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Fixed Point Theorem for Contractions of Rational Type in Ordered 2-Metric Spaces

Research Article

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Abstract: In this paper, we prove some fixed point theorems for mappings involving rational expression in the framework of 2-metric spaces endowed with a partial order using a class of pairs of functions satisfying certain assumptions.

Keywords: Fixed point, 2-metric space, contractions, partially ordered set, altering distance function.

1. Introduction

Fixed point theory is one of the well-known traditional theories in mathematics that has a broad set of applications. In 1922, Polish mathematician Stephan Banach published his famous contraction principle. Since then, this principle has been extended and generalized in several ways either by using the contractive condition or by imposing some additional conditions on an ambient space. From inspiration of this work, several mathematicians heavily studied this field. For example, the work of Kannan [23], Chatterjea [7], Berinde [5], Ciric [8], Geraghty [17], Meir and Keeler [29], Suzuki [39] and so forth.

There were some generalizations of a metric such as a 2-metric, a D-metric, a G-metric, a cone metric, and a complex-valued metric. The notion of a 2-metric has been introduced by Gähler in [14]. Note that a 2-metric is not a continuous function of its variables, whereas an ordinary metric is. This led Dhage to introduce the notion of a D-metric in [10]. But in [31] Mustafa and Sims showed that most of topological properties of D-metric were not correct. In [30] Mustafa and Sims introduced the notion of a G-metric to overcome flaws of a D-metric. After that, many fixed point theorems on G-metric spaces have been stated. However, it was shown in [21] and [37] that in several situations fixed point results in G-metric spaces can be in fact deduced from fixed point theorems in metric or quasi-metric spaces.

In [18] Huang and Zhang defined the notion of a cone metric. After that, many authors extended some fixed point theorems on metric spaces to cone metric spaces. However, it was shown later by various authors that in several cases the fixed point results in cone metric spaces can be obtained by reducing them to their standard metric counterparts; for example, see [11, 12, 22, 26]. In [3] Azam, Fisher and Khan have introduced the notion of a complex-valued metric and some fixed point theorems have been stated. But in [38] Sastry, Naidu and Bekeshie showed that some fixed point theorems recently generalized to complex-valued metric spaces are consequences of their counterparts in the setting of metric spaces and hence are redundant.

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Note that in the above generalizations, only a 2-metric space has not been known to be topologically equivalent to an ordinary metric. Then there was no easy relationship between results obtained in 2-metric spaces and metric spaces. In particular, the fixed point theorems on 2-metric spaces and metric spaces may be unrelated easily. For the fixed point theorems on 2-metric spaces, the readers may refer to [2, 9, 13–15, 19, 20, 27, 28, 41].

The aim of this paper is to establish some fixed point theorems for mappings involving rational expression in the framework of 2-metric spaces endowed with a partial order using a class of pairs of functions satisfying certain assumptions.

2. Mathematical Preliminaries

First we recall some notions and lemmas which will be useful in what follows. The notion of a 2-metric space was studied by Gähler in [14].

Definition 2.1 ([14]). *Let X be a nonempty set. A real valued function d on X^3 is said to a 2-metric if, for all $x,\ y,\ z,\ a \in X$, the following conditions hold:*

(d1) To each pair of distinct points x, y in X, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$;

(d2) $d(x, y, z) = 0$ if at least two of x, y, z are equal;

(d3) $d(x,\ y,\ z) = d(x,\ z,\ y) = d(y,\ x,\ z) = d(y,\ z,\ x) = d(z,\ x,\ y) = d(z,\ y,\ x)$;

(d4) $d(x,\ y,\ z) \leq d(x,\ y,\ a) + d(x,\ a,\ z) + d(a,\ y,\ z)$.

Then $(X,\ d)$ is called a 2-metric space, which will be sometimes denoted by X if there is no confusion. Every member $x \in X$ is called a point in X. Geometrically a 2-metric $d(x,\ y,\ z)$ represents the area of a triangle with vertices x, y and z.

Definition 2.2 ([14]). *Let $(X,\ d)$ be a 2-metric space and $a,\ b \in X$, $r = 0$. The set $B(a,\ b,\ r) = \{x \in X\ : d(a,\ b,\ x) < r\}$ is called a 2-ball centered at a and b with radius r. The topology generated by the collection of all 2-balls as a sub-basis is called a 2-metric topology on X.*

Definition 2.3 ([19]). *A sequence $\{x_n\}$ in a 2-metric space $(X,\ d)$ is said to be convergent to a point $x \in X$, if $\lim\limits_{n\to\infty} d(x_n,\ x,\ u) = 0$ for all $u \in X$.*

Definition 2.4 ([19]). *A sequence $\{x_n\}$ in a 2-metric space $(X,\ d)$ is said to be Cauchy sequence if for all $z \in X$, $\lim\limits_{n\to\infty} d(x_n,\ x_m,\ z) = 0$.*

Definition 2.5 ([19]). *A sequence $\{x_n\}$ in a 2-metric space $(X,\ d)$ is said to be complete if every Cauchy sequence in X is convergent.*

Definition 2.6 ([27]). *A 2-metric space $(X,\ d)$ is said to be compact if every sequence in X has a convergent subsequence.*

Lemma 2.7 ([27]). *Every 2-metric space is a T_1-space.*

Lemma 2.8 ([27]). *$\lim\limits_{n\to\infty} x_n = x$ in a 2-metric space $(X,\ d)$ if and only if $\lim\limits_{n\to\infty} x_n = x$ in the 2-metric topological space X.*

Lemma 2.9 ([27]). *If $f : X \to Y$ is a continuous map from a 2-metric space X to a 2-metric space Y, then $\lim\limits_{n\to\infty} x_n = x$ in X implies $\lim\limits_{n\to\infty} Tx_n = Tx$ in Y.*

Remark 2.10.

(1). It is straightforward from Definition 2.1 that every 2-metric is non-negative and every 2-metric space contains at least three distinct points.

(2). A 2-metric $d(x,\ y,\ z)$ is sequentially continuous in one argument. Moreover, if a 2-metric $d(x,\ y,\ z)$ is sequentially continuous in two arguments, then it is sequentially continuous in all three arguments [32].

(3). A convergent sequence in a 2-metric space need not be a Cauchy sequence [32].

