International Journal of Mathematics And its Applications
Volume 4, Issue 2–C (2016), 149–169.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Leibniz Type AQ-Functional Equation in Non-Archimedian Fuzzy Normed Spaces

Research Article

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Abstract: In this paper, we establish the generalized Ulam - Hyers stability of a Leibniz type Additive-Quadratic functional equation

$$f(x-t)+f(y-t)+f(z-t)=3f\left(\frac{x+y+z}{3}-t\right)+f\left(\frac{2x-y-z}{3}\right)+f\left(\frac{-x+2y-z}{3}\right)+f\left(\frac{-x-y+2z}{3}\right)$$

in Non-Archimedian Fuzzy normed space, using direct and fixed point methods.

MSC: 39B52, 32B72, 32B82.

Keywords: Additive functional equations, Quadratic functional equation, Mixed type functional equation, Ulam - Hyers stability, Non-Archimedian Fuzzy normed space, fixed point.

1. Introduction

In mathematics, a functional equation is any equation that specifies a function in implicit form. Often, the equation relates the value of a function (or functions) at some point with its values at other points. For instance, properties of functions can be determined by considering the types of functional equations they satisfy. But the theory of functional equations is relatively young. The beginning of a theory of functional equations is connected with the work of an excellent specialist in this field, Hungarian mathematician J. Aczel. The stability problem for functional equations first was planed in 1940 by Ulam [46]:

When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?.

If the problem accepts a solution, we say that the equation is stable. This phenomenon is called $Ulam-Hyers\ stability$ and has been extensively investigated for different functional equations. Let $(G,.)$ be a groupoid and let $(Y,.)$ be a groupoid with the metric ρ. The following definition of stability of the equation of additive homomorphism from G to Y

$$f(x+y)=f(x)+f(y) \tag{1}$$

is formulated.

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Definition 1.1. *Equation (1) stable in the Hyers-Ulam sence if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every function $f : G \to Y$ fulfilling*

$$\rho\left(f(x+y), f(x)+f(y)\right) \leq \delta, \qquad x, y \in G$$

there exists a solution g of (1) satisfying

$$\rho\left(f(x), g(x)\right) \leq \epsilon, \qquad x \in G.$$

The study of stability problems for functional equations concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [19]. It was further generalized and excellent results obtained by number of authors [2, 18, 37, 38, 40]. Its solutions via various forms of functional equations like additive, quadratic, cubic, quartic, mixed type functional equations which involves only these types of functional equations were discussed. We refer the interested readers for more information on such problems to the monographs [1, 4, 6–14, 20–22, 36, 48]. The generalized Ulam-Hyers stability of various types of functions equations (via.) fuzzy normed space, Non-Archimedean space were discussed in [15–17, 25–28, 43, 44, 47]. Recently, Matina J. Rassias et. al., [35] introduced the Leibniz type additive-quadratic functional equation of the form

$$f(x-t)+f(y-t)+f(z-t) = 3f\left(\frac{x+y+z}{3}-t\right)+f\left(\frac{2x-y-z}{3}\right)+f\left(\frac{-x+2y-z}{3}\right)+f\left(\frac{-x-y+2z}{3}\right) \tag{2}$$

and obtained its general solution and generalized Ulam - Hyers stability of Leibniz AQ - mixed type functional equation in quasi-beta normed space using direct and fixed point methods. The solution of the Leibniz type additive and quadratic functional equation (2) is given in the following lemmas.

Lemma 1.2 ([35])**.** *If an odd function $f : X \to Y$ satisfies the functional equation (2) then f is additive.*

Lemma 1.3 ([35])**.** *If an even function $f : X \to Y$ satisfies the functional equation (2) then f is quadratic.*

In this paper, the authors investigate the Generalized Ulam-Hyers stability of a Leibniz type additive and quadratic functional equation (2) in of non-Archimedean fuzzy normed spaces using direct and fixed point methods.

2. Preliminaries

It is to be noted that, Mirmostafaee and Moslehian [25] initiate a notion of a non-Archimedean fuzzy norm and studied the stability of the Cauchy equation in the context of non-Archimedean fuzzy spaces. They presented an interdisciplinary relation between the theory of fuzzy spaces, the theory of non- Archimedean spaces, and the theory of functional equations. During the last three decades, theory of non-Archimedean spaces has prolonged the interest of physicists for their research, in particular, in problems coming from quantum physics, $p-$adic strings, and superstrings. One may note that $|n| \leq 1$ in each valuation field, every triangle is isosceles and there may be no unit vector in a non-Archimedean normed space (cf [23]). These facts show that the non-Archimedean framework is of special interest. Fuzzy set theory is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. It has also very useful applications in various fields, such that population dynamics, chaos control, computer programming, nonlinear dynamical systems, nonlinear operators, statistical convergence, etc. [28, 42, 45]. The fuzzy topology proves to be a very valuable tool to deal with such situations where the use of classical theories breaks down. The most fascinating application of fuzzy topology in quantum particle physics arises in string and E-infinity theory of EI Naschie [30]- [34]. The definition of non-Archimedean fuzzy normed spaces was given in [27].

Definition 2.1. *Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}$, such that for any $a, b \in \mathbb{K}$, we have*

(NA1) $|a| \geq 0$ *and equality holds if and only if* $a = 0$*;*

(NA2) $|ab| = |a|\ |b|$*;*

(NA3) $|a + b| \leq \max\{|a|, |b|\}$.

The condition $(NA3)$ is called the strong triangle inequality. Clearly, $|1| = |-1| = |1|$ and $n \leq 1$ for all $n \geq \mathbb{N}$. We always assume in addition that $|\cdot|$ is non trivial, i.e., that

(NA4) *there is an $a_0 \in \mathbb{K}$, such that $|a_0| \neq 0, 1$.*

The most important examples of non-Archimedean spaces are $p-$adic numbers.

Example 2.2. *Let p be a prime number. For any nonzero rational number x, there exists a unique integer n_x, such that $x = \frac{a}{b}p^n x$, where a and b are integers not divisible by p. Then $|x|_p = p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$ which is called the $p-$adic number field. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum\limits_{k \geq n_x}^{\infty} a_k p^k$, where $|a_k| \leq p - 1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $\left|\sum\limits_{k \geq n_x}^{\infty} a_k p^k\right|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field (see [41]). Note that if $p > 2$ then $|2^n|_p = 1$ for each integer n but $|2|_2 < 1$.*

Now we give the definition of a non-Archimedean fuzzy normed space.

Definition 2.3. *Let X be a linear space over a non-Archimedean field $\mathbb{K}$. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be a non-Archimedean fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*

(NAF1) $N(x, c) = 0$ *for all* $c \leq 0$;

(NAF2) $x = 0$ *if and only if* $N(x, c) = 1$ *for all* $c > 0$;

(NAF3) $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$;

(NAF4) $N(x + y, \max\{s, t\}) \geq min\{N(x, s), N(y, t)\}$;

(NAF5) $\lim\limits_{t\to\infty} N(x, t) = 1$.

