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Solutions of BSM Equation Using Fourier Transform Method

Research Article

Jigna V.Panchal[1] and A.K.Desai[2]*

1 Department of Mathematics, Indus University, Ahmedabad, India.
2 Department of Mathematics, Gujarat University, Ahmedabad, India.

Abstract: In Mathematical Finance call option is the stock market device and pricing an option is a key part.The solution of Black-Scholes-Merton (BSM) Partial Differential Equation gives the theoretical value of an option(Call/Put). It is also very useful application for online trading platform. In the present paper we have applied Fourier Transform Method to solve the equation for Log payoff function and Modified Log payoff function, which are the boundary conditions for the BSM partial differential equation. Also we have observed and shown that averages of these two payoff functions will give exactly the average of two solutions.

MSC: 91B24, 91B28.

Keywords: Black-Scholes-Merton Model, Partial Differential Equation, Call/Put option, Fourier Transform Method.

1. Introduction

In Mathematical Finance, the Black-Scholes-Merton equation is a Partial Differential Equation to find the value of European Call/Put option. Suppose $C(S,t)$ is the value of European call option. The following most basic equation, which appears in almost every basic book on the subject, e.g. [1–4];

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} - rC = 0 \tag{1}$$

is known as a Black-Scholes-Merton Partial Differential Equation. Where,

S is Spot price of asset

X is Exercise price or strike price

T is Total period of time

r is Risk free interest rate

σ is Volatility

$t \in [0, T]$ and $C(S,t) = 0$ for all t

$C(S,t) \to S$ as $S \to \infty$.

* *E-mail: jignapach@gmail.com*

Consider the European call option whose final payoff at the expiry time T is given by a function f of the final spot price S (we note that in the literature it is often denoted by many as S_T) which is assumed to be a continuous function, that need not be differentiable. Also we demand,

$$\lim_{t\to T_-} C(S,t) = f(S).$$

We can convert the Black-Scholes-Merton Partial Differential Equation to the heat equation using the following substitutions:

$$\begin{aligned} y &= T-t,\\ x &= \ln\left(\frac{S}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t) \text{ and}\\ D(x,y) &= e^{r(T-t)}C(S,t). \end{aligned}$$

These substitutions also convert the above mentioned boundary condition,

$$\lim_{t\to T_-} C(S,t) = f(S)$$

to the initial condition,

$$\lim_{y\to 0^+} D(x,y) = f(Xe^x).$$

Thus the Black-Scholes-Merton Partial Differential Equation gets converted to the following Heat equation and with the following initial condition:

$$\frac{\partial D}{\partial y} = \frac{\sigma^2}{2}\frac{\partial^2 D}{\partial x^2} \text{ with } \lim_{y\to 0^+} D(x,y) = f(Xe^x). \tag{2}$$

Many people have solved same problem using the Method of separation of variables and Laplace Transform Method [6]. Here we use Fourier Transform Method to solve the problem. Applying Fourier Transform on Heat equation we get,

$$\begin{aligned} \frac{\partial}{\partial y}F(D) + \frac{\sigma^2\lambda^2}{2}F(D) &= 0\\ F(D) &= C_1 e^{-\frac{\sigma^2\lambda^2}{2}y}. \end{aligned}$$

Now we get, $F(D(x,0)) = G(\lambda)$ because $D(x,0) = f(Xe^x)$. Here G is the Fourier Transform of f, so that C_1 is determined and we now have:

$$F(D) = G(\lambda)e^{-\frac{\sigma^2\lambda^2}{2}y}.$$

Taking inverse Fourier Transform on both the sides we get,

$$D(x,y) = F^{-1}\left(G(\lambda)e^{-\frac{\sigma^2\lambda^2}{2}y}\right)$$

Now, using convolution theorem and the facts, that

$$F^{-1}(G(\lambda)) = f(Xe^x) \text{ and } F^{-1}\left(e^{-\frac{\sigma^2\lambda^2}{2}y}\right) = \frac{1}{\sigma\sqrt{y}}e^{\frac{-x^2}{2\sigma^2 y}}$$

we get,

$$\begin{aligned} D(x,y) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\nu)\frac{1}{\sigma\sqrt{y}}e^{\frac{-(x-\nu)^2}{2\sigma^2 y}}\,\mathrm{d}\nu\\ \therefore D(x,y) &= \frac{1}{\sigma\sqrt{2\pi y}}\int_{-\infty}^{\infty} f(\nu)e^{\frac{-(x-\nu)^2}{2\sigma^2 y}}\,\mathrm{d}\nu. \end{aligned} \tag{3}$$