(4). In a 2-metric space $(X,\ d)$, every convergent sequence is a Cauchy sequence if d is continuous [32].

(5). There exists a 2-metric space $(X,\ d)$ such that every convergent sequence is a Cauchy sequence but d is not continuous [32].

Khan et al. [25] initiated the use of control function that alter distance between two points in a metric space, which they called an altering distance function

Definition 2.11 ([25]). *A function $\varphi : [0,\ \infty) \to [0,\ \infty)$ is called altering distance function if the following conditions are satisfied:*

(a1). φ is monotone increasing and continuous,

(a2). $\varphi(t) = 0$ if and only if $t = 0$.

In [4], Bergiz et al. introduced the notion of pair of generalized altering distance functions as follows.

Definition 2.12 ([4]). *The pair $(\varphi,\ \phi)$, where $\varphi,\ \phi : [0, \infty) \to [0, \infty)$ is called a pair of generalized altering distance functions if the following conditions are satisfied:*

(b1). φ is continuous;

(b2). φ is non-decreasing;

(b3). $\lim_{n\to\infty} \phi(t_n) = 0 \Rightarrow \lim_{n\to\infty} t_n = 0$.

The condition (b3) was introduced by Moradi and Farajzadeh in [32]. Notice that the above conditions do not determine the values of $\varphi(0)$ and $\phi(0)$.

In the recent work, Agarwal et al. [1] introduced the following family of function.

Definition 2.13 ([1]). *We will denote by $\mathcal{F}$ the family of all pairs $(\varphi,\ \phi)$, where $\varphi,\ \phi : [0,\ \infty) \to [0,\ \infty)$ are functions satisfying the following three conditions.*

(F1). φ is non-decreasing;

(F2). if there exists $t_0 \in [0,\ \infty)$ such that $\phi\,(t_0) = 0$, then $t_0 = 0$ and $\varphi^{-1}\,(0) = \{0\}$.

(F3). if $\{a_k\},\ \{b_k\} \subset [0, \infty)$ are sequences such that $\{a_k\} \to L,\ \{b_k\} \to L$ and verifying $L < \{b_k\}$ and $\varphi\,(b_k) \leq (\varphi - \phi)\,(a_k)$ for all k, then $L = 0$.

In this paper, we consider the following class of pairs of functions ??.

Definition 2.14 ([34]). *A pair of functions $(\varphi,\ \phi)$ is said to belong to the class $\mathfrak{F}$, if they satisfy the following conditions:*

(c1). $\varphi,\ \phi : [0,\ \infty) \to [0,\ \infty)$;

(c2). for $t,\ s\ \in\ [0,\ \infty)$, $\varphi(t)=\phi(s)$ *then* $t=s$;

(c3). for $\{t_n\}$ *and* $\{s_n\}$ *sequence in* $[0,\ \infty)$ *such that* $\lim\limits_{n\to\infty} t_n\ =\lim\limits_{n\to\infty} s_n\ =a$, *if* $\varphi(t_n)=\phi(s_n)$ *for any* $n\in\mathbb{N}$, *then* $a=0$.

Notice that, if a pair $(\varphi,\ \phi)$ verifies (F1) and (F2), then the pair $(\varphi,\ \phi=\varphi-\phi)$ satisfies (c1) and (c2). Furthermore, if $(\varphi,\ \phi=\varphi-\phi)$ satisfies (c3), then $(\varphi,\ \phi)$ satisfies (F3).

Remark 2.15 ([34]). *Note that, if* $(\varphi,\ \phi)\in\mathfrak{F}$ *and* $\varphi(t)=\phi(t)$, *then* $t=0$, *since we can take* $t_n=s_n=\ t$ *for any* $n\in\mathbb{N}$ *and by (c3) we deduce* $t=0$.

Example 2.16.

(a). The conditions (c1)-(c3) of the above definition are fulfilled for the functions $\varphi,\ \phi\ :\ [0,\ \infty)\ \to\ [0,\ \infty)$ *defined by* $\varphi\left(t\right)=\ln\left(\frac{5t+1}{12}\right)$ *and* $\phi\left(t\right)=\ln\left(\frac{3t+1}{12}\right)$ *for all* $t\in[0,\ \infty)$.

(b). The conditions (c1)-(c3) of the above definition are fulfilled for the functions $\varphi,\ \phi\ :\ [0,\ \infty)\ \to\ [0,\ \infty)$ *defined by* $\varphi\left(t\right)=\ln\left(t+\frac{1}{2}\right)$ *and* $\phi\left(t\right)=\ln\left(\frac{t}{2}+\frac{1}{2}\right)$ *for all* $t\in[0,\ \infty)$.

In the sequel, we present some interesting examples of pairs of functions belonging to the class $\mathfrak{F}$ which will be very important in our study.

Example 2.17 ([34]). *Let* $\varphi\ :[0,\infty)\to[0,\infty)$ *be a continuous and increasing function such that* $\varphi(t)=0$ *if and only if* $t=0$ *(these functions are known in the literature as altering distance functions).*

Let $\phi\ :[0,\infty)\to[0,\infty)$ be a non-decreasing function such that $\phi(t)=0$ if and only if $t=0$ and suppose that $\phi=\varphi$. Then the pair $(\varphi,\varphi-\phi)\in\mathfrak{F}$. In fact, it is clear that $(\varphi,\varphi-\phi)$ satisfy (c1). To prove (c2), suppose that $t,\ s\ \in\ [0,\infty)$ and $\varphi(t)=(\varphi-\phi)(s)$. Then, from

$$\varphi(t)=\varphi(s)-\phi(s)=\varphi(s) \tag{1}$$

and taking into account the increasing character of φ, we can deduce that $t=s$. In order to prove (c3), we suppose that