A non-Archimedean fuzzy norm is a pair (X, N) where X be a linear space and N is non-Archimedean fuzzy norm on X. If $(NAF4)$ holds then

(NAF6) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$.

Recall that a classical vector space over the complex or real field satisfying $(NAF1) - (NAF5)$ is called a fuzzy normed space in the literature. We repeatedly use the fact $N(-x, t) = N(x, t),\quad x \in X, t > 0$, which is deduced from $(NAF3)$. It is easy to see that $(NAF4)$ is equivalent to the following condition:

(NAF7) $N(x + y, t) \geq \min\{N(x, t), N(y, t)\}$.

Example 2.4. *Let $(X, ||\cdot||)$ be a non-Archimedean normed space. Then*

$$N(x,t) = \begin{cases} \dfrac{t}{t+\|x\|}, & t>0, \quad x \in X, \\ 0, & t \le 0, \quad x \in X \end{cases}$$

Then (X, N) is a non-Archimedean fuzzy normed space.

Example 2.5. *Let $(X, ||\cdot||)$ be a non-Archimedean normed space. Then*

$$N(x,t) = \begin{cases} 0, & t \le ||x||, \\ 1, & t > ||x||. \end{cases}$$

Then (X, N) is a non-Archimedean fuzzy normed space.

Definition 2.6. *Let (X, N) be a non-Archimedean fuzzy normed space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N - \lim_{n\to\infty} x_n = x$.*

Definition 2.7. *A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \ge n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$. Due to this fact that*

$$N(x_n - x_m, t) \ge \min\{N(x_{j+1} - x_j, t)/m \le j \le n-1, \quad n > m\},$$

a sequence $\{x_n\}$ is Cauchy if and only if $\lim_{n\to\infty} N(x_{n+1} - x_n, t) = 1$ for all $t > 0$.

Definition 2.8. *Every convergent sequence in a non-Archimedean fuzzy normed space is a Cauchy sequence. If every Cauchy sequence is convergent, then the non-Archimedean fuzzy normed space is called a non-Archimedean fuzzy Banach space.*

Here after, throughout this paper, assume that $\mathbb{K}$ non-Archimedean field, X be vector space over $\mathbb{K}$, (Y, N') be a non-Archimedean fuzzy Banach space over $\mathbb{K}$ and (Z, N') be an (Archimedean or non-Archimedean) fuzzy normed space. Also we use the following notation for a given mapping $f : X \to Y$ such that

$$Df(x,y,z,t) = f(x-t)+f(y-t)+f(z-t)-3f\left(\frac{x+y+z}{3}-t\right)-f\left(\frac{2x-y-z}{3}\right)-f\left(\frac{-x+2y-z}{3}\right)-f\left(\frac{-x-y+2z}{3}\right)$$

for all $x, y, z, t \in X$.

3. Stability of the Functional Equation (2) Using Direct Method

In this section, the non-Archimedean fuzzy stability of a Leibniz type AQ -functional equation (2) is provided using direct method. The following theorem provide the stability result of (2) for f is odd function.

Theorem 3.1. *Let $\kappa = \pm 1$ be fixed and let $\vartheta : X^4 \to Z$ be a mapping such that for some d with $0 < \left(\frac{d}{2}\right)^{\kappa} < 1$*

$$N'\left(\vartheta\left(2^{\kappa n}x, 2^{\kappa n}x, 2^{\kappa n}x, 2^{\kappa n}x\right), r\right) \ge N'\left(d^{\kappa n}\vartheta\left(x,x,x,x\right), r\right) \tag{3}$$

for all $x \in X$, all $r > 0$, and

$$\lim_{n\to\infty} N'\left(\vartheta\left(2^{\kappa n}x, 2^{\kappa n}y, 2^{\kappa n}z, 2^{\kappa n}t\right), 2^{\kappa n}r\right) = 1 \tag{4}$$

for all $x, y, z, t \in X$ and all $r > 0$. Suppose that an odd function $f_a : X \to Y$ satisfies the inequality

$$N\left(D\ f_a(x,y,z,t), r\right) \geq N'\left(\vartheta(x,y,z,t), r\right) \tag{5}$$

for all $x, y, z, t \in X$ and all $r > 0$. Then the limit

$$A(x) = N - \lim_{n\to\infty} \frac{f_a(2^{\kappa n}x)}{2^{\kappa n}} \tag{6}$$

exists for all $x \in X$ and the mapping $A : X \to Y$ is a unique additive mapping satisfying (2) and

$$N\left(f_a(x) - A(x), r\right) \geq N'\left(\vartheta(2x, x, 0, 0), r|2-d|\right) \tag{7}$$

for all $x \in X$ and all $r > 0$.

Proof. First assume $\kappa = 1$. Replacing (x, y, z, t) by $(2x, x, 0, 0)$ in (5) and using oddness of f_a, we get

$$N\left(f_a(2x) - 2f_a(x), r\right) \geq N'\left(\vartheta(2x, x, 0, 0), r\right), \quad \forall\ x \in X, r > 0. \tag{8}$$

Using $(NAF3)$ in (8), we arrive,

$$N\left(\frac{f_a(2x)}{2} - f_a(x), \frac{r}{2}\right) \geq N'\left(\vartheta(2x, x, 0, 0), r\right), \quad \forall\ x \in X, r > 0. \tag{9}$$

Replacing x by $2^n x$ in (9), we obtain

$$N\left(\frac{f_a(2^{n+1}x)}{2} - f_a(2^n x), \frac{r}{2}\right) \geq N'\left(\vartheta(2^{(n+1)}x, 2^n x, 0, 0), r\right), \quad \forall\ x \in X, r > 0. \tag{10}$$

Using (3) and $(NAF3)$ in (10), we have

$$N\left(\frac{f_a(2^{n+1}x)}{2} - f_a(2^n x), \frac{r}{2}\right) \geq N'\left(\vartheta(2x, x, 0, 0), \frac{r}{d^n}\right), \quad \forall\ x \in X, r > 0. \tag{11}$$

One can easy to verify from (11), that

$$N\left(\frac{f_a(2^{n+1}x)}{2^{n+1}} - \frac{f_a(2^n x)}{2^n}, \frac{r}{2\cdot 2^n}\right) \geq N'\left(\vartheta(2x, x, 0, 0), \frac{r}{d^n}\right), \quad \forall\ x \in X, r > 0. \tag{12}$$

Replacing r by $d^n r$ in (12), we obtain

$$N\left(\frac{f_a(2^{n+1}x)}{2^{n+1}} - \frac{f_a(2^n x)}{2^n}, \frac{d^n\ r}{2\cdot 2^n}\right) \geq N'\left(\vartheta(2x, x, 0, 0), r\right), \quad \forall\ x \in X, r > 0. \tag{13}$$