2. Solution of the Problem Using Different Payoff Functions

2.1. Plain Vanilla Payoff:

This is very basic and commonly used payoff function. Many financial organizations use this as a payoff. The very first BSM formula was derived using this [3]. Now we consider Plain Vanilla Payoff function which is as:

$$f(S) = \max\{S - X,\ 0\} = \begin{cases} S - X & \text{if } S \geq X \\ 0 & \text{if } S \leq X \end{cases}$$

$$\therefore f(Xe^x) = \max\{X(e^x - 1),\ 0\} = \begin{cases} X(e^x - 1) & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

The solution of Plain Vanilla is as follows:

$$\therefore C(S,t) = SN(d_1) - Xe^{-r(T-t)}N(d_2) \tag{4}$$

where,

$$d_1 = \frac{x + \sigma^2 y}{\sigma\sqrt{y}} = \frac{\ln\left(\frac{S}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\left(\frac{S}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

Which is also derived using Fourier Transform Method in [7].

2.2. Log Payoff:

Paul Wilmott has discussed BSM formula for Log payoff function [4]. Other than this several types of option pricing formulas have been derived with different payoff functions [2]. For Log Payoff function the solution of BSM equation is derived with all details in [7] using Fourier Transform Method. For the sake of continuity we have added certain points. Now we consider the payoff function which is known as Log payoff, which is as:

$$f(S) = \max\left\{\ln\left(\frac{S}{X}\right),\ 0\right\} = \begin{cases} \ln\left(\frac{S}{X}\right) & \text{if } S \geq X \\ 0 & \text{if } S \leq X \end{cases}$$

$$\therefore f(Xe^x) = \max\{x,\ 0\} = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

$$\therefore D(x,y) = \frac{1}{\sigma\sqrt{2\pi y}} \int_0^\infty \nu e^{-\frac{(x-\nu)^2}{2\sigma^2 y}} \mathrm{d}\nu. \text{ Taking,}$$

$$Z = \frac{\nu - x}{\sigma\sqrt{y}}$$

we get,

$$\begin{aligned} D(x,y) &= \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sigma\sqrt{y}}}^\infty (x + \sigma\sqrt{y}Z)e^{-\frac{Z^2}{2}} \mathrm{d}Z \\ &= \frac{x}{\sqrt{2\pi}} \int_{-\frac{x}{\sigma\sqrt{y}}}^\infty e^{-\frac{Z^2}{2}} \mathrm{d}Z + \sigma\sqrt{\frac{y}{2\pi}} \int_{-\frac{x}{\sigma\sqrt{y}}}^\infty Ze^{-\frac{Z^2}{2}} \mathrm{d}Z \\ &= \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sigma\sqrt{y}}} e^{-\frac{t^2}{2}} \mathrm{d}t + \sigma\sqrt{\frac{y}{2\pi}} e^{-\frac{x^2}{2\sigma^2 y}} \end{aligned}$$

$$\therefore D(x,y) = xN(d) + \sigma\sqrt{\frac{y}{2\pi}}e^{-\frac{x^2}{2\sigma^2 y}}$$

where, $d = \frac{x}{\sigma\sqrt{y}}$

$$\therefore C(S,t) = e^{-r(T-t)}\left[\ln\left(\frac{S}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right]N(d) + \frac{1}{\sqrt{2\pi}}e^{-r(T-t)}\sigma\sqrt{T-t}e^{-\frac{d^2}{2}} \tag{5}$$

where,

$$d = \frac{\ln\left(\frac{S}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