$$\varphi(t_n)=\varphi(s_n)-\phi(s_n)=\varphi(s_n) \tag{2}$$

for any $n\in\mathbb{N}$, where $t_n,\ s_n\in[0,\infty)$ and $\lim\limits_{n\to\infty} t_n=\lim\limits_{n\to\infty} s_n=a$. Taking $n\to\infty$ in (2), we infer that $\lim\limits_{n\to\infty}\phi(s_n)=0$. Let us suppose that $a>0$. Since $\lim\limits_{n\to\infty} s_n=a>0$, we can find $\epsilon>0$ and a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $s_{n_k}>\epsilon$ for any $k\in\mathbb{N}$. As ϕ is non-decreasing, we have $\phi(s_{n_k})>\phi(\epsilon)$ for any $k\in\mathbb{N}$ and, consequently, $\lim\limits_{n\to\infty}\phi(s_{n_k})=\phi(\epsilon)$. This contradicts the fact that $\lim\limits_{n\to\infty}\phi(s_{n_k})=0$. Therefore, $a=0$. This proves that $(\varphi,\varphi-\phi)\in\mathfrak{F}$. An interesting particular case is when φ is the identity mapping, $\varphi=1_{[0,\infty)}$ and $\phi:[0,\infty)\to[0,\infty)$ is a non-decreasing function such that $\phi(t)=0$ if and only if $t=0$ and $\phi(t)=t$ for any $t\in[0,\infty)$.

Example 2.18 ([3]). *Let* S *be the class of functions defined by*

$$S=\{\alpha:\ [0,\infty)\to[0,\ 1)\ :\{\alpha(t_n)\to1\Rightarrow\ t_n\to0\}\}.$$

Let us consider the pairs of functions $(1_{[0,\infty)},\ \alpha1_{[0,\infty)})$, *where* $\alpha\in S$ *and* $\alpha1_{[0,\infty)}$ *is defined by* $\left(\alpha1_{[0,\infty)}\right)(t)=\alpha\left(t\right)t$, *for* $t\in[0,\infty)$. *Then* $(1_{[0,\infty)},\ \alpha1_{[0,\infty)})\in\mathfrak{F}$.

It is clear that the pairs $(1_{[0,\infty)},\ \alpha 1_{[0,\infty)})$, with $\alpha \in S$ satisfy (c1). To prove (c2), from $1_{[0,\infty)}(t) \leq\ \alpha 1_{[0,\infty)}(s)$ for $t,\ s \in [0,\infty)$, we infer, since $\alpha : [0,\infty) \to [0,\ 1)$, that $t \leq \alpha(s)s < s$ and, consequently, $(1_{[0,\infty)},\ \alpha 1_{[0,\infty)})$ satisfies (c2).

In order to prove (c3), we suppose that $1_{[0,\infty)}(t_n) = t_n \leq 1_{[0,\infty)}(s_n) = \alpha(s_n)s_n$ for any $n \in \mathbb{N}$, where $t_n, s_n \in [0,\infty)$ and $\lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} s_n = a$. Let us suppose that $a > 0$. Since $\lim\limits_{n\to\infty} s_n = a > 0$, we can find a subsequence $\{s_{n_k}\}$ such that $s_{n_k} > 0$ for any $k \in \mathbb{N}$. Now as

$$t_n \leq \alpha\left(s_n\right) s_n \leq s_n \quad \text{for any} \quad n \in \mathbb{N}, \tag{3}$$

In particular, we have

$$t_{n_k} \leq \alpha\left(s_{n_k}\right) s_{n_k} \leq s_{n_k} \quad \text{for any} \quad k \in \mathbb{N}, \tag{4}$$

and since $s_{n_k} > 0$ for any $k \in \mathbb{N}$,

$$\frac{t_{n_k}}{s_{n_k}} \leq \alpha\left(s_{n_k}\right) \leq 1 \tag{5}$$

Taking $k \to \infty$ in the last inequality, we obtain

$$\lim_{k\to\infty} \alpha\left(s_{n_k}\right) = 1 \tag{6}$$

Finally, since $\alpha \in S$, we infer that $\lim\limits_{k\to\infty} s_{n_k} = 0$ and this contradicts the fact that $\lim\limits_{n\to\infty} s_n = a > 0$. Therefore, $a = 0$. This proves that $(1_{[0,\infty)},\ \alpha 1_{[0,\infty)}) \in \mathfrak{F}$ for $\alpha \in S$.

Remark 2.19 ([34])**.** *Suppose that $g\ : [0,\infty) \to [0,\infty)$ is an increasing function and $(\varphi,\phi) \in \mathfrak{F}$. Then it is easily seen that the pair $(g \circ \varphi, g \circ \phi) \in \mathfrak{F}$.*

Definition 2.20 ([6])**.** *Let $(X,\ \leq)$ is a partially ordered set and $f\ : X \to X$ is said to be monotone non-decreasing if for all $x,\ y\ \in X$,*

$$x \leq y \ \Rightarrow\ fx \leq fy. \tag{7}$$

3. Main Results

Now, we give our main results.

Theorem 3.1. *Let $(X,\ \preceq)$ is a partially ordered set. Suppose that there exist a 2-metric d on X such that be a complete 2-metric space. Let $f : X \to X$ be a non-decreasing mapping such that there exists a pair of functions $(\varphi,\phi) \in \mathfrak{F}$ satisfying*

$$\varphi\left(d\left(fx,\ fy,a\right)\right) \leq \max\left\{\phi\left(d\left(x,\ y,a\right)\right), \phi\left(\frac{d\left(y,\ fy,a\right)\left[1+d\left(x,\ fx,a\right)\right]}{1+d\left(fx,\ fy,a\right)}\right)\right\}, \tag{8}$$

for all $a \in X$ and for all comparable elements $x,\ y \in X$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$, for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof. If $x_0 = fx_0$, then the proof is finished. Suppose now that that $x_0 \prec fx_0$. Since f is non-decreasing function, we have

$$x_0 \prec fx_0 \preceq f^2x_0 \preceq f^3x_0 \preceq \cdots \preceq f^{n-1}x_0 \preceq f^nx_0 \preceq \ldots \tag{9}$$

Put

$$x_{n+1} = fx_n,\ \forall\, n \geq 0. \tag{10}$$

For simplicity, we set $d_n\left(a\right) = d\left(x_n, x_{n+1}, a\right)$ for all $a \in X$.