It is easy to see that

$$\frac{f_a(2^n x)}{2^n} - f_a(x) = \sum_{i=0}^{n-1}\left[\frac{f_a(2^{i+1}x)}{2^{i+1}} - \frac{f_a(2^i x)}{2^i}\right], \quad \forall\ x \in X. \tag{14}$$

It follows from (13) and (14), we have

$$\begin{aligned} N\left(\frac{f_a(2^n x)}{2^n} - f_a(x), \sum_{i=0}^{n-1}\frac{d^i\ r}{2\cdot 2^i}\right) &\geq min \bigcup_{i=0}^{n-1}\left\{N\left(\frac{f_a(2^{i+1}x)}{2^{i+1}} - \frac{f_a(2^i x)}{2^i}, \frac{d^i\ r}{2\cdot 2^i}\right)\right\} \\ &\geq min \bigcup_{i=0}^{n-1}\left\{N'\left(\vartheta(2x, x, 0, 0), r\right)\right\} \\ &\geq N'\left(\vartheta(2x, x, 0, 0), r\right), \quad \forall\ x \in X, r > 0. \end{aligned} \tag{15}$$

Replacing x by $2^m x$ in (15) and using (3), $(NAF3)$, we get

$$N\left(\frac{f_a(2^{n+m}x)}{2^{n+m}} - \frac{f_a(2^m x)}{2^m}, \sum_{i=0}^{n-1} \frac{d^i\ r}{2\cdot 2^{i+m}}\right) \geq N'\left(\vartheta(2x, x, 0, 0), \frac{r}{d^m}\right) \tag{16}$$

. for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Replacing r by $d^m r$ in (16), we get

$$N\left(\frac{f_a(2^{n+m}x)}{2^{n+m}} - \frac{f_a(2^m x)}{2^m}, \sum_{i=0}^{n-1} \frac{d^i\ r}{2\cdot 2^{i+m}}\right) \geq N'\left(\vartheta(2x, x, 0, 0), r\right) \tag{17}$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. It follows from (17) that

$$N\left(\frac{f_a(2^{n+m}x)}{2^{n+m}} - \frac{f_a(2^m x)}{2^m}, r\right) \geq N'\left(\vartheta(2x, x, 0, 0), \frac{r}{\sum_{i=m}^{m+n-1} \frac{d^i}{2\cdot 2^i}}\right) \tag{18}$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Since $0 < d < 2$ and $\sum_{i=0}^{n} \left(\frac{d}{2}\right)^i < \infty$, using $(NAF5)$ implies that $\left\{\frac{f_a(2^n x)}{2^n}\right\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a non-Archimedean fuzzy Banach space, this sequence converges to some point $A(x) \in Y$. So, we can define a mapping $A : X \to Y$ by

$$A(x) = N - \lim_{n\to\infty} \frac{f_a(2^n x)}{2^n}, \quad \forall\ x \in X.$$

Putting $m = 0$ in (18), we get

$$N\left(\frac{f_a(2^n x)}{2^n} - f_a(x), r\right) \geq N'\left(\vartheta(2x, x, 0, 0), \frac{r}{\sum_{i=0}^{n-1} \frac{d^i}{2\cdot 2^i}}\right), \ \forall\ x \in X, r > 0. \tag{19}$$

Letting $n \to \infty$ in (19) and using $(NAF5)$, we arrive

$$N\left(f_a(x) - A(x), r\right) \geq N'\left(\vartheta(2x, x, 0, 0), r(2-d)\right) \quad \forall\ x \in X, r > 0.$$

To prove A satisfies (2), replacing (x, y, z, t) by $(2^n x, 2^n y, 2^n z, 2^n t)$ in (5), respectively , we obtain

$$N\left(\frac{1}{2^n}(Df_a(2^n x, 2^n y, 2^n z, 2^n t)), r\right) \geq N'\left(\vartheta(2^n x, 2^n y, 2^n z, 2^n t), 2^n r\right) \tag{20}$$

for all $x, y, z, t \in X$ and all $r > 0$. Now,

$$N\left(A(x-t) + A(y-t) + A(z-t) - 3A\left(\frac{x+y+z}{3} - t\right) - A\left(\frac{2x-y-z}{3}\right) - A\left(\frac{-x+2y-z}{3}\right) - A\left(\frac{-x-y+2z}{3}\right), r\right)$$

$$\begin{aligned}
\geq min \Bigg\{ & N\left(A(x-t) - \frac{1}{2^n} f_a(2^n(x-t)), \frac{r}{8}\right), \\
& N\left(A(y-t) - \frac{1}{2^n} f_a(2^n(y-t)), \frac{r}{8}\right), N\left(A(z-t) - \frac{1}{2^n} f_a(2^n(z-t)), \frac{r}{8}\right) \\
& N\left(-3A\left(\frac{x+y+z}{3} - t\right) + \frac{3}{2^n} f_a\left(\frac{2^n(x+y+z)}{3} - 2^n t\right), \frac{r}{8}\right), \\
& N\left(-A\left(\frac{2x-y-z}{3}\right) + \frac{1}{2^n} f_a\left(\frac{2^n(2x-y-z)}{3}\right), \frac{r}{8}\right), \\
& N\left(-A\left(\frac{-x+2y-z}{3}\right) + \frac{1}{2^n} f_a\left(\frac{2^n(-x+2y-z)}{3}\right), \frac{r}{8}\right), \\
& N\left(-A\left(\frac{-x-y+2z}{3}\right) + \frac{1}{2^n} f_a\left(\frac{2^n(-x-y+2z)}{3}\right), \frac{r}{8}\right), \\
& N\Bigg(\frac{1}{2^n} f_a(2^n(x-t)) + \frac{1}{2^n} f_a(2^n(y-t)) + \frac{1}{2^n} f_a(2^n(z-t)) - \frac{3}{2^n} f_a\left(\frac{2^n(x+y+z)}{3} - 2^n t\right) \\
& \quad - \frac{1}{2^n} f_a\left(\frac{2^n(2x-y-z)}{3}\right) - \frac{1}{2^n} f_a\left(\frac{2^n(-x+2y-z)}{3}\right) - \frac{1}{2^n} f_a\left(\frac{2^n(-x-y+2z)}{3}\right), \frac{r}{8}\Bigg)\Bigg\}
\end{aligned} \tag{21}$$

for all $x, y, z, t \in X$ and all $r > 0$. Using (20) and $(NAF5)$ in (21), we arrive

