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Invariant Submanifold of $(4k, 2k)$ Structure Manifold

Research Article

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Abstract: In this paper, we have studied various properties of a $(4k, 2k)$ structure manifold and its invariant submanifold, where k is positive integer. Under two different assumptions, the nature of induced structure ψ, has also been discussed.

Keywords: Invariant submanifold, Nijenhuis tensor, projection operators and complementary distributions.

1. Introduction

Let V^m be a C^∞ m-dimensional Riemannian manifold imbedded in a C^∞ n-dimensional Riemannian manifold M^n, where $m < n$. The imbedding being denoted by

$$f \; : \; V^m \longrightarrow M^n$$

Let B be the mapping induced by f i.e. $B{=}df$

$$df \; : \; T(V) \longrightarrow T(M)$$

Let $T\ (V{,}M)$ be the set of all vectors tangent to the submanifold $f(V)$. It is well known that

$$B \; : \; T(V) \longrightarrow T(V,\ M)$$

Is an isomorphism. The set of all vectors normal to $f\ (V)$ forms a vector bundle over $f\ (V)$, which we shall denote by $N\ (V, M)$. We call $N\ (V,\ M)$ the normal bundle of V^m. The vector bundle induced by f from $N\ (V,\ M)$ is denoted by $N\ (V)$. We denote by $C\ :\ N(V)\ \longrightarrow\ N\ (V,\ M)$ the natural isomorphism and by $\eta^r_s\ (V)$ the space of all C^∞ tensor fields of type $(r,\ s)$ associated with $N(V)$. Thus $\zeta^0_0(V) = \eta^0_0\ (V)$ is the space of all C^∞ functions defined on V^m while an element of $\eta^1_0\ (V)$ is a C^∞ vector field normal to V^m and an element of $\zeta^1_0(V)$ is a C^∞ vector field tangential to V^m.

Let $\bar{X}$ and $\bar{Y}$ be vector fields defined along $f(V)$ and $\tilde{X}$, $\tilde{Y}$ be the local extensions of $\bar{X}$ and $\bar{Y}$ respectively. Then $\left[\tilde{X},\ \tilde{Y}\right]$ is a vector field tangential to M^n and its restriction $\left[\tilde{X},\ \tilde{Y}\right] \Big/ f(V)$to $f(V)$ is determined independently of the choice of these local extension $\tilde{X}$ and $\tilde{Y}$. Thus $\left[\bar{X},\ \bar{Y}\right]$ is defined as

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$$\left[\bar{X},\ \bar{Y}\right]\ =\ \left[\tilde{X},\ \tilde{Y}\right]\Big/ f\left(V\right) \tag{1}$$

Since B is an isomorphism

$$[BX,\ BY]\ =\ B\ [X,Y]\ \text{ for all }\ X,Y\ \in\ \zeta_0^1\left(V\right) \tag{2}$$

Let $\bar{G}$ be the Riemannain metric tensor of M^n, we define g and $g*$ on V^m and $N(V)$ respectively as

$$g\left(X_1,\ X_2\right)\ =\ \tilde{G}\ \left(BX_1,\ BX_2\right)\ f,\ \text{and} \tag{3}$$

$$g^*\left(N_1,\ N_2\right)\ =\ \tilde{G}\ \left(CN_1,\ CN_2\right) \tag{4}$$

For all $X_1,\ X_2\ \in\ \ \zeta_0^1\left(V\right)$ and $N_1,\ N_2 \in\ \ \eta_0^1\ \left(V\right)$. It can be verified that g and g^* are the induced metrics on V^m and $N(V)$ respectively. Let $\tilde{\nabla}$ be the Riemannian connection determined by $\tilde{G}$ in M^n, then $\tilde{\nabla}$ induces a connection ∇ in $f\left(V\right)$ defined by

$$\nabla_{\bar{X}}\bar{Y} =\ \tilde{\nabla}_{\tilde{X}}\tilde{Y}\Big/ f\left(V\right) \tag{5}$$

where $\bar{X}$ and $\bar{Y}$ are arbitrary C^∞ vector fields defined along $f\left(V\right)$ and tangential to $f\left(V\right)$. Let us suppose that M^n is a $(4k,\ 2k)$ structure manifold with structure tensor $\tilde{\psi}$ of type (1, 1) satisfying

$$\tilde{\psi}^{4k}\ +\ \tilde{\psi}^{2k}\ =\ 0 \tag{6}$$

Let $\tilde{L}$ and $\tilde{M}$ be the complementary distributions corresponding to the projection operators

$$\tilde{l}\ =\ -\ \tilde{\psi}^{2k}, \qquad \tilde{m}\ =\ I+\ \tilde{\psi}^{2k} \tag{7}$$

where I denotes the identity operator. From and , we have

$$(a)\ \ \tilde{l}+\tilde{m}=\ I;\ \ (b)\ \ \tilde{l}^2\ =\ \tilde{l};\ \ (c)\ \ \tilde{m}^2\ =\ \tilde{m};\ \ (d)\ \ \tilde{l}\,\tilde{m}\ =\ \tilde{m}\,\tilde{l}\ =\ 0 \tag{8}$$

