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Fixed Point Theorem For Fuzzy Mapping

Research Article

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Abstract: In this paper we state and prove fixed fuzzy point theorem for fuzzy mapping in complete metric space.

Keywords: Fuzzy mapping, Fixed point, Hausdorf distance.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [7] in 1965. The idea of an intuitionistic fuzzy set is due to Atanassov [4–6] and [2] has defined the concept of fuzzy topological spaces induced by Chang [1]. In 1981, Heilpern [9] first introduced the concept of fuzzy contractive mappings and proved a fixed point theorem for these mappings in metric linear spaces. His result is a generalization of the fixed point theorem for point-to-set-maps of Nadler [8]. Estruct and Vidal [10] give a fixed point theorem for fuzzy contraction mappings over a complete metric space which is a generalization of the given Heilpern's fixed point theorem.

2. Preliminaries

Definition 2.1 ([9]). *Let $A, B \in W(X), \alpha \in [0,1]$. Define*

$$P_\alpha(A,B) = \inf_{x\in A_\alpha, y\in B_\alpha} d(x,y),$$

$$D_\alpha(A,B) = dist(A_\alpha, B_\alpha)$$

$$D(A,B) = \sup_\alpha D_\alpha(A,B)$$

Whenever dist is Hausdorf distance, The function p_α is called a α-space, D_α is a α-distance, and D is a distance between$A$ and$B$. It is easy to see that p_α is nondecreasing function of α.

Definition 2.2 ([9]). *Let $A, B \in W(X)$. An approximate uantity A is more accurate than B, denoted $A \subset B$, iff $A(x) \leq B(x)$, for each $x \in X$. It easy to see that relation $\subset$ is a partial order determined on the family $W(X)$.*

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Definition 2.3 ([9]). *Let X be an arbitrary set and Y any metric linear space. F is called a fuzzy mapping iff F is mapping from the set X into $W(Y)$, i.e., $F(x) \in W(Y)$ for each $x \in X$. A fuzzy mapping F is a fuzzy subset on $X \times Y$ with membership function $F(x, y)$. The function value $F(x, y)$ is the grade of membership of y in $F(x)$.*

Definition 2.4 ([9]). *Let $A, B \in W(X), \alpha \in [0,1]$. Define*

$$\begin{aligned} p_\alpha(A,B) &= \inf\left\{d(x,y) : x \in A_\alpha, y \in B_\alpha\right\} \\ D_\alpha(A,B) &= H(A_\alpha, B_\alpha), \\ D(A,B) &= \sup_\alpha D_\alpha(A,B) \end{aligned}$$

Where H is the Hausdorf distance. For $x \in X$ we write $p_\alpha(x,B)$ instead of $P_\alpha(\{x\},B)$.

Definition 2.5 ([10]). *Let X be metric space and $\alpha \in [0,1]$. Consider the following family $W_\alpha(X)$:*

$$W_\alpha(X) = \left\{A \in I^X : A_\alpha \text{ is nonempty compact and convex}\right\}.$$

We need the following lemma. Let (X, d) be a metric space.

Lemma 2.6 ([9]). *Let $x \in X$, $A \in W(X)$, and $\{x\}$ be a fuzzy set with membership function of set $\{x\}$. If $\{x\} \subset A$, then $p_\alpha(x,A) = 0$ for each $\alpha \in [0,1]$..*

Proof. If $\{x\} \subset A$, $x \in A_\alpha$ for each $\alpha \in [0,1]$.

$$p_\alpha(x,A) = \inf_{y \in A_\alpha} d(x,y) = 0.$$

□

Lemma 2.7 ([9]). *$p_\alpha(x,A) \leq d(x,y) + p_\alpha(y,A)$ for any $x, y \in X$.*

Proof.

$$\begin{aligned} p_\alpha(x,A) &= \inf_{z \in A_\alpha} d(x,z) \leq \inf_{z \in A_\alpha} (d(x,y) + d(y,z)) \\ &= d(x,y) + p_\alpha(y,A). \end{aligned}$$

□

Lemma 2.8 ([9]). *If $\{x_0\} \subset A$, then $p_\alpha(x_0,B) \leq D_\alpha(A,B)$ for each $B \in W(X)$.*

Proof.

$$\begin{aligned} p_\alpha(x_0,B) &= \inf_{y \in B_\alpha} d(x_0,y) \leq \sup_{x \in A_\alpha} \inf_{y \in B_\alpha} d(x,y) \\ &\leq D_\alpha(A,B) \end{aligned}$$

□

Lemma 2.9 ([3]). *Let (X,d) be a complete metric space, $F : X \to W(X)$ be a fuzzy map and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \in F(x_0)$.*

3. Main Result

Theorem 3.1. *Let $\alpha \in (0,1]$ and (X,d) be a complete metric space. Let F be a continuous fuzzy mapping from X into $W_\alpha(X)$ satisfying the following condition: There exist $K \in [0,\infty) \to [0,\infty)$, $K(0)=0$, $K(t)<t$ for all $t \in (0,\infty)$ and K is non-decreasing such that*

$$D_\alpha\left(F(x),F(y)\right) \leq K \max\left\{\tfrac{1}{2}d(x,y), p_\alpha(y,F(x)), \frac{p_\alpha(x,F(y))+p_\alpha(x,F(x))}{2}\right\} \tag{1}$$

and $\sum\limits_{n=1}^{\infty}\left[\left(\frac{K}{2}\right)^n d(x_0,x_1)\right] < \infty$. Then there exists $x \in X$ such that x_α is a fixed fuzzy point of F.