$$\begin{aligned} & N\left(A(x-t)+A(y-t)+A(z-t)-3A\left(\frac{x+y+z}{3}-t\right)\right. \\ & \quad \left. -A\left(\frac{2x-y-z}{3}\right)-A\left(\frac{-x+2y-z}{3}\right)-A\left(\frac{-x-y+2z}{3}\right), r\right) \\ & \quad \geq min\left\{1,1,1,1,1,1,1, N'\left(\vartheta(2^n x, 2^n y, 2^n z, 2^n t), 2^n r\right)\right\} \\ & \quad \geq N'\left(\vartheta(2^n x, 2^n y, 2^n z, 2^n t), 2^n r\right) \end{aligned} \tag{22}$$

for all $x, y, z, t \in X$ and all $r > 0$. Letting $n \to \infty$ in (22) and using (4), we see that

$$\begin{aligned} N\left(A(x-t)+A(y-t)+A(z-t)-3A\left(\frac{x+y+z}{3}-t\right)-A\left(\frac{2x-y-z}{3}\right)\right. \\ \left. -A\left(\frac{-x+2y-z}{3}\right)-A\left(\frac{-x-y+2z}{3}\right), r\right) = 1 \end{aligned}$$

for all $x, y, z, t \in X$ and all $r > 0$. Using $(NAF2)$ in the above inequality, we get

$$A(x-t)+A(y-t)+A(z-t)-3A\left(\frac{x+y+z}{3}-t\right) = A\left(\frac{2x-y-z}{3}\right)+A\left(\frac{-x+2y-z}{3}\right)+A\left(\frac{-x-y+2z}{3}\right)$$

for all $x, y, z, t \in X$. Hence A satisfies the functional equation (2). In order to prove $A(x)$ is unique, let $A'(x)$ be another additive function satisfying (2) and (6). Hence,

$$\begin{aligned} N(A(x)-A'(x), r) & \geq min\left\{N\left(A(2^n x)-\frac{f_a(2^n x)}{2^n}, \frac{2^n r}{2}\right), N\left(\frac{f_a((2^n x)}{2^n}-A'(2^n x), \frac{2^n r}{2}\right)\right\} \\ & \geq N'\left(\vartheta(2^{n+1}x, 2^n x, 0, 0), \frac{r\ 2^n(2-d)}{2}\right) \\ & \geq N'\left(\vartheta(2x, x, 0, 0), \frac{r\ 2^n(2-d)}{2d^n}\right), \quad \forall\ x \in X, r > 0. \end{aligned}$$

Since $\lim_{n\to\infty} \frac{r\ 2^n(2-d)}{2d^n} = \infty$, we obtain

$$\lim_{n\to\infty} N'\left(\vartheta(2x, x, 0, 0), \frac{r\ 2^n(2-d)}{2d^n}\right) = 1.$$

Thus

$$N(A(x)-A'(x), r) = 1, \quad \forall\ x \in X, r > 0,$$

Hence $A(x) = A'(x)$. Therefore $A(x)$ is unique. For $\kappa = -1$, we can prove the result by a similar method. This completes the proof of the theorem. □

From Theorem 3.1, we obtain the following corollary concerning the stability for the functional equation (2).

Corollary 3.2. *Suppose that a odd function $f_a : X \to Y$ satisfies the inequality*

$$N\left(D\ f_a(x,y,z,t), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon\left(||x||^s+||y||^s+||z||^s+||t||^s\right), r\right), \\ N'\left(\epsilon\left\{||x||^s\ ||y||^s\ ||z||^s\ ||t||^s+\left(||x||^{4s}+||y||^{4s}+||z||^{4s}+||t||^{4s}\right)\right\}, r\right), \end{cases} \tag{23}$$

for all $r > 0$ and all $x, y, z, t \in X$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$N\left(f_a(x) - A(x), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), \\ N'\left((1+2^s)\epsilon||x||^s, r|2-2^s|\right), & s < 1 \quad or \quad s > 1; \\ N'\left((1+2^s)\epsilon||x||^{4s}, r|2-2^{4s}|\right), & s < \frac{1}{4} \quad or \quad s > \frac{1}{4}; \end{cases} \tag{24}$$

for all $x \in X$ and all $r > 0$.

Proof. Setting

$$\vartheta(x, y, z, t) = \begin{cases} \epsilon, \\ \epsilon\left(||x||^s + ||y||^s + ||z||^s + ||t||^s\right), \\ \epsilon\left\{||x||^s \ ||y||^s \ ||z||^s \ ||t||^s + \left(||x||^{4s} + ||y||^{4s} + ||z||^{4s} + ||t||^{4s}\right)\right\}, \end{cases}$$

then the corollary is followed from Theorem 3.1. If we define

$$d = \begin{cases} 1, \\ 2^s, \\ 2^{4s}. \end{cases}$$

□

Example 3.3. *Let X be a normed space and N and N' be non-archimedian fuzzy norms on X and $\mathbb{R}$ defined by*

$$N\left(x, r\right) \ = \begin{cases} \dfrac{r}{r + \|x\|} & r > 0, \quad x \in X, \\ 0, & r \leq 0, \quad x \in X. \end{cases} \tag{25}$$

$$N'\left(x, r\right) \ = \begin{cases} \dfrac{r}{r + \|x\|} & r > 0, \quad x \in \mathbb{R}, \\ 0, & r \leq 0, \quad x \in \mathbb{R}. \end{cases} \tag{26}$$

Let $\vartheta : (0, \infty) \to (0, \infty)$ be a function such that $\vartheta\left(2l\right) \ < \ d\,\vartheta\left(l\right)$ for all $l > 0$ and $0 < d < 2$. Define

$$\begin{aligned} \beta\left(x, y, z, t\right) = \vartheta\left(\|x - t\|\right) + \vartheta\left(\|y - t\|\right) + \vartheta\left(\|z - t\|\right) - 3\vartheta\left(\left\|\frac{x + y + z}{3} - t\right\|\right) \\ - \vartheta\left(\left\|\frac{2x - y - z}{3}\right\|\right) - \vartheta\left(\left\|\frac{-x + 2y - z}{3}\right\|\right) + \vartheta\left(\left\|\frac{-x - y + 2z}{3}\right\|\right) \end{aligned}$$

for all $x, y, z, t \in X$. Let $x_0 \in X$ be a unit vector and define $f_a : X \to X$ by $f_a\left(x\right) \ = \ x + \vartheta\left(\|x\|\right)x_0$. Now for any $x, y, z, t \in X$ and $r > 0$, we have

$$\begin{aligned} N\ \left(Df_a(x, y, z, t), r\right) &= \frac{r}{r + \|\beta\left(x, y, z, t\right)\| \cdot \|x_0\|} \\ &\geq \frac{r}{r + \|\beta\left(x, y, z, t\right)\|} \\ &= N'\left(\beta\left(x, y, z, t\right), r\right). \end{aligned}$$

For any $x, y, z, t \in X$ and $r > 0$, we have

$$\begin{aligned} N'\left(\beta\left(2x, 2y, 2z, 2t\right), r\right) &= \frac{r}{r + \beta\left(2x, 2y, 2z, 2t\right)} \\ &\geq \frac{r}{r + d\ \beta\left(x, y, z, t\right)} \\ &= N'\left(d\ \beta\left(x, y, z, t\right), r\right). \end{aligned}$$

Hence the inequalities (3) and (5) are satisfied. Using Theorem 3.1, there exists a unique additive mapping $A : X \to Y$ *such that*

$$N\left(A\left(x\right)-f_{a}\left(x\right),\ r\right)\ \geq\ N'\left(\frac{\beta\left(2x,x,0,0\right)}{|2-d|},r\right)$$

$x \in X$ *and* $r > 0$.

