International Journal of Mathematics And its Applications
Volume 4, Issue 2–C (2016), 85–87.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Strictly Scattered Spaces

Research Article

M.Sreeja[1]*

1 Department of Mathematics, P.T.M. Government College, Perinthalmanna, Kerala, India.

Abstract: We introduce a particular type of scattered space called strictly scattered space and completely determine them.

MSC: 54A05, 54G12.

Keywords: Scattered spaces, Strictly scattered spaces.

1. Introduction

A space is said to be scattered if every nonempty subset of it contains a point isolated with respect to the relative topology. The study of scattered spaces is having unlimited importance, because they have a plenty of interesting properties. For example, in [2], it is proved that every topological space is a closed continuous image of a scattered space. V.Kannan and M.Rajagopalan had worked on these spaces, which can be seen in [1], [2], [3].

In this paper, we introduce the concept of a particular type of scattered space, called strictly scattered space and study its properties. Let us start with some preliminaries.

2. Preliminaries

Definition 2.1 ([4])**.** *A reflexive transitive relation R on a set X is called a pre-order on X. The ordered pair (X, R) is called a pre-ordered set.*

Definition 2.2. *A linearly ordered set is well ordered if every nonempty subset has a first element.*

Definition 2.3 ([1])**.** *A topological space (X, T) is said to be scattered if every nonempty subset of it contains a point isolated with respect to the relative topology.*

Definition 2.4 ([1])**.** *Let X be a topological space. Let $A \subset X$. We define $A^o = A; A^1 = \{x/x \in A \text{ and } x \in cl(A - \{x\})\}$. If α is an ordinal and A^α has been defined, then we put $A^{\alpha+1} = (A^\alpha)^1$. If α is a limit ordinal and A^β has been defined for all ordinals $\beta < \alpha$, then we put $A^\alpha = \cap_{\beta<\alpha} A^\beta$. If α is an ordinal we call A^α the αth derivative of A. A topological space X is said to have a derived length if $X^\alpha = \emptyset$ for some ordinal α. In this case, the derived length of X is defined as the least ordinal α so that $X^\alpha = \emptyset$.*

* E-mail: sreejabijusankar@gmail.com

Definition 2.5 ([4]). *Let T be a topology on a set X. Then we can associate with it a pre-order $\leq$ on X such that $a \leq b$ if and only if every open set containing b contains a.*

Definition 2.6 ([4]). *Let $(X, \leq)$ be a pre-ordered set. Let $x \in X$. Then define $\{\bar{x}\} = \{y : x \leq y\}$ and $\{\hat{x}\} = \{y : y \leq x\}$. Denote the smallest topology on X for which the sets $\{\bar{x}\}$ are closed for every x in X by $\mu(\leq)$ and the smallest topology on X for which the sets $\{\hat{x}\}$ are open for every x in X by $\nu(\leq)$.*

Lemma 2.7 ([4]). *A topology T on X has the associated pre-order $\leq$ on X if and only if*
$\mu(\leq) \subset T \subset \nu(\leq)$

Lemma 2.8 ([4]). *If $(X, \leq)$ is a linearly ordered set, then $\mu(\leq) = \nu(\leq)$ if and only if every non maximum element has a successor.*

Now let us state the main result.

3. Strictly Scattered Spaces

We will start with the definition of a strictly scattered space.

Definition 3.1. *A topological space (X, T) is said to be stricty scattered if every nonempty subset of it contains one unique isolated point with respect to the relative topology.*

The following lemma gives an example of a strictly scattered space.

Lemma 3.2. *A well-ordered set with the initial segment topology is a strictly scattered space.*

Proof. Let X be a well-ordered set and T be the initial segment topology on X. Let $A \subset X$ and $a \in A$ be the first element of A. Take $b \in A$ as the first element of $A - \{a\}$. Let U be the initial segment of b in X. Then U is open in X and $U \cap A = \{a\}$. i.e., a is an isolated point of A with respect to the relative topology on A.
Now let c be any element of $A - \{a, b\}$. Then $a < b < c$ and so every open set of X containing c contains a and so that c will not be an isolated point of A with respect to the relative topology.
thus a is a unique isolated point of A, and since A is arbitrary, (X, T) is strictly scattered. □

Let us state the main theorem.

Theorem 3.3. *A topological space is strictly scattered if and only if it is homeomorphic to the initial segment topology on a well ordered set.*

Proof. Let (X, T) be a strictly scattered topological space. Define a linear order on X as follows. If $a, b \in X$, then $a < b$ if and only if a is the unique isolated point of the subspace $\{a, b\}$ of X.
Then the unique isolated point of any subspace A will be the first element of A. For, let $a \in A$ be the unique isolated point of A and suppose there exist $b \in A$ such that $b < a$. Then b is the unique isolated point of the subspace $\{a, b\}$, which implies that every open set of X containing a contains b, which contradicts the fact that a is an isolated point of A. Thus X is well ordered. Now let $a \in X$ be a non maximum element of X and let A be the set of all successors of a in X. Take the first element b of A, then b is the immediate successor of a, for if $a < c < b$, then $c \in A$ and $c < b$, which is a contradiction. Since X satisfies the property that every non maximum element has an immediate successor, by [4], $\mu(R) = \nu(R)$ . .i.e., there exist one and only one topology on X with this property. By Lemma 3.2, a well-ordered set with initial segment topology satisfies this property. $\therefore X$ is homeomorphic to the initial segment topology on X. □

Remark 3.4. *By Theorem 3.3 it is clear that there is a well ordered set associated with a strictly scattered space. Thus we can associate an ordinal number $Ord(X)$ with every strictly scattered space.*

Corollary 3.5. *Let (X, T) be a strictly scattered space. Then the derived length of (X, T) is equal to $Ord(X)$.*

Proof. By Theorem 3.3, (X, T) is homeomorphic to the initial segment topology on the well-ordered set $(X, \leq)$. Let α be an ordinal number such that $(X, \leq) \cong \alpha$. Now, let $a_1 \in X$ be the unique isolated point of X. Then $X^1 = X - \{a_1\}$. Let $a_2 \in X^1$ be the unique isolated point of X^1. Then $X^2 = X^1 - \{a_2\} = X - \{a_1, a_2\}$. Continuing like this, we get $X^n = X - \{a_1, a_2, a_3, ...a_n\}$ where $a_i \in X^{i-1}$ is the unique isolated point of X^{i-1} for any successor ordinal n. Thus it is clear that $X^\alpha = \emptyset$ if and only if $\alpha = Ord(X)$. □

4. Acknowledgment

I am very much thankful to my supervisor Dr. P. T. Ramachandran for his helps, advises, suggestions and criticisms during the preparation of this paper.

References

[1] V.Kannan And M.Rajagopalan, *On Scattered Spaces*, Proceedings of The American Mathematical Society, 43(2)(1974), 402-408.

[2] V.Kannan And M.Rajagopalan, *Scattered spaces II*, Illinois Journal of Mathematics, 21(4)(1977), 735-751.

[3] M.Rajagopalan, *Scattered Spaces III*, Journal of The Indian Math. Soc., 41(1977), 405-427.

[4] Susan J.Andima and W.J.Thron, *Order Inducd Topological Properties*, Pacific Journal of Mathematics, 75(2)(1978), 297-318.