Let D_l and D_m be the subspaces inherited by complementary projection operators l and m respectively. We define

$$D_l\ =\ \{X\ \in\ T_p\left(V\right)\ :\ \ lX\ =\ X,\ mX\ =\ 0\}$$

$$D_m\ =\ \{X\ \in\ T_p\left(V\right)\ :\ \ mX\ =\ X,\ lX\ =\ 0\}$$

Thus $T_p\left(V\right)\ =\ D_l\ +\ D_m$. Also $Ker\ l\ =\ \{X\ :\ lX\ = 0\}\ =\ D_m$; $Ker\ m\ =\ \{X\ :\ mX\ = 0\}\ =\ D_l$ at each point p of $f\left(V\right)$.

2. Invariant Submanifold of $(4k,\ 2k)$ Structure Manifold

We call V^m to be invariant submanifold of M^n if the tangent space $T^p\left(f\left(V\right)\right)$ of $f\left(V\right)$ is invariant by the linear mapping $\tilde{\psi}$ at each point p of $f\left(V\right)$. Thus

$$\tilde{\psi}BX\ =\ B\psi X, \tag{9}$$

for all $X\ \in\ \zeta_0^1\left(V\right)$ and ψ being a (1, 1) tensor field in V^m.

Theorem 2.1. *Let $\tilde{N}$ and N be the Nijenhuis tensors determined by $\tilde{\psi}$ and ψ in M^n and V^m respectively, then*

$$\tilde{N}\ (BX,\ BY)\ =\ BN\ (X,Y),\ \textit{for all}\ X,\ Y\ \in\ \zeta_0^1\,(V). \tag{10}$$

Proof. We have, by using (2) and (10)

$$\begin{aligned}
\tilde{N}\ (BX,\ BY)\ &=\ \left[\tilde{\psi}BX,\ \tilde{\psi}BY\right]\ +\ \tilde{\psi}^2\,[BX,\ BY]\ -\ \tilde{\psi}\left[\tilde{\psi}BX,\ BY\right]\ -\ \tilde{\psi}\ \left[BX,\ \tilde{\psi}BY\right]\\
&=\ [B\psi X,\ B\psi Y]\ +\ \tilde{\psi}^2 B\,[X,\ Y] -\ \tilde{\psi}\,[B\psi X,\ BY]\ -\ \tilde{\psi}\ [BX,\ B\psi Y]\\
&=\ B\,[\psi X,\ \psi Y]\ +\ B\psi^2\,[X,\ Y]\ -\ \tilde{\psi}B\ [\psi X,\ Y] -\ \tilde{\psi}B\,[X,\psi Y]\\
&=\ B\,\{[\psi X,\ \psi Y]\ +\ \psi^2\,[X,\ Y]\ -\ \psi\ [\psi X,\ Y]\ -\psi\,[X,\psi Y]\}\\
&= BX\ + B\psi^3\ X
\end{aligned} \tag{11}$$

□

3. Distribution $\tilde{M}$ Never Being Tangential to $f\,(V)$

Theorem 3.1. *If the distribution $\tilde{M}$ is never tangential to $f\,(V)$, then*

$$\tilde{m}\ (BX)\ =\ 0\ \textit{ for all }\ X \in\ \zeta_0^1\ (V) \tag{12}$$

and the induced structure ψ on V^m satisfies

$$\psi^{2k} =\ -I. \tag{13}$$

Proof. If possible $\tilde{m}\ (BX)\ \neq\ 0$. From (10) we get

$$\tilde{\psi}^{2k}BX\ =\ B\psi^{2k}\ X; \tag{14}$$

from (1) and (14)

$$\begin{aligned}
\tilde{m}\ (BX)\ &=\ \left(I\ +\tilde{\psi}^{2k}\right)\ BX\\
&=\ BX\ + B\psi^{2k}\ X\\
\tilde{m}\ (BX)\ &=\ B\left[X\ +\ \psi^{2k}X\right]
\end{aligned} \tag{15}$$

This relation shows that $\tilde{m}\ (BX)$ is tangential to $f\,(V)$ which contradicts the hypothesis. Thus $\tilde{m}\ (BX)= 0$. Using this result in (15) and remembering that B is an isomorphism, we get $\psi^{2k}\ =\ -I$. □