The following theorem and corollary provide the stability result of (2) for f is even function. The proof is similar tracing to that of Theorem 3.1 and Corollary 3.2. Hence the details of the proof is omitted.

Theorem 3.4. *Let* $\kappa = \pm 1$ *be fixed and let* $\vartheta : X^4 \to Z$ *be a mapping such that for some* d *with* $0 < \left(\frac{d}{4}\right)^{\kappa} < 1$

$$N'\left(\vartheta\left(2^{\kappa}x,2^{\kappa}x,2^{\kappa}x,2^{\kappa}x\right),r\right) \geq N'\left(d^{\kappa}\vartheta\left(x,x,x,x\right),r\right) \tag{27}$$

for all $x \in X$ *and all* $d > 0$, *and*

$$\lim_{n\to\infty} N'\left(\vartheta\left(2^{\kappa n}x,2^{\kappa n}y,2^{\kappa n}z,2^{\kappa n}t\right),4^{\kappa n}r\right) = 1 \tag{28}$$

for all $x, y, z, t \in X$ *and all* $r > 0$. *Suppose that a even function* $f_q : X \to Y$ *satisfies the inequality*

$$N\left(D\ f_q(x,y,z,t),r\right) \geq N'\left(\vartheta(x,y,z,t),r\right) \tag{29}$$

for all $x, y, z, t \in X$ *and all* $r > 0$. *Then the limit*

$$Q(x) = N - \lim_{n\to\infty} \frac{f_q(2^{\beta n}x)}{4^{\beta n}} \tag{30}$$

exists for all $x \in X$ *and the mapping* $Q : X \to Y$ *is a unique quadratic mapping satisfying (2) and*

$$N\left(f_q(x) - Q(x),r\right) \geq N'\left(\vartheta(2x,x,0,0),r|4-d|\right) \tag{31}$$

for all $x \in X$ *and all* $r > 0$.

Corollary 3.5. *Suppose that a even function* $f_q : X \to Y$ *satisfies the inequality*

$$N\left(D\ f_q(x,y,z,t),r\right) \geq \begin{cases} N'\left(\epsilon,r\right), \\ N'\left(\epsilon\left(||x||^s+||y||^s+||z||^s+||t||^s\right),r\right), \\ N'\left(\epsilon\left\{||x||^s\ ||y||^s\ ||z||^s\ ||t||^s+\left(||x||^{4s}+||y||^{4s}+||z||^{4s}+||t||^{4s}\right)\right\},r\right), \end{cases} \tag{32}$$

for all $x, y, z, t \in X$ *and all* $r > 0$, *where* ϵ, s *are constants with* $\epsilon > 0$. *Then there exists a unique quadratic mapping* $Q : X \to Y$ *such that*

$$N\left(f_q(x) - Q(x),r\right) \geq \begin{cases} N'\left(\epsilon,r\right), \\ N'\left(\epsilon(2^s+1)||x||^s,r|4-2^s|\right), & s<2 \quad or \quad s>2; \\ N'\left(\epsilon(2^{4s}+1)||x||^{4s},r|4-2^{4s}|\right), & s<\frac{1}{2} \quad or \quad s>\frac{1}{2}; \end{cases} \tag{33}$$

for all $x \in X$ *and all* $r > 0$.

Example 3.6. *Let X be a normed space and N and N' be non-archimedian fuzzy norms on X and $\mathbb{R}$ defined by (25),(26). Let $\vartheta : (0,\infty) \to (0,\infty)$ be a function such that $\vartheta(2l) < d\,\vartheta(l)$ for all $l > 0$ and $0 < d < 4$. Define*

$$\begin{aligned}\beta(x,y,z,t) = \vartheta(\|x-t\|) + \vartheta(\|y-t\|) + \vartheta(\|z-t\|) - 3\vartheta\left(\left\|\frac{x+y+z}{3} - t\right\|\right)\\ - \vartheta\left(\left\|\frac{2x-y-z}{3}\right\|\right) - \vartheta\left(\left\|\frac{-x+2y-z}{3}\right\|\right) + \vartheta\left(\left\|\frac{-x-y+2z}{3}\right\|\right)\end{aligned}$$

for all $x,y,z,t \in X$. Let $x_0 \in X$ be a unit vector and define $f_q : X \to X$ by $f_q(x) = x + \vartheta(\|x\|)\,x_0$. Now for any $x,y,z,t \in X$ and $r > 0$, we have

$$\begin{aligned}N\,(Df_q(x,y,z,t),r) &= \frac{r}{r + \|\beta(x,y,z,t)\| \cdot \|x_0\|}\\ &\geq \frac{r}{r + \|\beta(x,y,z,t)\|}\\ &= N'(\beta(x,y,z,t), r).\end{aligned}$$

For any $x,y,z,t \in X$ and $r > 0$, we have

$$\begin{aligned}N'(\beta(2x,2y,2z,2t), r) &= \frac{r}{r + \beta(2x,2y,2z,2t)}\\ &\geq \frac{r}{r + d\;\beta(x,y,z,t)}\\ &= N'(d\;\beta(x,y,z,t), r).\end{aligned}$$

Hence the inequalities (27) and (29) are satisfied. Using Theorem 3.4, there exists a unique additive mapping $A : X \to Y$ such that

$$N\,(A(x) - f_q(x),\; r) \;\geq\; N'\left(\frac{\beta(2x,x,0,0)}{|4-d|}, r\right)$$

$x \in X$ and $r > 0$.

Theorem 3.7. *Let $\kappa = \pm 1$ be fixed and let $\vartheta : X^4 \to Z$ be a mapping such that for some d with $0 < \left(\frac{d}{2}\right)^{\kappa} < 1$ and satisfying (3),(4),(27) and (28). Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\,(D\;f(x,y,z,t), r) \geq N'(\vartheta(x,y,z,t), r),\;\; \forall\; x,y,z,t \in X, r > 0. \tag{34}$$

Then there exists a unique additive mapping $A : X \to Y$ and unique quadratic mapping $Q : X \to Y$ satisfying (2) and

$$N\,(f(x) - A(x) - Q(x), r) \geq N_3(\vartheta(2x,x,0,0), r) \tag{35}$$

where

$$N_3(\vartheta(2x,x,0,0), r) = \min\{N_1(\vartheta(2x,x,0,0), 2r|2-d|), N_2(\vartheta(2x,x,0,0), 2r|4-d|)\} \tag{36}$$

for all $x \in X$ and all $r > 0$.