Proof. Let $x_0 \in X$ and suppose that there exists $x_1 \in (F(x_0))_\alpha$. Since $(F(x_1))_\alpha$ is a nonempty compact subset of X, then there exists $x_2 \in (F(x_0))_\alpha$ such that $d\left(x_1,x_2\right) = p_\alpha(x_1,F(x_1)) \leq D_\alpha(F(x_0),F(x_1))$. By Lemma 2.8. By induction we construct a sequence $\{x_n\}$ in X such that $x_n \in (F(x_{n-1}))_\alpha$ and $d(x_n,x_{n+1}) \leq D_\alpha(F(x_n),F(x_{n-1}))$. Nothing that K is non-decreasing and using inequality (1), we have

$$\begin{aligned}
d(x_n,x_{n+1}) &\leq D_\alpha(F(x_n),F(x_{n-1})) \\
&\leq K\left(\max\left\{\frac{1}{2}d(x_{n-1},x_n), p_\alpha(x_n,F(x_{n-1})), \frac{p_\alpha(x_{n-1},F(x_n))+p_\alpha(x_{n-1},F(x_{n-1}))}{2}\right\}\right) \\
&\leq K\left(\max\left\{\frac{1}{2}d(x_{n-1},x_n), d(x_n,x_n), \frac{d(x_{n-1},x_{n+1})+d(x_{n-1},x_n)}{2}\right\}\right) \\
&\leq K\left(\max\left\{\frac{1}{2}d(x_{n-1},x_n), \frac{d(x_{n+1},x_n)}{2}\right\}\right) \\
&\leq \frac{1}{2}K\left(\max\left\{d(x_{n-1},x_n), d(x_{n+1},x_n)\right\}\right)
\end{aligned} \tag{2}$$

Suppose $d(x_n,x_{n+1}) > d(x_{n-1},x_n)$ for some n. Then from (2) and $K(t)<t$ for all $t \in (0,\infty)$, we have

$$\begin{aligned}
d(x_n,x_{n+1}) &\leq \frac{1}{2}Kd(x_n,x_{n+1}) \\
d(x_n,x_{n+1}) &\leq \frac{1}{2}d(x_n,x_{n+1})
\end{aligned}$$

This is a contradiction. Therefore we have

$$\begin{aligned}
d(x_n,x_{n+1}) &\leq \frac{1}{2}Kd(x_{n-1},x_n) \\
&\leq \frac{1}{2}K\left\{\frac{K}{2}d(x_{n-2},x_{n-1})\right\} \\
&\leq \frac{1}{2}K\left\{\frac{K}{2}d(x_{n-2},x_{n-1})\right\} \\
&\leq \frac{K}{2}\frac{K}{2}\left\{\frac{K}{2}d(x_{n-3},x_{n-2})\right\}\vdots \\
&\leq \frac{K^n}{2^n}d(x_0,x_1)
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
d(x_n,x_{n+m}) &\leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{n+m-1},x_{n+m}) \\
&\leq \left(\frac{K}{2}\right)^n d(x_0,x_1) + \left(\frac{K}{2}\right)^{n+1} d(x_0,x_1) + \cdots + \left(\frac{K}{2}\right)^{n+m-1} d(x_0,x_1) \\
&\leq \left[\left(\frac{K}{2}\right)^n + \left(\frac{K}{2}\right)^{n+1} + \cdots + \left(\frac{K}{2}\right)^{n+m-1}\right] d(x_0,x_1) \\
&\leq \sum_{i=n}^{n+m-1}\left[\left(\frac{K}{2}\right)^i d(x_0,x_1)\right]
\end{aligned}$$

Since $\sum\limits_{n=1}^{\infty}\left[\left(\frac{K}{2}\right)^n d(x_0,x_1)\right] < \infty$. So there exists r such that

$$\leq \sum_{i=n}^{n+m-1}\left[\left(\frac{K}{2}\right)^i d(x_0,x_1)\right] = r < \infty$$

Therefore $\{x_n\}$ is a Cauchy sequence in X. Suppose $\{x_n\}$ converges to $x \in X$. Now, by Lemma 2.6 and 2.7

$$\begin{aligned} p_\alpha(x, F(x)) &\leq d(x,x_n) + p_\alpha(x_n, F(x)) \\ &\leq d(x,x_n) + D_\alpha(F(x_{n-1}), F(x)) \\ &\leq d(x,x_n) + \frac{K}{2} d(x_{n-1}, x) \end{aligned}$$

Consequently, $p_\alpha(x, F(x)) = 0$ and by Lemma 2.6, $x_\alpha \subset F(x)$. Hence x_α is a fixed fuzzy point of F. □

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