Step I: We will show that $\lim_{n\to\infty} d_n(a) = 0$. If there exists $n \geq 1$ such that $x_n = x_{n+1}$, then from (9), $fx_n = x_{n+1} = x_n$, that is x_n is a fixed point of f and the proof is finished. Now suppose that $x_n \neq x_{n+1}$, that is $d(x_n, x_{n+1}, a) \neq 0$, for all $n \geq 1$. Since $x_{n-1} \prec x_n$ for all $n \geq 1$, applying the contractive condition (8) and using (10), we have

$$\begin{aligned}\varphi(d_n(a)) &= \varphi(d(fx_{n-1},\ fx_n, a)) \\ &\leq \max\left\{\phi(d(x_{n-1},\ x_n, a)), \phi\left(\frac{d(x_n,\ fx_n, a)[1 + d(x_{n-1},\ fx_{n-1}, a)]}{1 + d(fx_{n-1},\ fx_n, a)}\right)\right\} \\ &= \max\left\{\phi(d_{n-1}(a)), \phi\left(\frac{d_n(a)[1 + d_{n-1}(a)]}{1 + d_n(a)}\right)\right\}\end{aligned} \tag{11}$$

Now, we can distinguish two cases.

Case I. Consider

$$\max\left\{\phi(d_{n-1}(a)), \phi\left(\frac{d_n(a)[1 + d_{n-1}(a)]}{1 + d_n(a)}\right)\right\} = \phi(d_{n-1}(a)) \tag{12}$$

In this case from (11), we have

$$\varphi(d_n(a)) \leq \phi(d_{n-1}(a)) \tag{13}$$

Since $(\varphi, \phi) \in \mathfrak{F}$, we deduce that $d_n(a) \leq d_{n-1}(a)$.

Case II. If

$$\max\left\{\phi(d_{n-1}(a)), \phi\left(\frac{d_n(a)[1 + d_{n-1}(a)]}{1 + d_n(a)}\right)\right\} = \phi\left(\frac{d_n(a)[1 + d_{n-1}(a)]}{1 + d_n(a)}\right) \tag{14}$$

In this case from (11), we have

$$\varphi(d_n(a)) \leq \phi\left(\frac{d_n(a)[1 + d_{n-1}(a)]}{1 + d_n(a)}\right) \tag{15}$$

Since $(\varphi, \phi) \in \mathfrak{F}$ we get

$$d_n(a) \leq \frac{d_n(a)[1 + d_{n-1}(a)]}{1 + d_n(a)}$$

Since $d_n(a) \neq 0$, from the last inequality it follows that

$$d_n(a) \leq d_{n-1}(a)$$

In both cases, we conclude that the sequence $\{d_n(a)\}$ is a decreasing sequence of non-negative real numbers and is bounded below, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d_n(a) = r. \tag{16}$$

Now, we shall show that $r = 0$. Denote

$$A = \{n \in \mathbb{N}\ :\ n \text{ satisfies } (12)\}, \text{ and } B = \{n \in \mathbb{N}\ :\ n \text{ satisfies } (14)\}. \tag{17}$$

We note that the following.

1. If $Card\ A = \infty$, then from (11), we can find infinitely natural numbers n satisfying inequality (13) and since $\lim_{n\to\infty} d_n(a) = \lim_{n\to\infty} d_{n-1}(a) = r$ and $(\varphi, \phi) \in \mathfrak{F}$ we have $r = 0$.

2. If $Card\ B = \infty$, then from (11), we can find infinitely many $n \in \mathbb{N}$ satisfying inequality (15). Since $(\varphi, \phi) \in \mathfrak{F}$ and using the similar argument to the one used in Case II, we obtain

$$d_n(a) \leq \frac{d_n(a)[1 + d_{n-1}(a)]}{1 + d_n(a)}$$

for infinitely many $n \in \mathbb{N}$. Letting the limit as $n \to \infty$ and taking into account that $\lim_{n\to\infty} d_n(a) = \lim_{n\to\infty} d_{n-1}(a) = r$, we deduce that $r \leq r(1 + r)/(1 + r)$ and consequently, we obtain $r = 0$.

Therefore, in both cases we have

$$\lim_{n\to\infty} d_n(a) = r = 0. \tag{18}$$

Step II: We will show that $d_n(x_m) = 0$, $\forall\, n, m \in \mathbb{N}$. As $\{d_n(a)\}$ is decreasing, if $d_{n-1}(a) = 0$, then $d_n(a) = 0$. Since from condition (d1) of Definition 2.1, $d_0(x_0) = 0$, we have $d_n(x_0) = 0$ for all $n \in \mathbb{N}$. Since $d_{m-1}(x_m) = 0$, we have

$$d_n(x_m) = 0 \tag{19}$$

for all $n \geq m-1$. For $0 \leq n < m-1$, we have $m-1 \geq n+1$ and from (18), we have

$$d_{m-1}(x_{n+1}) = d_{m-1}(x_n) = 0 \tag{20}$$

It implies that

$$\begin{aligned} d_n(x_m) &= d(x_n, x_{n+1}, x_m) \\ &\leq d(x_n, x_{n+1}, x_{m-1}) + d(x_n, x_{m-1}, x_m) + d(x_{m-1}, x_{n+1}, x_m) \\ &= d_n(x_{m-1}) + d_{m-1}(x_n) + d_{m-1}(x_{n+1}) \\ &= d_n(x_{m-1}) \end{aligned}$$

Since $d_n(x_{n+1}) = 0$, from the above inequality, we have

$$d_n(x_m) \leq d_n(x_{m-1}) \leq d_n(x_{m-2}) \leq \cdots \leq d_n(x_{n+1}) = 0 \tag{21}$$

for all $0 \leq n < m-1$. From (19) and (21), we have

$$d_n(x_m) = 0 \tag{22}$$

for all $n, m \in \mathbb{N}$.