Proof. Clearly $|4| \leq |2| \leq d$. Let $f_o(x) = \dfrac{f_a(x) - f_a(-x)}{2}$ for all $x \in X$. Then $f_o(0) = 0$ and $f_o(-x) = -f_o(x)$ for all $x \in X$. Hence

$$\begin{aligned}N\,(D\;f_o(x,y,z,t), r) &\geq \min\left\{N'(D\;f_a(x,y,z,t), r), N'(D\;f_a(-x,-y,-z,-t), r)\right\}\\ &\geq \min\left\{N'(\vartheta(x,y,z,t), r), N'(\vartheta(-x,-y,-z,-t), r)\right\}\end{aligned} \tag{37}$$

for all $x, y, z, t \in X$ and all $r > 0$. Let

$$N_1(\vartheta(x,y,z,t),r) = \min\left\{N'\left(\vartheta(x,y,z,t),r\right), N'\left(\vartheta(-x,-y,-z,-t),r\right)\right\} \tag{38}$$

for all $x, y, z, t \in X$ and all $r > 0$. By Theorems 3.1, there exists a unique additive mapping $A : X \to Y$ such that

$$N\left(f_o(x) - A(x), r\right) \geq N_1\left(\vartheta(2x,x,0,0), r|2-d|\right) \tag{39}$$

for all $x \in X$ and all $r > 0$. Also, let $f_e(x) = \dfrac{f_q(x) + f_q(-x)}{2}$ for all $x \in X$. Then $f_e(0) = 0$ and $f_e(-x) = f_e(x)$ for all $x \in X$. Hence

$$\begin{aligned} N\left(D\ f_e(x,y,z,t), r\right) &= N\left(D\ f_q(x,y,z,t) - D\ f_q(-x,-y,-z,-t), 2r\right) \\ &\geq \min\left\{N'\left(\vartheta(x,y,z,t),r\right), N'\left(\vartheta(-x,-y,-z,-t),r\right)\right\} \end{aligned} \tag{40}$$

for all $x, y, z, t \in X$ and all $r > 0$. Let

$$N_2(\vartheta(x,y,z,t),r) = \min\left\{N'\left(\vartheta(x,y,z,t),r\right), N'\left(\vartheta(-x,-y,-z,-t),r\right)\right\} \tag{41}$$

for all $x, y, z, t \in X$ and all $r > 0$. By Theorem 3.4, there exists a unique quadratic mapping $Q : X \to Y$ such that

$$N\left(f_e(x) - Q(x), r\right) \geq N_2\left(\vartheta(2x,x,0,0), r|4-d|\right) \tag{42}$$

for all $x \in X$ and all $r > 0$. Define

$$f(x) = f_e(x) + f_o(x) \tag{43}$$

for all $x \in x$. From (35),(38) and (39), we arrive

$$\begin{aligned} N\left(f(x) - A(x) - Q(x), r\right) &= N\left(f_e(x) + f_o(x) - A(x) - Q(x), r\right) \\ &\geq \min\left\{N\left(f_o(x) - A(x), \frac{r}{2}\right), N\left(f_e(x) - Q(x), \frac{r}{2}\right)\right\} \\ &\geq \min\left\{N_1\left(\vartheta(2x,x,0,0), 2r|2-d|\right), N_2\left(\vartheta(2x,x,0,0), 2r|4-d|\right)\right\} \\ &= N_3\left(\vartheta(2x,x,0,0), r\right) \end{aligned}$$

where

$$N_3\left(\vartheta(2x,x,0,0), r\right) = \min\left\{N_1\left(\vartheta(2x,x,0,0), 2r|2-d|\right), N_2\left(\vartheta(2x,x,0,0), 2r|4-d|\right)\right\} \tag{44}$$

for all $x \in X$ and all $r > 0$. Hence the theorem is proved. $\square$

The following corollary is the immediate consequence of Corollaries 3.2, 3.4 and Theorem 3.5 concerning the stability for the functional equation (2).

Corollary 3.8. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(D\ f(x,y,z,t), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon\left(||x||^s + ||y||^s + ||z||^s + ||t||^s\right), r\right), \\ N'\left(\epsilon\left\{||x||^s\ ||y||^s\ ||z||^s\ ||t||^s + \left(||x||^{4s} + ||y||^{4s} + ||z||^{4s} + ||t||^{4s}\right)\right\}, r\right), \end{cases} \tag{45}$$

for all $x, y, z, t \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - A(x) - Q(x), r\right) \geq \begin{cases} N_3\left(\epsilon, r\right), \\ N_3\left(\epsilon(2^s+1)||x||^s, r\right), & s \neq 1, 2; \\ N_3\left(\epsilon(2^{4s}+1)||x||^{4s}, r\right), & s \neq \frac{1}{4}, \frac{1}{2}; \end{cases} \tag{46}$$

for all $x \in X$ and all $r > 0$.

Example 3.9. *Let X be a normed space and N and N' be non-archimedian fuzzy norms on X and $\mathbb{R}$ defined by (25),(26). Let $\vartheta : (0, \infty) \to (0, \infty)$ be a function such that $\vartheta(2l) < d\,\vartheta(l)$ for all $l > 0$ and $0 < d < 2$. Define*

$$\begin{aligned}\beta(x, y, z, t) = {} & \vartheta(\|x-t\|) + \vartheta(\|y-t\|) + \vartheta(\|z-t\|) - 3\vartheta\left(\left\|\frac{x+y+z}{3} - t\right\|\right) \\ & - \vartheta\left(\left\|\frac{2x-y-z}{3}\right\|\right) - \vartheta\left(\left\|\frac{-x+2y-z}{3}\right\|\right) + \vartheta\left(\left\|\frac{-x-y+2z}{3}\right\|\right)\end{aligned}$$

for all $x, y, z, t \in X$. Let $x_0 \in X$ be a unit vector and define $f : X \to X$ by $f(x) = x + \vartheta(\|x\|)\, x_0$. Now for any $x, y, z, t \in X$ and $r > 0$, we have

$$\begin{aligned} N\left(Df(x,y,z,t), r\right) &= \frac{r}{r + \|\beta(x,y,z,t)\| \cdot \|x_0\|} \\ &\geq \frac{r}{r + \|\beta(x,y,z,t)\|} \\ &= N'\left(\beta(x,y,z,t), r\right). \end{aligned}$$

For any $x, y, z, t \in X$ and $r > 0$, we have

$$\begin{aligned} N'\left(\beta(2x,2y,2z,2t), r\right) &= \frac{r}{r + \beta(2x,2y,2z,2t)} \\ &\geq \frac{r}{r + d\,\beta(x,y,z,t)} \\ &= N'\left(d\,\beta(x,y,z,t), r\right). \end{aligned}$$

Hence the inequalities (3), (27) and (34) are satisfied. Using Theorem 3.7, there exists a unique additive mapping $A : X \to Y$ and quadratic mapping $Q : X \to Y$ such that

$$N\left(A(x) - Q(x) - f(x),\ r\right) \ \geq\ N_3\left(\beta(2x, x, 0, 0), r\right)$$

$x \in X$ and $r > 0$.

