = L \cap M$, where $L \in \tau$ and M is a semi*-$\mathcal{I}$-closed set in X.*

Theorem 2.34. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $V \subseteq X$. Then V is a $s^{\star}C_{\mathcal{I}}$-set in X if and only if $V = G \cap s^{\star}_{\mathcal{I}}cl(V)$ for an open set G in X.*

Proof. If V is a $s^{\star}C_{\mathcal{I}}$-set, then $V = G \cap M$ for an open set G and a semi*-$\mathcal{I}$-closed set M. Then $V \subseteq M$ and so $V \subseteq s^{\star}_{\mathcal{I}}cl(V) \subseteq M$. It follows that $V = V \cap s^{\star}_{\mathcal{I}}cl(V) = G \cap M \cap s^{\star}_{\mathcal{I}}cl(V) = G \cap s^{\star}_{\mathcal{I}}cl(V)$. Conversely, it is enough to prove that $s^{\star}_{\mathcal{I}}cl(V)$ is a semi*-$\mathcal{I}$-closed set. Any semi*-$\mathcal{I}$-closed set containing V contains $s^{\star}_{\mathcal{I}}cl(V)$ also and any semi*-$\mathcal{I}$-closed set containing $s^{\star}_{\mathcal{I}}cl(V)$ contains V. Hence $s^{\star}_{\mathcal{I}}cl(V) = s^{\star}_{\mathcal{I}}cl(s^{\star}_{\mathcal{I}}cl(V)) = s^{\star}_{\mathcal{I}}cl(V) \cup int(cl^{\star}(s^{\star}_{\mathcal{I}}cl(V)))$ and thus $int(cl^{\star}(s^{\star}_{\mathcal{I}}cl(V))) \subseteq s^{\star}_{\mathcal{I}}cl(V)$. Thus $s^{\star}_{\mathcal{I}}cl(V)$ is semi*-$\mathcal{I}$-closed. □

Theorem 2.35. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. The following properties are equivalent.*

(1). A is a semi*-$\mathcal{I}$-closed set in X.

(2). A is a $s^{\star}C_{\mathcal{I}}$-set and a $gs^{\star}_{\mathcal{I}}$-closed set in X.

Proof. (1) $\Rightarrow$ (2): It follows from the fact that any semi*-$\mathcal{I}$-closed set in X is a $s^{\star}C_{\mathcal{I}}$-set and a $gs^{\star}_{\mathcal{I}}$-closed set in X.
(2) $\Rightarrow$ (1): Suppose that A is a $s^{\star}C_{\mathcal{I}}$-set and a $gs^{\star}_{\mathcal{I}}$-closed set in X. Since A is a $s^{\star}C_{\mathcal{I}}$-set, then by Theorem 2.34, $A = G \cap s^{\star}_{\mathcal{I}}cl(A)$ for an open set G in $(X, \tau, \mathcal{I})$. Since $A \subseteq G$ and A is $gs^{\star}_{\mathcal{I}}$-closed in X, we have $s^{\star}_{\mathcal{I}}cl(A) \subseteq G$. It follows that $s^{\star}_{\mathcal{I}}cl(A) = A$ and hence A is semi*-$\mathcal{I}$-closed. □

Proposition 2.36. *Every $g^{\#}$-closed set is $\alpha g^{\#}$-closed but not conversely.*

Proof. It follows from the fact that $cl_{\alpha}(A) \subseteq cl(A)$ for any subset A of X. □

Example 2.37. *In Example 2.6, it is clear that $\{b\}$ is $\alpha g^{\#}$-closed but not $g^{\#}$-closed.*

3. αg-$\mathcal{I}$-locally Closed Sets

Definition 3.1. *A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is called an αg-$\mathcal{I}$-locally closed set (briefly, αg-$\mathcal{I}$-LC) if A=U$\cap$V where U is αg-open and V is $\star$-closed.*

Definition 3.2. *[13] A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is called a weakly $\mathcal{I}$-locally closed set (briefly, weakly $\mathcal{I}$-LC) if A=U$\cap$V where U is open and V is $\star$-closed.*

Proposition 3.3. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A a subset of X. Then the following hold.*

(1). If A is αg-open, then A is αg-$\mathcal{I}$-LC-set.

(2). If A is $\star$-closed, then A is αg-$\mathcal{I}$-LC-set.

(3). If A is a weakly $\mathcal{I}$-LC-set, then A is an αg-$\mathcal{I}$-LC-set.

The converses of the above Proposition 3.3 need not be true as shown in the following Examples.

Example 3.4.

(1). In Example 2.3, $\{b\}$ is an αg-$\mathcal{I}$-LC-set but not $\star$-closed.