Step III: Next we shall prove that $d(x_i, x_j, x_k) = 0$ for all $i, j, k \in \mathbb{N}$. Without loss of generality, we may assume that $i = j$, it follows that

$$\begin{aligned} d(x_i, x_j, x_k) &= d(x_i, x_j, x_{i+1}) + d(x_i, x_{i+1}, x_k) + d(x_{i+1}, x_j, x_k) \\ &= d_i(x_j) + d_i(x_k) + d(x_{i+1}, x_j, x_k) \\ &= d(x_{i+1}, x_j, x_k) \end{aligned}$$

Similarly,

$$d(x_{i+1}, x_j, x_k) = d(x_{i+2}, x_j, x_k)$$

Inductively, we have

$$d(x_i, x_j, x_k) = d(x_{j-1}, x_j, x_k) = d_j(x_k) = 0$$

This proves that for all $i, j, k \in \mathbb{N}$,

$$d(x_i, x_j, x_k) = 0. \tag{23}$$

Step IV: Now, we will show that $\{x_n\}$ is a Cauchy sequence. Suppose the contrary. Then there exist $a \in X$ and $\epsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \ \text{ and } \ d\left(x_{m_i}, x_{n_i}, a\right) \geq \epsilon \tag{24}$$

This means that

$$d\left(x_{m_i}, x_{n_i-1}, a\right) < \epsilon \tag{25}$$

Using (**??**), (25) and the rectangle inequality (that holds for a 2-metric space), we have

$$\begin{aligned}
\epsilon &\leq d\left(x_{m_i}, x_{n_i}, a\right) \\
&\leq d\left(x_{m_i}, x_{n_i}, x_{n_i-1}\right) + d\left(x_{m_i}, x_{n_i-1}, a\right) + d\left(x_{n_i-1}, x_{n_i}, a\right) \\
&\leq \epsilon + d_{n_i-1}\left(a\right) + d_{n_i-1}\left(x_{m_i}\right)
\end{aligned} \tag{26}$$

On letting $i \to \infty$ in (26) and using (18), (22) we get

$$\lim_{i\to\infty} d\left(x_{m_i}, x_{n_i}, a\right) = \epsilon \tag{27}$$

It follows from (27) that

$$\begin{aligned}
0 &< d\left(x_{m_i}, x_{n_i}, a\right) - d\left(x_{m_i}, x_{n_i-1}, a\right) \\
&\leq d\left(x_{m_i}, x_{n_i}, x_{n_i-1}\right) + d\left(x_{m_i}, x_{n_i-1}, a\right) + d\left(x_{n_i-1}, x_{n_i}, a\right) - d\left(x_{m_i}, x_{n_i-1}, a\right) \\
&= d\left(x_{m_i}, x_{n_i}, x_{n_i-1}\right) + d\left(x_{n_i-1}, x_{n_i}, a\right)
\end{aligned} \tag{28}$$

On making $i \to \infty$, we immediately obtain that:

$$\lim_{i\to\infty} d\left(x_{m_i}, x_{n_i-1}, a\right) = \epsilon \tag{29}$$

Note that

$$\left|d\left(x_{m_i-1},\ x_{n_i-1}, a\right) - d\left(x_{m_i},\ x_{n_i}, a\right)\right| \leq d_{x_{m_i-1}}\left(\ x_{n_i-1}\right) + d_{x_{m_i-1}}\left(a\right) + d_{n_i-1}\left(x_{m_i}\right) + d_{n_i-1}\left(a\right)$$

On letting $i \to \infty$, in these inequalities and by using inequalities (**??**), (**??**), (**??**), and (**??**), we obtain;

$$\lim_{i\to\infty} d\left(x_{m_i-1},\ x_{n_i-1}, a\right) = \epsilon \tag{30}$$

Now using contractive condition (8), we get

$$\begin{aligned}
\varphi\left(d\left(x_{m_i}, x_{n_i}, a\right)\right) &= \varphi\left(d\left(fx_{m_i-1}, fx_{n_i-1}, a\right)\right) \\
&\leq \max\left\{\phi\left(d\left(x_{m_i-1},\ x_{n_i-1}, a\right)\right), \phi\left(\frac{d\left(x_{n_i-1},\ fx_{n_i-1}, a\right)\left[1 + d\left(x_{m_i-1},\ fx_{m_i-1}, a\right)\right]}{1 + d\left(fx_{m_i-1},\ fx_{n_i-1}, a\right)}\right)\right\} \\
&\leq \max\left\{\phi\left(d\left(x_{m_i-1},\ x_{n_i-1}, a\right)\right), \phi\left(\frac{d\left(x_{n_i-1}, x_{n_i}, a\right)\left[1 + d\left(x_{m_i-1},\ x_{m_i}, a\right)\right]}{1 + d\left(x_{m_i},\ x_{n_i}, a\right)}\right)\right\}
\end{aligned} \tag{31}$$

Let us put

$$C = \{i \in \mathbb{N} : \varphi\left(d\left(x_{m_i}, x_{n_i}, a\right)\right) \leq \phi\left(d\left(x_{m_i-1},\ x_{n_i-1}, a\right)\right)\}, \tag{32}$$

$$D = \left\{i \in \mathbb{N} : \varphi\left(d\left(x_{m_i}, x_{n_i}, a\right)\right) \leq \phi\left(\frac{d\left(x_{n_i-1}, x_{n_i}, a\right)\left[1 + d\left(x_{m_i-1},\ x_{m_i}, a\right)\right]}{1 + d\left(x_{m_i},\ x_{n_i}, a\right)}\right)\right\} \tag{33}$$

By (31), we have $Card\ C = \infty$ or $Card\ D = \infty$. Let us suppose that $Card\ C = \infty$. Then there exists infinitely many $i \in \mathbb{N}$ satisfying

$$\varphi\left(d\left(x_{m_i}, x_{n_i}, a\right)\right) \leq \phi\left(d\left(x_{m_i-1},\ x_{n_i-1}, a\right)\right)$$

and since $(\varphi, \phi) \in \mathfrak{F}$, by letting the limit as $i \to \infty$, we have

$$\lim_{i \to \infty} d\left(x_{m_i}, x_{n_i}, a\right) \leq \lim_{i \to \infty} d\left(x_{m_i-1},\ x_{n_i-1}, a\right)$$

We infer from (27) and (30) that $\epsilon = 0$. This is a contradiction. On the other hand, if $Card\ D = \infty$, then we can find infinitely many $i \in \mathbb{N}$ satisfying

$$\varphi\left(d\left(x_{m_i}, x_{n_i}, a\right)\right) \leq \phi\left(\frac{d\left(x_{n_i-1}, x_{n_i}, a\right)\left[1 + d\left(x_{m_i-1},\ x_{m_i}, a\right)\right]}{1 + d\left(x_{m_i},\ x_{n_i}, a\right)}\right)$$

and since $(\varphi, \phi) \in \mathfrak{F}$ we obtain

$$d\left(x_{m_i}, x_{n_i}, a\right) \leq \frac{d\left(x_{n_i-1}, x_{n_i}, a\right)\left[1 + d\left(x_{m_i-1},\ x_{m_i}, a\right)\right]}{1 + d\left(x_{m_i},\ x_{n_i}, a\right)}$$

Taking $i \to \infty$ and using (18) and (27) we obtain $\epsilon \leq 0$, which is a contradiction. Therefore, in both the cases, we obtain a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that $\lim_{n \to \infty} x_n = u$.