4. Stability of the Functional Equation (2) Using Fixed Point Method

In this section, the authors presented generalized Ulam - Hyers stability of the functional equation (2) in non-Archimedian Fuzzy normed space using fixed point method. Now we will recall the fundamental results in fixed point theory.

Theorem 4.1 (Banach's contraction principle). *Let (X, d) be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is*

(A1) $d(Tx, Ty) \leq Ld(x, y)$ *for some (Lipschitz constant)* $L < 1$. *Then,*

(i) The mapping T *has one and only fixed point* $x^* = T(x^*)$;

(ii)The fixed point for each given element x^* *is globally attractive, that is*

(A2) $\lim_{n\to\infty} T^n x = x^*$, *for any starting point* $x \in X$;

(iii) One has the following estimation inequalities:

(A3) $d(T^n x, x^*) \leq \frac{1}{1-L} \quad d(T^n x, T^{n+1}x), \forall \ \ n \geq 0, \forall \ \ x \in X$;

(A4) $d(x, x^*) \leq \frac{1}{1-L} \quad d(x, x^*), \forall \ \ x \in X$.

Theorem 4.2 (The alternative of fixed point [24]). *Suppose that for a complete generalized metric space* (X, d) *and a strictly contractive mapping* $T : X \to X$ *with Lipschitz constant* L. *Then, for each given element* $x \in X$, *either*

(B1) $d(T^n x, T^{n+1}x) = \infty \quad \forall \ \ n \geq 0$, *or*

(B2) there exists a natural number n_0 *such that:*

(i) $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \geq n_0$;

(ii)The sequence $(T^n x)$ *is convergent to a fixed point* y^* *of* T

(iii) y^* *is the unique fixed point of* T *in the set* $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;

(iv) $d(y^*, y) \leq \frac{1}{1-L} \quad d(y, Ty)$ *for all* $y \in Y$.

For to prove the stability result, we define the following:

μ_i is a constant such that

$$\mu_i = \begin{cases} 2 & if \ \ i = 0 \\ \frac{1}{2} & if \ \ i = 1 \end{cases},$$

and Ω is the set such that

$$\Omega = \{g | g : X \to Y, g(0) = 0\}.$$

The following theorem provide the stability result of (2) for f is odd function in fixed point method.

Theorem 4.3. *Let* $f_a : X \to Y$ *be a mapping for which there exist a function* $\vartheta : X^4 \to Z$ *with the condition*

$$\lim_{n\to\infty} N'\left(\vartheta\left(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n t\right), \mu_i^n r\right) = 1, \quad \forall \ x, y, z, t \in X, r > 0 \tag{47}$$

and satisfying the functional inequality

$$N\left(D \ f_a(x, y, z, t), r\right) \geq N'\left(\vartheta(x, y, z, t), r\right), \ \forall \ x, y, z, t \in X, r > 0. \tag{48}$$

If there exists $L = L(i)$ *such that the function*

$$x \to \beta(x) = \vartheta\left(x, \frac{x}{2}, 0, 0\right),$$

has the property

$$N'\left(L\frac{\beta(\mu_i x)}{\mu_i}, r\right) = N'\left(\beta(x), r\right), \ \forall \ x \in X, r > 0. \tag{49}$$

Then there exists a unique additive function $A : X \to Y$ *satisfying the functional equation (2) and*

$$N\left(f_a(x) - A(x), r\right) \geq N'\left(\beta(x), \frac{L^{1-i}}{1-L}r\right), \ \forall \ x \in X, r > 0. \tag{50}$$

Proof. Let d be a general metric on Ω, such that

$$d(g,h) = inf\left\{K \in (0,\infty) | N\left(g(x)-h(x),r\right) \geq N'\left(\beta(x),Kr\right), x \in X\right\}.$$

It is easy to see that (Ω, d) is complete. Define $T : \Omega \to \Omega$ by $Tg(x) = \dfrac{1}{\mu_i} g(\mu_i x)$, for all $x \in X$. For $g, h \in \Omega$, we have

$$\begin{aligned} d(g,h) = K \quad & \Rightarrow \quad N\left(g(x)-h(x),r\right) \geq N'\left(\beta(x),Kr\right) \\ & \Rightarrow \quad N\left(\frac{g(\mu_i x)}{\mu_i} - \frac{h(\mu_i x)}{\mu_i}, r\right) \geq N'\left(\beta(\mu_i x), K\mu_i r\right) \\ & \Rightarrow \quad N\left(Tg(x)-Th(x),r\right) \geq N'\left(\beta(x),KLr\right) \\ & \Rightarrow \quad d\left(Tg(x),Th(x)\right) \leq KL \\ & \Rightarrow \quad d\left(Tg,Th\right) \leq Ld(g,h) \qquad \forall g,h \in \Omega. \end{aligned}$$

Therefore T is strictly contractive mapping on Ω with Lipschitz constant L. Replacing (x, y, z, t) by $(2x, x, 0, 0)$ in (48) and using oddness of f_a, we get

$$N\left(f_a(2x)-2f_a(x),r\right) \geq N'\left(\vartheta(2x,x,0,0),r\right), \qquad \forall \quad x \in X, r > 0. \tag{51}$$

Using $(NAF2)$ in (51), we arrive

$$N\left(\frac{f_a(2x)}{2} - f_a(x), r\right) \geq N'\left(\vartheta(2x,x,0,0),2r\right), \qquad \forall \quad x \in X, r > 0. \tag{52}$$

With the help of (49), when $i = 0$, it follows from (52), that

$$N\left(\frac{f_a(2x)}{2} - f_a(x), r\right) \geq N'\left(\beta(x),Lr\right) \Rightarrow d(Tf_a, f_a) \leq L = L^1 = L^{1-i}. \tag{53}$$

Replacing x by $\dfrac{x}{2}$ in (51), we obtain

$$N\left(f_a(x) - 2f_a\left(\frac{x}{2}\right), r\right) \geq N'\left(\vartheta\left(x,\frac{x}{2},0,0\right),r\right), \qquad \forall \quad x \in X, r > 0. \tag{54}$$