Theorem 3.2. *Let $\tilde{M}$ be never tangential to $f\,(V)$, then*

$$\underset{\tilde{m}}{\tilde{N}}\ (BX,\ BY)\ =\ 0. \tag{16}$$

Proof. We have

$$\underset{\tilde{m}}{\tilde{N}}\ (BX,\ BY)\ =\ [\tilde{m}\ BX,\ \tilde{m}BY]\ +\ \tilde{m}^2\,[BX,\ BY] -\ \tilde{m}\ [\tilde{m}BX,\ BY]\ -\ \tilde{m}\ [BX,\ \tilde{m}BY] \tag{17}$$

Using (2), (8) (c) and (12), we get (16). □

Theorem 3.3. *Let $\tilde{M}$ be never tangential to $f(V)$, then*

$$\underset{\tilde{l}}{\tilde{N}}\ (BX,\ BY)\ =\ 0. \tag{18}$$

Proof. We have

$$\underset{\tilde{l}}{\tilde{N}}\ (BX,\ BY)\ =\ \left[\tilde{l}\,BX,\ \tilde{l}\,BY\right]\ +\ \tilde{l}^2\ [BX,\ BY]\ -\ \tilde{l}\ \left[\tilde{l}\,BX,\ BY\right] -\ \tilde{l}\ \left[BX,\ \tilde{l}\,BY\right] \tag{19}$$

Using (2), (8) (a), (b) and (12) in (19); we get (18) □

Theorem 3.4. *Let $\tilde{M}$ be never tangential to $f(V)$. Define*

$$\tilde{H}\ \left(\tilde{X}, \tilde{Y}\right)\ =\ \tilde{N}\ \left(\tilde{X}, \tilde{Y}\right)\ -\ \tilde{N}\ \left(\tilde{m}\tilde{X}, \tilde{Y}\right)\ -\ \tilde{N}\ \left(\tilde{X},\ \tilde{m}\tilde{Y}\right) + \tilde{N}\ \left(\tilde{m}\tilde{X},\ \tilde{m}\tilde{Y}\right) \tag{20}$$

For all $\tilde{X},\ \tilde{Y}\ \in\ \zeta_0^1\ (M)$, then

$$\tilde{H}\ (BX, BY)\ =\ BN\ (X, Y). \tag{21}$$

Proof. Using $\tilde{X}\ =\ BX,\ \ \tilde{Y} = BY$ and (11), (12) in (20), we get (21). □

4. Distribution $\tilde{M}$ Always Being Tangential to $f(V)$

Theorem 4.1. *Let $\tilde{M}$ be always tangential to $f(V)$, then*

$$(a)\ \ \tilde{m}\ (BX)\ =\ Bm\ X;\ \ (b)\ \ \tilde{l}\ (BX)\ =\ Bl\ X. \tag{22}$$

Proof. From (15), We get (22) (a). Also

$$l\ =\ -\ \psi^{2k} \tag{23}$$

$$lX\ =\ -\psi^{2k}\ X$$

$$BlX\ =\ -B\ \psi^{2k}\ X \tag{24}$$

Using (10) in (24)

$$BlX\ =\ -\ \tilde{\psi}^{2k}\ BX\ =\ \tilde{l}\ (BX), \tag{25}$$

which is (22) (b). □

Theorem 4.2. *Let $\tilde{M}$ be always tangential to $f(V)$, then l and m satisfy*

$$(a)\ \ l + m = I;\ \ (b)\ \ lm = ml = 0;\ \ (c)\ \ l^2 = l;\ \ (d)\ \ m^2 = m. \tag{26}$$

Proof. Using (8) and (22) We get the results. □

Theorem 4.3. *If $\tilde{M}$ is always tangential to $f(V)$, then*

$$\psi^{4k}\ +\ \psi^{2k} = 0. \tag{27}$$

Proof. From (10)

$$\tilde{\psi}^{4k}\ BX\ =\ B\,\psi^{4k}\ X \tag{28}$$

Using (1) in (28)

$$-\tilde{\psi}^{2k}\ BX\ =\ B\,\psi^{4k}\ X$$
$$-B\psi^{2k}X\ =\ B\,\psi^{4k}\ X\ \text{ or}$$
$$\psi^{4k}+\ \psi^{2k}\ =0$$

which is (27). □

Theorem 4.4. *If $\tilde{M}$ Is always tangential to $f\,(V)$ then as in* (20)

$$\tilde{H}\ (BX,BY)\ =\ BH\ (X,Y) \tag{29}$$

Proof. From (20) we get

$$\tilde{H}\ (BX,BY)=\tilde{N}\ (BX,BY)-\tilde{N}\ (\tilde{m}BX,BY)-\tilde{N}\ (BX,\ \tilde{m}BY)\ +\ \tilde{N}\ (\tilde{m}BX,\ \tilde{m}BY) \tag{30}$$

Using (22) (a) and (11) in (30) we get (29). □

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