Step V: Next, we will show that u is a fixed point of f. Since $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \leq u$. By the contractive condition (8), we obtain for any $n \in \mathbb{N}$,

$$\varphi\left(d\left(fu,\ fx_n, a\right)\right) \leq max\left\{\phi\left(d\left(u,\ x_n, a\right)\right), \phi\left(\frac{d\left(x_n,\ fx_n, a\right)\left[1 + d\left(u,\ fu, a\right)\right]}{1 + d\left(fu,\ fx_n, a\right)}\right)\right\} \tag{34}$$

Put

$$E = \{n \in \mathbb{N} : \varphi\left(d\left(fu,\ fx_n, a\right)\right) \leq \phi\left(d\left(u,\ x_n, a\right)\right)\} \tag{35}$$

$$F = \left\{n \in \mathbb{N} : \varphi\left(d\left(fu,\ fx_n, a\right)\right) \leq \phi\left(\frac{d\left(x_n,\ fx_n, a\right)\left[1 + d\left(u,\ fu, a\right)\right]}{1 + d\left(fu,\ fx_n, a\right)}\right)\right\} \tag{36}$$

Now we can distinguish two cases again.

1. If $Card\ E = \infty$, then from (34), we can find infinitely natural numbers n satisfying inequality $\varphi\left(d\left(fu,\ fx_n, a\right)\right) \leq \phi\left(d\left(u,\ x_n, a\right)\right)$ and since $\lim_{n \to \infty} x_n = u$ and $(\varphi, \phi) \in \mathfrak{F}$ we obtain $\lim_{n \to \infty} d\left(fu,\ fx_n, a\right) = 0$. Thus $\lim_{n \to \infty} fx_n = fu$, where, to simplify our assumptions, we will denote the subsequence by the same symbol fx_n. By (10), we have $\lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} fx_n = fu$. The uniqueness of the limit, we have $fu = u$.

2. If $Card\ F = \infty$, then from (34), we can find infinitely many $n \in \mathbb{N}$ satisfying inequality

$$\varphi\left(d\left(fu,\ fx_n, a\right)\right) \leq \phi\left(\frac{d\left(x_n,\ fx_n, a\right)\left[1 + d\left(u,\ fu, a\right)\right]}{1 + d\left(fu,\ fx_n, a\right)}\right).$$

Again to simplify our considerations, we will denote the subsequence by the same symbol fx_n. Since $(\varphi, \phi) \in \mathfrak{F}$ we deduce that

$$d\left(fu,\ x_{n+1}, a\right) \leq \frac{d\left(x_n,\ x_{n+1}, a\right)\left[1 + d\left(u,\ fu, a\right)\right]}{1 + d\left(fu,\ x_{n+1}, a\right)}$$

for infinitely many $n \in \mathbb{N}$. Letting $n \to \infty$ and taking into account that $\lim\limits_{n\to\infty} d\left(x_n,\ x_{n+1}, a\right) = 0$, we deduce that $\lim\limits_{n\to\infty} d\left(fu,\ x_{n+1}, a\right) = 0$ and consequently, we obtain $\lim\limits_{n\to\infty} x_{n+1} = fu$. The uniqueness of the limit, $fu = u$.

From the above both case, we deduce that u is a fixed point of f. This completes the proof of the theorem. $\square$

In what follows, we prove a sufficient condition for the uniqueness of the fixed point in Theorem 3.1.

Theorem 3.2. *Suppose that:*

(1). Hypothesis of Theorem 3.1 hold.

(2). For each $x, y \in X$, there exists $z \in X$ that is comparable to x and y.

Then f has a unique fixed point.

Proof. As in the proof of Theorem 3.1, we see that f has a fixed point. Now we prove that the uniqueness of the fixe point of f. Let u and v be two fixed points of f. We consider the following two cases:

Case.1 u is comparable to v. Then $f^n u$ is comparable to $f^n v$ for all $n \in \mathbb{N}$. For all $a \in X$, applying (8), we have

$$\varphi\left(d\left(f^n u,\ f^n v,\ a\right)\right) \leq \max\left\{\phi\left(d\left(f^{n-1}u,\ f^{n-1}v,\ a\right)\right), \phi\left(\frac{d\left(f^{n-1}v,\ f^n v,\ a\right)\left[1 + d\left(f^{n-1}u,\ f^n u,\ a\right)\right]}{1 + d\left(f^n u,\ f^n v,\ a\right)}\right)\right\} \tag{37}$$

Put

$$G = \left\{n \in \mathbb{N} : \varphi\left(d\left(f^n u,\ f^n v,\ a\right)\right) \leq \phi\left(d\left(f^{n-1}u,\ f^{n-1}v,\ a\right)\right)\right\} \tag{38}$$

$$H = \left\{n \in \mathbb{N} : \varphi\left(d\left(f^n u,\ f^n v,\ a\right)\right) \leq \phi\left(\frac{d\left(f^{n-1}v,\ f^n v,\ a\right)\left[1 + d\left(f^{n-1}u,\ f^n u, a\right)\right]}{1 + d\left(f^n u,\ f^n v,\ a\right)}\right)\right\} \tag{39}$$

Now we remark the following:

1. If $Card\ G = \infty$, then from (37), we can find infinitely natural numbers n satisfying inequality $\varphi\left(d\left(f^n u,\ f^n v,\ a\right)\right) \leq \phi\left(d\left(f^{n-1}u,\ f^{n-1}v,\ a\right)\right)$. Since $(\varphi, \phi) \in \mathfrak{F}$, it follows that $d\left(u,\ v,\ a\right) \leq\ d\left(u,\ v,\ a\right)$ and so $u = v$.