2.3. Modified Log Payoff:

H.V.Dedania and S.J.Ghevariya have discussed BSM formula for Modified Log Payoff function [5]. Which is closely related to Log Payoff function and close to the celebrated Plain Vanilla Payoff function. Other than this several types of option pricing formulas have been derived with different payoff functions [2]. Now we consider the payoff function which is known as Modified Log Payoff, which is as:

$$f(S) = \max\left\{S\ln\left(\frac{S}{X}\right),\ 0\right\} = \begin{cases} S\ln\left(\frac{S}{X}\right) & \text{if } S \geq X \\ 0 & \text{if } S \leq X \end{cases}$$

$$\therefore f(Xe^x) = \max\{xXe^x,\ 0\} = \begin{cases} xXe^x & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

$$\therefore D(x,y) = \frac{X}{\sigma\sqrt{2\pi y}}\int_0^\infty \nu e^\nu e^{-\frac{(x-\nu)^2}{2\sigma^2 y}}\,\mathrm{d}\nu. \text{ Taking}$$

$$Z = \frac{\nu - x}{\sigma\sqrt{y}}$$

we get,

$$\begin{aligned}
D(x,y) &= \frac{X}{\sqrt{2\pi}}\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty (x+\sigma\sqrt{y}Z)e^{x+\sigma\sqrt{y}Z}e^{-\frac{Z^2}{2}}\,\mathrm{d}Z \\
&= \frac{x}{\sqrt{2\pi}}\left[xe^x\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty e^{x+\sigma\sqrt{y}Z}e^{-\frac{Z^2}{2}}\,\mathrm{d}Z + \sigma\sqrt{y}e^x\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty Ze^{x+\sigma\sqrt{y}Z}e^{-\frac{Z^2}{2}}\,\mathrm{d}Z\right] \\
&= \frac{X}{\sqrt{2\pi}}\left[xe^{x+\frac{\sigma^2 y}{2}}\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty e^{-\frac{(Z^2-2\sigma\sqrt{y}Z+\sigma^2 y)}{2}}\,\mathrm{d}Z + \sigma\sqrt{y}e^{x+\frac{\sigma^2 y}{2}}\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty e^{-\frac{(Z^2-2\sigma\sqrt{y}Z+\sigma^2 y)}{2}}\,\mathrm{d}Z\right] \\
&= \frac{X}{\sqrt{2\pi}}\left[xe^{x+\frac{\sigma^2 y}{2}}\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty e^{-\frac{(Z-\sigma\sqrt{y})^2}{2}}\,\mathrm{d}Z + \sigma\sqrt{y}e^{x+\frac{\sigma^2 y}{2}}\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty e^{-\frac{(Z^2-\sigma\sqrt{y})^2}{2}}\,\mathrm{d}Z\right] \\
&= \frac{X}{\sqrt{2\pi}}\left[xe^{x+\frac{\sigma^2 y}{2}}\int_{-\frac{x+\sigma^2 y}{\sigma\sqrt{y}}}^\infty e^{-\frac{t^2}{2}}\,\mathrm{d}t + \sigma\sqrt{y}e^{x+\frac{\sigma^2 y}{2}}\int_{-\frac{x+\sigma^2 y}{\sigma\sqrt{y}}}^\infty (t+\sigma\sqrt{y})e^{-\frac{(t^2}{2}}\,\mathrm{d}t\right] \\
&= \frac{X}{\sqrt{2\pi}}\left[e^x\left(xe^{\frac{\sigma^2 y}{2}} + \sigma^2 y\right)\int_{-\frac{x+\sigma^2 y}{\sigma\sqrt{y}}}^\infty e^{-\frac{t^2}{2}}\,\mathrm{d}t + \sigma\sqrt{y}e^{x+\frac{\sigma^2 y}{2}}\int_{-\frac{x+\sigma^2 y}{\sigma\sqrt{y}}}^\infty te^{-\frac{t^2}{2}}\,\mathrm{d}t\right] \\
&= \frac{X}{\sqrt{2\pi}}e^{x+\frac{\sigma^2 y}{2}}\left[(x+\sigma^2 y)\int_{-\infty}^{-\frac{x+\sigma^2 y}{\sigma\sqrt{y}}} e^{-\frac{t^2}{2}}\,\mathrm{d}t + \sigma\sqrt{y}e^{\frac{-1}{2}\left(\frac{x+\sigma^2 y}{\sigma\sqrt{y}}\right)^2}\right]
\end{aligned}$$