(2). Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ and $\mathcal{I} = \{\phi, \{c\}\}$. It is clear that $\{c\}$ is an αg-$\mathcal{I}$-LC-set but not αg-open.

Example 3.5. *In Example 2.3, $\{b\}$ is an αg-$\mathcal{I}$-LC-set but not a weakly $\mathcal{I}$-LC-set.*

Theorem 3.6. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A is an αg-$\mathcal{I}$-LC-set and B is a $\star$-closed set, then A$\cap$B is an αg-$\mathcal{I}$-LC-set.*

Proof. Let B be $\star$-closed, then A$\cap$B=(U$\cap$V)$\cap$B=U$\cap$(V$\cap$B), where V$\cap$B is $\star$-closed. Hence A$\cap$B is an αg-$\mathcal{I}$-LC-set. □

Theorem 3.7. *A subset of an ideal topological space (X, τ, $\mathcal{I}$) is $\star$-closed if and only if it is*

(1). weakly $\mathcal{I}$-LC and $\mathcal{I}_g$-closed. [11]

(2). αg-$\mathcal{I}$-LC and $\mathcal{I}_{g\#}$-closed.

Proof. (2) Necessity is trivial. We prove only sufficiency. Let A be αg-$\mathcal{I}$-LC-set and $\mathcal{I}_{g\#}$-closed set. Since A is αg-$\mathcal{I}$-LC, A=U$\cap$V, where U is αg-open and V is $\star$-closed. So, we have A=U$\cap$V$\subseteq$U. Since A is $\mathcal{I}_{g\#}$-closed, A$^\star\subseteq$U. Also since A = U$\cap$V$\subseteq$V and V is $\star$-closed, we have A$^\star\subseteq$V. Consequently, $A^\star \subseteq U \cap V = A$ and hence A is $\star$-closed. □

Remark 3.8.

(1). The notions of weakly $\mathcal{I}$-LC-set and $\mathcal{I}_g$-closed set are independent [11].

(2). The notions of αg-$\mathcal{I}$-LC-set and $\mathcal{I}_{g\#}$-closed set are independent.

Example 3.9. *In Example 2.6, $\{a\}$ is αg-$\mathcal{I}$-LC-set but not $\mathcal{I}_{g\#}$-closed.*

Example 3.10. *In Example 2.6, it is clear that $\{a, c, d\}$ is $\mathcal{I}_{g\#}$-closed set but not αg-$\mathcal{I}$-LC-set.*

Definition 3.11. *Let A be a subset of a topological space (X, τ). Then the αg-kernel of the set A, denoted by αg-ker(A), is the intersection of all αg-open supersets of A.*

Definition 3.12. *A subset A of a topological space (X, τ) is called $\Lambda_{\alpha g}$-set if A=αg-ker(A).*

Definition 3.13. *A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is called $\lambda_{\alpha g}$-$\mathcal{I}$-closed if A=L∩F where L is a $\Lambda_{\alpha g}$-set and F is $\star$-closed.*

Lemma 3.14.

(1). Every $\star$-closed set is $\lambda_{\alpha g}$-$\mathcal{I}$-closed but not conversely.

(2). Every $\Lambda_{\alpha g}$-set is $\lambda_{\alpha g}$-$\mathcal{I}$-closed but not conversely.

(3). Every αg-$\mathcal{I}$-LC-set is $\lambda_{\alpha g}$-$\mathcal{I}$-closed.

Example 3.15. *In Example 2.3, $\{b\}$ is $\lambda_{\alpha g}$-$\mathcal{I}$-closed but not $\star$-closed.*

Example 3.16. *Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ and $\mathcal{I} = \{\phi, \{c\}\}$. It is clear that $\{c\}$ is $\lambda_{\alpha g}$-$\mathcal{I}$-closed but not a $\Lambda_{\alpha g}$-set.*

Remark 3.17. *It is easily observed from Examples 3.15 and 3.16, that the concepts of $\Lambda_{\alpha g}$-set and $\star$-closed set are independent for $\{b\}$ is a $\Lambda_{\alpha g}$-set but not a $\star$-closed set whereas $\{c\}$ is $\star$-closed but not a $\Lambda_{\alpha g}$-set.*

Lemma 3.18. *For a subset A of an ideal topological space (X, τ, $\mathcal{I}$), the following are equivalent.*

(1). A is $\lambda_{\alpha g}$-$\mathcal{I}$-closed.

(2). A=L∩cl*(A) where L is a $\Lambda_{\alpha g}$-set.

(3). A=αg-ker(A)∩cl*(A).