2. If $Card\ H = \infty$, then from (37), we can find infinitely natural numbers n satisfying inequality

$$\varphi\left(d\left(f^n u,\ f^n v,\ a\right)\right) \leq \phi\left(\frac{d\left(f^{n-1}v,\ f^n v,\ a\right)\left[1 + d\left(f^{n-1}u,\ f^n u,\ a\right)\right]}{1 + d\left(f^n u,\ f^n v,\ a\right)}\right).$$

Then since $(\varphi, \phi) \in \mathfrak{F}$, we have $d\left(u,\ v,\ a\right) \leq 0$ and so $u = v$.

Therefore, in both cases we proved that $u = v$.

Case.2 u is not comparable to v. Then there exists $z \in X$ that is comparable to u and v. Now, we can define the sequence $\{z_n\}$ in X as follows: $z_0 = z, \quad fz_n = z_{n+1}, \quad \forall\, n \in \mathbb{N}$. Since f is non-decreasing we have,

$$z_0 \preceq z_n \preceq z_{n+1} \text{ and } \lim_{n\to\infty} d(z_n,\ z_{n+1},\ a) = 0. \tag{40}$$

As $u \preceq z_n$, putting $x = u$ and $y = z_n$ in the contractive condition (8), we get

$$\begin{aligned}\varphi(d(u,\ z_{n+1},\ a)) &= \varphi(d(fu,\ fz_n,\ a)) \\ &\leq \max\left\{\phi(d(u,\ z_n,\ a)), \phi\left(\frac{d(z_n,\ fz_n,\ a)[1 + d(u,\ fu,\ a)]}{1 + d(fu,\ fz_n,\ a)}\right)\right\} \\ &= max\left\{\phi(d(u,\ z_n,\ a)), \phi\left(\frac{d(z_n,\ z_{n+1},\ a)}{1 + d(u,\ z_{n+1},\ a)}\right)\right\}\end{aligned} \tag{41}$$

Let us denote

$$I = \{n \in \mathbb{N} : \varphi(d(u,\ z_{n+1},\ a)) \leq \phi(d(u,\ z_n,\ a))\} \tag{42}$$

$$J = \left\{n \in \mathbb{N} : \varphi(d(u,\ z_{n+1},\ a)) \leq \phi\left(\frac{d(z_n,\ z_{n+1},\ a)}{1 + d(u,\ z_{n+1},\ a)}\right)\right\} \tag{43}$$

Now we remark following again.

1. If $Card\ I = \infty$, then from (41), we can find infinitely natural numbers n satisfying inequality $\varphi(d(u,\ z_{n+1},\ a)) \leq \phi(d(u,\ z_n,\ a))$. Since $(\varphi, \phi) \in \mathfrak{F}$, it follows that the sequence $\{d(u,\ z_{n+1},\ a)\}$ is non-increasing and it has a limit $l \geq 0$. Since $\lim_{n\to\infty} d(u,\ z_{n+1},\ a) = \lim_{n\to\infty} d(u,\ z_n,\ a) = l$ and $(\varphi,\ \phi) \in \mathfrak{F}$, we obtain $l = 0$. Hence $\lim_{n\to\infty} d(u,\ z_{n+1},\ a) = 0$.
2. If $Card\ J = \infty$, then from (41), we can find infinitely natural numbers n satisfying inequality

$$\varphi(d(u,\ z_{n+1},\ a)) \leq \phi\left(\frac{d(z_n,\ z_{n+1},\ a)}{1 + d(u,\ z_{n+1},\ a)}\right).$$

   Then since $(\phi,\ \varphi) \in \mathfrak{F}$, we have

$$d(u,\ z_{n+1}, a) \leq \frac{d(z_n,\ z_{n+1},\ a)}{1 + d(u,\ z_{n+1},\ a)}$$

   Taking $n \to \infty$ and using (40), we have $\lim_{n\to\infty} d(u,\ z_{n+1},\ a) = 0$.

Therefore, in both cases we proved that $\lim_{n\to\infty} d(u,\ z_{n+1},\ a) = 0$, that is, $\lim_{n\to\infty} z_{n+1} = u$. In the same way it can be deduced that $\lim_{n\to\infty} z_{n+1} = v$. By Lemma 2.7, we get $u = v$. That is, the fixed point is unique. □

By Theorem 3.1, we obtain the following corollaries.

Corollary 3.3. *Let $(X,\ \preceq)$ is a partially ordered set. Suppose that there exist a 2-metric d on X such that be a complete 2-metric space. Let $f : X \to X$ be a non-decreasing mapping satisfying*

$$d(fx,\ fy,\ a) \leq \alpha d(x,\ y,\ a) + \beta\frac{d(y,\ fy,\ a)[1 + d(x,\ fx,\ a)]}{1 + d(fx,\ fy,\ a)}, \tag{44}$$

for all $a \in X$ and for all comparable elements $x,\ y \in X$, where $\alpha,\ \beta > 0$ and $\alpha + \beta < 1$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$, for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof. Since

$$\begin{aligned} d(fx,\ fy,\ a) &\leq \alpha d(x,\ y,\ a) + \beta \frac{d(y,\ fy,\ a)\left[1 + d(x,\ fx,\ a)\right]}{1 + d(fx,\ fy,\ a)} \\ &\leq (\alpha + \beta) \max\left\{ d(x,\ y,\ a),\ \frac{d(y,\ fy,\ a)\left[1 + d(x,\ fx,\ a)\right]}{1 + d(fx,\ fy,\ a)} \right\} \\ &= \max\left\{ \gamma d(x,\ y,\ a),\ \gamma \frac{d(y,\ fy,\ a)\left[1 + d(x,\ fx,\ a)\right]}{1 + d(fx,\ fy,\ a)} \right\} \end{aligned}$$

for all $a \in X$ and for all comparable elements $x,\ y \in X$, where $\gamma = \alpha + \beta < 1$. This condition is a particular case of the contractive condition appearing in Theorem 3.1 with the pair of functions $(\varphi,\ \phi) = (1_{[0,\ \infty)}\ ,\ \gamma 1_{[0,\ \infty)}) \in \mathfrak{F}$, given by $\varphi = 1_{[0,\ \infty)}$ and $\phi = \gamma 1_{[0,\ \infty)}$, where $\gamma \in S = \{\gamma : [0,\ \infty) \to [0,\ 1) : \{\gamma(t_n) \to 1 \Rightarrow t_n \to 0\}\}$ (see Example 2.18). Furthermore, we relaxed the requirement of the continuity of mapping to prove the results. $\square$