$$\therefore D(x,y) = \frac{X}{\sqrt{2\pi}}e^{x+\frac{\sigma^2 y}{2}}\left[\left(x+\sigma^2 y\right)N(d) + \sigma\sqrt{y}e^{\frac{-d^2}{2}}\right]$$

where, $d = \frac{x+\sigma^2 y}{\sigma\sqrt{y}}$

$$\therefore C(S,t) = S\left[\ln\left(\frac{S}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)N(d) + \frac{1}{\sqrt{2\pi}}\sigma\sqrt{T-t}e^{-\frac{d^2}{2}}\right] \tag{6}$$

where,

$$d = \frac{\ln\left(\frac{S}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Now, we consider average of two Payoff functions Log and Modified Log, i.e.

$$f(S) = \begin{cases} \frac{\ln\left(\frac{S}{X}\right) + S\ln\left(\frac{S}{X}\right)}{2} & \text{if } S \geq X \\ 0 & \text{if } S \leq X \end{cases}$$

$$\therefore f(Xe^x) = \begin{cases} \frac{xXe^x + x}{2} & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

$$\begin{aligned}
\therefore D(x,y) &= \frac{1}{2\sigma\sqrt{2\pi y}}\int_0^\infty (X\nu e^\nu + \nu)e^{-\frac{(x-\nu)^2}{2\sigma^2 y}}\,\mathrm{d}\nu \\
&= \frac{X}{2\sigma\sqrt{2\pi y}}\int_0^\infty \nu e^\nu e^{-\frac{(x-\nu)^2}{2\sigma^2 y}}\,\mathrm{d}\nu + \frac{1}{2\sigma\sqrt{2\pi y}}\int_0^\infty \nu e^{-\frac{(x-\nu)^2}{2\sigma^2 y}}\,\mathrm{d}\nu
\end{aligned}$$