Lemma 3.19. *A subset A⊆(X, τ, $\mathcal{I}$) is $\mathcal{I}_{g\#}$-closed if and only if cl*(A)⊆ αg-ker(A).*

Proof. Suppose that A ⊆ X is an $\mathcal{I}_{g\#}$-closed set. Suppose x ∉ αg-ker(A). Then there exists an αg-open set U containing A such that x ∉ U. Since A is an $\mathcal{I}_{g\#}$-closed set, A ⊆ U and U is αg-open implies that cl*(A) ⊆ U and so x ∉ cl*(A). Therefore cl*(A) ⊆ αg-ker(A).
Conversely, suppose cl*(A) ⊆ αg-ker(A). If A ⊆ U and U is αg-open, then cl*(A) ⊆ αg-ker(A) ⊆ U. Therefore, A is $\mathcal{I}_{g\#}$-closed. □

Theorem 3.20. *For a subset A of an ideal topological space (X, τ, $\mathcal{I}$), the following are equivalent.*

(1). A is $\star$-closed.

(2). A is $\mathcal{I}_{g\#}$-closed and αg-$\mathcal{I}$-LC.

(3). A is $\mathcal{I}_{g\#}$-closed and $\lambda_{\alpha g}$-$\mathcal{I}$-closed.

Proof. (1)⇒(2)⇒(3) Obvious.
(3)⇒(1) Since A is $\mathcal{I}_{g\#}$-closed, by Lemma 3.19, cl*(A)⊆αg-ker(A). Since A is $\lambda_{\alpha g}$-$\mathcal{I}$-closed, by Lemma 3.18, A=αg-ker(A)∩cl*(A)=cl*(A). Hence A is $\star$-closed. □

The following two Examples show that the concepts of $\mathcal{I}_{g\#}$-closedness and $\lambda_{\alpha g}$-$\mathcal{I}$-closedness are independent.

Example 3.21. *In Example 3.10, $\{a, c, d\}$ is $\mathcal{I}_{g\#}$-closed but not $\lambda_{\alpha g}$-$\mathcal{I}$-closed.*

Example 3.22. *In Example 2.3, $\{b\}$ is $\lambda_{\alpha g}$-$\mathcal{I}$-closed but not $\mathcal{I}_{g\#}$-closed.*

4. Decompositions of $\star$-continuity

Definition 4.1. *A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is said to be $\star$-continuous [11] (resp. $\mathcal{I}_g$-continuous [11], αg-$\mathcal{I}$-LC-continuous, $\lambda_{\alpha g}$-$\mathcal{I}$-continuous, $\mathcal{I}_{g\#}$-continuous, weakly $\mathcal{I}$-LC-continuous [13]) if $f^{-1}(A)$ is $\star$-closed (resp. $\mathcal{I}_g$-closed, αg-$\mathcal{I}$-LC-set, $\lambda_{\alpha g}$-$\mathcal{I}$-closed, $\mathcal{I}_{g\#}$-closed, weakly $\mathcal{I}$-LC-set) in $(X, \tau, \mathcal{I})$ for every closed set A of (Y, σ).*

Theorem 4.2. *A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is $\star$-continuous if and only if it is*

(1). weakly $\mathcal{I}$-LC-continuous and $\mathcal{I}_g$-continuous [11].

(2). αg-$\mathcal{I}$-LC-continuous and $\mathcal{I}_{g\#}$-continuous.

Proof. It is an immediate consequence of Theorem 3.7. □

Theorem 4.3. *For a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$, the following are equivalent.*

(1). f is $\star$-continuous.

(2). f is $\mathcal{I}_{g\#}$-continuous and αg-$\mathcal{I}$-LC-continuous.

(3). f is $\mathcal{I}_{g\#}$-continuous and $\lambda_{\alpha g}$-$\mathcal{I}$-continuous.

Proof. It is an immediate consequence of Theorem 3.20. □

References

[1] A.Acikgoz and S.Yuksel, *Some new sets and decompositions of $A_{\mathcal{I}-R}$-continuity, α-$\mathcal{I}$-continuity, continuity via idealization*, Acta Math. Hungar., 114(1-2)(2007), 79-89.

[2] J.Dontchev, M.Ganster and T.Noiri, *Unified operation approach of generalized closed sets via topological ideals*, Math. Japonica, 49(1999), 395-401.

[3] J.Dontchev, M.Ganster and D.Rose, *Ideal resolvability*, Topology and its Applications, 93(1999), 1-16.

[4] E.Ekici, *On $\mathcal{AC}_{\mathcal{I}}$-sets, $\mathcal{BC}_{\mathcal{I}}$-sets, $\beta^{\star}_{\mathcal{I}}$-open sets and decompositions of continuity in ideal topological spaces*, Creat. Math. Inform, 20(1)(2011), 47-54.