Corollary 3.4. *Let $(X,\ \preceq)$ is a partially ordered set. Suppose that there exist a 2-metric d on X such that be a complete 2-metric space. Let $f : X \to X$ be a non-decreasing mapping such that there exists a pair of functions $(\varphi,\ \phi) \in \mathfrak{F}$, satisfying*

$$\varphi(d(fx,\ fy,\ a)) \leq \phi(d(x,\ y,\ a)), \tag{45}$$

for all $a \in X$ and for all comparable elements $x,\ y \in X$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$, for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Corollary 3.5. *Let $(X,\ \preceq)$ is a partially ordered set. Suppose that there exist a 2-metric d on X such that be a complete 2-metric space. Let $f : X \to X$ be a non-decreasing mapping such that there exists a pair of functions $(\varphi,\ \phi) \in \mathfrak{F}$, satisfying*

$$\varphi(d(fx,\ fy,\ a)) \leq \phi\left(\frac{d(y,\ fy,\ a)\left[1 + d(x,\ fx,\ a)\right]}{1 + d(fx,\ fy,\ a)}\right), \tag{46}$$

for all $a \in X$ and for all comparable elements $x,\ y \in X$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$, for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Taking into account Example 2.17, we have the following corollary.

Corollary 3.6. *Let $(X,\ \preceq)$ is a partially ordered set. Suppose that there exist a 2-metric d on X such that be a complete 2-metric space. Let $f : X \to X$ be a non-decreasing mapping such that there exists a pair of functions $(\varphi,\ \phi) \in \mathfrak{F}$ satisfying*

$$\begin{aligned} \varphi(d(fx,\ fy,\ a)) \leq \max\Bigg\{ &\varphi(d(x,\ y,\ a)) - \phi(d(x,\ y,\ a)), \\ &\varphi\left(\frac{d(y,\ fy,\ a)\left[1 + d(x,\ fx,\ a)\right]}{1 + d(fx,\ fy,\ a)}\right) - \phi\left(\frac{d(y,\ fy,\ a)\left[1 + d(x,\ fx,\ a)\right]}{1 + d(fx,\ fy,\ a)}\right)\Bigg\}, \end{aligned} \tag{47}$$

for all $a \in X$ and for all comparable elements $x,\ y \in X$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$, for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Corollary 3.6 has the following consequences.

Corollary 3.7. *Let $(X,\ \preceq)$ is a partially ordered set. Suppose that there exist a 2-metric d on X such that be a complete 2-metric space. Let $f : X \to X$ be a non-decreasing mapping such that there exists a pair of functions $(\varphi,\ \phi) \in \mathfrak{F}$ satisfying*

$$\varphi(d(fx,\ fy,\ a)) \leq \varphi(d(x,\ y,\ a)) - \phi(d(x,\ y,\ a)) \tag{48}$$

for all $a \in X$ and for all comparable elements $x,\ y \in X$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$, for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Corollary 3.8. *Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist a 2-metric d on X such that be a complete 2-metric space. Let $f : X \to X$ be a non-decreasing mapping such that there exists a pair of functions $(\varphi, \phi) \in \mathfrak{F}$ satisfying*

$$\varphi\left(d\left(fx,\ fy,\ a\right)\right) \leq \varphi\left(\frac{d\left(y,\ fy,\ a\right)\left[1+d\left(x,\ fx,\ a\right)\right]}{1+d\left(fx,\ fy,\ a\right)}\right) - \phi\left(\frac{d\left(y,\ fy,\ a\right)\left[1+d\left(x,\ fx,\ a\right)\right]}{1+d\left(fx,\ fy,\ a\right)}\right) \tag{49}$$

for all $a \in X$ and for all comparable elements $x,\ y \in X$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$, for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Taking into account Example 2.18, we have the following corollary.

Corollary 3.9. *Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist a 2-metric d on X such that be a complete 2-metric space. Let $f : X \to X$ be a non-decreasing mapping such that there exists $\alpha \in S$ (see Example 2.18) satisfying*

$$d\left(fx,\ fy,\ a\right) \leq \max\left\{\alpha\left(d\left(x,\ y,\ a\right)\right)d\left(x,\ y,\ a\right), \alpha\left(\frac{d\left(y,\ fy,\ a\right)\left[1+d\left(x,\ fx,\ a\right)\right]}{1+d\left(fx,\ fy,\ a\right)}\right)\left(\frac{d\left(y,\ fy,\ a\right)\left[1+d\left(x,\ fx,\ a\right)\right]}{1+d\left(fx,\ fy,\ a\right)}\right)\right\} \tag{50}$$

for all $a \in X$ and for all comparable elements $x,\ y \in X$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$, for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

A consequence of Corollary 3.9 is the following corollary.

Corollary 3.10. *Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist a 2-metric d on X such that be a complete 2-metric space. Let $f : X \to X$ be a non-decreasing mapping such that there exists $\alpha \in S$ (see Example 2.18) satisfying*

$$d\left(fx,\ fy,\ a\right) \leq \alpha\left(d\left(x,\ y,\ a\right)\right)d\left(x,\ y,\ a\right) \tag{51}$$

for all $a \in X$ and for all comparable elements $x,\ y \in X$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$, for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Corollary 3.11. *Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist a 2-metric d on X such that be a complete 2-metric space. Let $f : X \to X$ be a non-decreasing mapping such that there exists $\alpha \in S$ (see Example 2.18) satisfying*

$$d\left(fx,\ fy,\ a\right) \leq \alpha\left(\frac{d\left(y,\ fy,\ a\right)\left[1+d\left(x,\ fx,\ a\right)\right]}{1+d\left(fx,\ fy,\ a\right)}\right)\left(\frac{d\left(y,\ fy,\ a\right)\left[1+d\left(x,\ fx,\ a\right)\right]}{1+d\left(fx,\ fy,\ a\right)}\right) \tag{52}$$

for all $a \in X$ and for all comparable elements $x,\ y \in X$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$, for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

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