Substituting,

$$Z = \frac{\nu - x}{\sigma\sqrt{y}}$$

we get,

$$\begin{aligned}
D(x,y) &= \frac{X}{2\sqrt{2\pi}}\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty (x+\sigma\sqrt{y}Z)e^{-\frac{Z^2}{2}}\,\mathrm{d}Z + \frac{1}{2\sqrt{2\pi}}\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty (x+\sigma\sqrt{y}Z)e^{-\frac{Z^2}{2}}\,\mathrm{d}Z \\
&= \frac{X}{2\sqrt{2\pi}}xe^x\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty e^{\sigma\sqrt{y}Z}e^{-\frac{Z^2}{2}}\,\mathrm{d}Z + \frac{X}{2\sqrt{2\pi}}\sigma\sqrt{y}e^x\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty Ze^{\sigma\sqrt{y}Z}e^{-\frac{Z^2}{2}}\,\mathrm{d}Z \\
&\quad + \frac{x}{2\sqrt{2\pi}}\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty e^{-\frac{Z^2}{2}}\,\mathrm{d}Z + \frac{\sigma\sqrt{y}}{2\sqrt{2\pi}}\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty Ze^{-\frac{Z^2}{2}}\,\mathrm{d}Z \\
&= \frac{X}{2\sqrt{2\pi}}xe^{x+\frac{\sigma^2 y}{2}}\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty e^{-\frac{(Z-\sigma\sqrt{y})^2}{2}}\,\mathrm{d}Z + \frac{X}{2\sqrt{2\pi}}\sigma\sqrt{y}e^{x+\frac{\sigma^2 y}{2}}\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty Ze^{-\frac{Z^2}{2}}\,\mathrm{d}Z \\
&\quad + \frac{\sigma\sqrt{y}}{2\sqrt{2\pi}}e^{-\frac{Z^2}{2\sigma^2 y}} \\
&= \frac{X}{2\sqrt{2\pi}}(x+\sigma^2 y)e^{x+\frac{\sigma^2 y}{2}}\int_{-\frac{x+\sigma^2 y}{\sigma\sqrt{y}}}^\infty e^{-\frac{t^2}{2}}\,\mathrm{d}t + \frac{X}{2\sqrt{2\pi}}\sigma\sqrt{y}e^{x+\frac{\sigma^2 y}{2}}\int_{-\frac{x+\sigma^2 y}{\sigma\sqrt{y}}}^\infty te^{-\frac{t^2}{2}}\,\mathrm{d}t \\
&\quad + \frac{x}{2\sqrt{2\pi}}\int_{-\frac{x}{\sigma\sqrt{y}}}^\infty e^{-\frac{t^2}{2}}\,\mathrm{d}t + \frac{\sigma\sqrt{y}}{2\sqrt{2\pi}}e^{-\frac{Z^2}{2\sigma^2 y}} \\
&= \frac{X}{2\sqrt{2\pi}}(x+\sigma^2 y)e^{x+\frac{\sigma^2 y}{2}}\int_{-\infty}^{\frac{x+\sigma^2 y}{\sigma\sqrt{y}}} e^{-\frac{t^2}{2}}\,\mathrm{d}t + \frac{X}{2\sqrt{2\pi}}\sigma\sqrt{y}e^{x+\frac{\sigma^2 y}{2}}e^{\frac{-1}{2}\left(\frac{x+\sigma^2 y}{\sigma\sqrt{y}}\right)^2} \\
&\quad + \frac{x}{2\sqrt{2\pi}}\int_{-\infty}^{\frac{x}{\sigma\sqrt{y}}} e^{-\frac{t^2}{2}}\,\mathrm{d}t + \frac{\sigma}{2}\sqrt{\frac{y}{2\pi}}e^{-\frac{x^2}{2\sigma^2 y}}
\end{aligned}$$

$$\therefore D(x,y) = \frac{X}{2}(x+\sigma^2 y)e^{x+\frac{\sigma^2 y}{2}}N(d_1) + \frac{X}{2\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} + \frac{X}{2}N(d_2) + \frac{\sigma}{2}\sqrt{\frac{y}{2\pi}}e^{-\frac{d_2^2}{2}}$$

where, $d_1 = \frac{x+\sigma^2 y}{\sigma\sqrt{y}}, d_2 = \frac{x}{\sigma\sqrt{y}}$ and $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{t^2}{2}}\,\mathrm{d}t$

$$\begin{aligned}
\therefore C(S,t) &= \frac{S}{2}\left[\ln\left(\frac{S}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]N(d_1) + \frac{S}{2\sqrt{2\pi}}e^{-r(T-t)}\sigma\sqrt{T-t}e^{-\frac{d_1^2}{2}} \\
&\quad + \frac{1}{2}e^{-r(T-t)}\left[\ln\left(\frac{S}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right]N(d_2) + \frac{1}{2\sqrt{2\pi}}e^{-r(T-t)}\sigma\sqrt{T-t}e^{-\frac{d_2^2}{2}}
\end{aligned}$$

where,

$$d_1 = \frac{x+\sigma^2 y}{\sigma\sqrt{y}} = \frac{\ln\left(\frac{S}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\left(\frac{S}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

One can see from equation (5) and (6) that, this is exactly the average of two solutions, i.e. solution using Log Payoff (5) and solution using Modified Log Payoff function (6).

3. Conclusion

The BSM equation is solved using Fourier Transform Method considering the stipulated boundary conditions (as three different Payoff function). This gives the fair value of an option. Using this, one can find the theoretical value of an option on variety of assets. We have opted for the notion of considering the averages of the definite Payoff functions. It is hoped that in the practical applications, where certain payoff functions are very close, sometimes averages also may be considered as a good alternative.

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