[5] E.Ekici and S.Ozen, *A generalized class of $\tau^{\star}$ in ideal spaces*, Filomat, 27(4)(2013), 529-535.

[6] E.Ekici and T.Noiri, *$\star$-hyperconnected ideal topological spaces*, Analele Stiintifice Ale Universitatii Al I. Cuza Din Iasi - Serie Noua-Matematica, Tomul LVIII, 1(2012), 121-129.

[7] E.Ekici and T.Noiri, *$\star$-extremally disconnected ideal topological spaces*, Acta Math. Hungar., 122(1-2)(2009), 81-90.

[8] E.Ekici, *On R-$\mathcal{I}$-open sets and $\mathcal{A}^{\star}_{\mathcal{I}}$-sets in ideal topological spaces*, Annals of the University of Craiova, Mathematics and Computer Science Series, 38(2)(2011), 26-31.

[9] T.R.Hamlett and D.Jankovic, *Compactness with respect to an ideal*, Boll. U. M. I., 7(4-B)(1990), 849-861.

[10] E.Hayashi, *Topologies defined by local properties*, Math. Ann., 156(1964), 205-215.

[11] V.Inthumathi, S.Krishnaprakash and M.Rajamani, *Strongly-$\mathcal{I}$-Locally closed sets and decompositions of $\star$-continuity*, Acta Math. Hungar., 130(4)(2011), 358-362.

[12] D.Jankovic and T.R.Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly, 97(4)(1990), 295-310.

[13] A.Keskin, S.Yuksel and T.Noiri, *Decompositions of $\mathcal{I}$-continuity and continuity*, Commun. Fac. Sci. Univ. Ank. Series A, 53(2004), 67-75.

[14] M.Khan, T.Noiri and M.Hussain, *On $s^{\star}g$-closed sets and $s^{\star}$-normal spaces*, J. Natur. Sci. Math., 48(1-2)(2008), 31-41.

[15] K.Kuratowski, *Topology*, Vol. I, Academic Press, New York, (1966).

[16] N.Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, 70(1963), 36-41.

[17] N.Levine, *Generalized closed sets in topology*, Rend. Circ. Mat. Palermo, 19(2)(1970), 89-96.

[18] H.Maki, R.Devi and K.Balachandran, *Associated topologies of generalized α-closed sets and α-generalized closed sets*, Mem. Fac. Sci. Kochi. Univ. Ser. A. Math., 15(1994), 51-63.

[19] A.S.Mashhour, M.E.Abd El-Monsef and S.N.El-Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt, 53(1982), 47-53.

[20] M.Murugalingam, *A study of semi generalized topology*, Ph.D Thesis, Manonmaniam Sundaranar University, Tirunelveli, Tamil Nadu, India, (2005).

[21] M.Navaneethakrishnan and J.Paulraj Joseph, *g-closed sets in ideal topological spaces*, Acta Math. Hungar., 119(4)(2008), 365-371.

[22] M.Navaneethakrishnan, J.Paulraj Joseph and D.Sivaraj, *$\mathcal{I}_g$-normal and $\mathcal{I}_g$-regular spaces*, Acta Math. Hungar., 125(2008), 327-340.

[23] R.L.Newcomb, *Topologies which are compact modulo an ideal*, Ph. D Dissertation, Univ. of Cal. at Santa Barbara, (1967).

[24] O.Njastad, *On some classes of nearly open sets*, Pacific J. Math., 15(1965), 961-970.

[25] K.C.Rao and K.Joseph, *Semi-star generalized closed sets*, Bull. Pure Appl. Sci., 19(E)(2)(2002), 281-290.

[26] V.Renuka Devi, D.Sivaraj and T.Tamizh Chelvam, *Codense and Completely codense ideals*, Acta Math. Hungar., 108(2005), 197-205.

[27] M.Sheik John, *A study on generalizations of closed sets and continuous maps in topological and bitopological spaces*, Ph. D Thesis, Bharathiar University, Coimbatore, (2002).

[28] M.H.Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc., 41(1937), 375-481.

[29] R.Vaidyanathaswamy, *Set Topology*, Chelsea Publishing Company, (1946).

[30] M.K.R.S.Veerakumar, *$\hat{g}$-closed sets intopological spaces*, Bull. Allah. Math. Soc., 18(2003), 99-112.

[31] M.K.R.S.Veerakumar, *$g^{\#}$-closed sets in topological spaces*, Mem. Fac. Sci. Kochi Univ. (Math.)., 24(2003), 1-13.