When $i = 1$, it follows from (54), that

$$N\left(f_a(x) - 2f_a(\frac{x}{2}), r\right) \geq N'\left(\beta(x),r\right) \Rightarrow d(f_a, T_{f_a}) \leq 1 = L^{1-i}. \tag{55}$$

Then from (53) and (55), we can conclude,

$$d(f_a, T_{f_a}) \leq L^{1-i} < \infty.$$

Now from the fixed point alternative in both cases, it follows that there exists a fixed point A of T in Ω such that

$$A(x) = N - \lim_{n\to\infty} \frac{f_a\left(\mu_i^n x\right)}{\mu_i^n}, \qquad \forall x \in X. \tag{56}$$

Replacing (x, y, z, t) by $(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n t)$ in (48), we arrive

$$N\left(\frac{1}{\mu_i^n} D\ f_a(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n t), r\right) \geq N'\left(\vartheta(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n t), \mu_i^n r\right), \qquad \forall \quad x \in X, r > 0.$$

By proceeding the same procedure in the Theorem 3.1 we can prove the function, $A : X \to Y$ is additive and it satisfies the functional equation (2). Since A is unique fixed point of T in the set

$$\Delta = \{f_a \in \Omega | d(f_a, A) < \infty\},$$

therefore A is a uniqe function such that

$$N\left(f_a(x) - A(x), r\right) \geq N'\left(\beta(x), Kr\right), \qquad \forall \quad x \in X, r > 0. \tag{57}$$

Again using the fixed point alternative, we obtain

$$\begin{aligned} & & d(f_a, A) \leq & \frac{1}{1-L} d(f_a, Tf_a) \\ \Rightarrow & & d(f_a, A) \leq & \frac{L^{1-i}}{1-L} \\ \Rightarrow & & N\left(f_a(x) - A(x), r\right) \geq & N'\left(\beta(x), \frac{L^{1-i}}{1-L} r\right). \end{aligned} \tag{58}$$

This completes the proof of the theorem. □

From Theorem 4.3, we obtain the following corollary concerning the stability for the functional equation (2).

Corollary 4.4. *Suppose that a odd function $f_a : X \to Y$ satisfies the inequality*

$$N\left(D\ f_a(x, y, z, t), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon\left(||x||^s + ||y||^s + ||z||^s + ||t||^s\right), r\right), \\ N'\left(\epsilon\left\{||x||^s\ ||y||^s\ ||z||^s\ ||t||^s + \left(||x||^{4s} + ||y||^{4s} + ||z||^{4s} + ||t||^{4s}\right)\right\}, r\right), \end{cases} \tag{59}$$

for all $x, y, z, t \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique aditive mapping $A : X \to Y$ such that

$$N\left(f_a(x) - A(x), r\right) \geq \begin{cases} N'\left(\epsilon, 2r\right) \\ N'\left(\epsilon ||x||^s \frac{(2^s+1)}{2^s}, \frac{2}{|2-2^s|} r\right), & s < 1 \quad or \quad s > 1; \\ N'\left(\epsilon ||x||^{4s} \frac{(2^{4s}+1)}{2^{4s}}, \frac{2}{|2-2^{4s}|} r\right), & s < \frac{1}{4} \quad or \quad \frac{1}{4}; \end{cases} \tag{60}$$

for all $x \in X$ and all $r > 0$.

Proof. Setting

$$\vartheta(x, y, z, t) = \begin{cases} \epsilon \\ \epsilon\left\{||x||^s + ||y||^s + ||z||^s + ||t||^s\right\}, \\ \epsilon\left\{||x||^s ||y||^s ||z||^s ||t||^s + \left[||x||^{4s} + ||y||^{4s} + ||z||^{4s} + ||t||^{4s}\right]\right\} \end{cases}$$

for all $x, y, z, t \in X$. Then,

$$N'\left(\vartheta(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n t), \mu_i^n r\right)$$
$$= \begin{cases} N'\left(\epsilon, \mu_i^n r\right) \\ N'\left(\epsilon\left\{||\mu_i^n x||^s + ||\mu_i^n y||^s + ||\mu_i^n z||^s + ||\mu_i^n t||^s\right\}, \mu_i^n r\right) \\ N'\left(\epsilon\Big\{||\mu_i^n x||^s\ ||\mu_i^n y||^s\ ||\mu_i^n z||^s\ ||\mu_i^n t||^s + \left[||\mu_i^n x||^{4s} + ||\mu_i^n y||^{4s} + ||\mu_i^n z||^{4s} + ||\mu_i^n t||^{4s}\right]\Big\}, \mu_i^n r\right) \end{cases}$$
$$= \begin{cases} N'\left(\epsilon, \mu_i^n r\right) \\ N'\left(\epsilon\left\{||x||^s + ||y||^s + ||z||^s + ||t||^s\right\}, (\mu_i^{1-s})^n r\right) \\ N'\left(\epsilon\left\{||x||^{4s} + ||y||^{4s} + ||z||^{4s} + ||t||^{4s}\right\} + ||x||^s\ ||y||^s\ ||z||^s\ ||t||^s, (\mu_i^{1-4s})^n r\right) \end{cases}$$
$$= \begin{cases} \to 1 \text{ as } \ n \to \infty \\ \to 1 \text{ as } \ n \to \infty \text{ for } \ s < 1 \text{ if } \ i = 0 \text{ and } \ s > 1 \text{ if } \ i = 1, \\ \to 1 \text{ as } \ n \to \infty \text{ for } \ s < \frac{1}{4} \text{ if } \ i = 0 \text{ and } \ s > \frac{1}{4} \text{ if } \ i = 1, \end{cases}$$

Thus, (47) is holds.

But we have $\beta(x) = \vartheta\left(x, \frac{x}{2}, 0, 0\right)$ has the property

$$N'\left(L\frac{1}{\mu_i}\beta(\mu_i x), r\right) = N'\left(\beta(x), r\right)\ \forall\ x \in X, r > 0.$$

Hence

$$N'\left(\beta(x), r\right) = N'\left(\vartheta\left(x, \frac{x}{2}, 0, 0\right), r\right) = \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon||x||^s\left(\dfrac{2^s+1}{2^s}\right), r\right), \\ N'\left(\epsilon||x||^{4s}\left(\dfrac{2^{4s}+1}{2^{4s}}\right), r\right). \end{cases}$$

Now,

$$N'\left(\frac{1}{\mu_i}\beta(\mu_i x), r\right) = \begin{cases} N'\left(\epsilon, \mu_i r\right), \\ N'\left(\epsilon||x||^s\left(\dfrac{2^s+1}{2^s}\right), \mu_i^{1-s} r\right), \\ N'\left(\epsilon||x||^{4s}\left(\dfrac{2^{4s}+1}{2^{4s}}\right), \mu_i^{1-4s} r\right) \end{cases} = \begin{cases} N'\left(\beta(x), \mu_i r\right), \\ N'\left(\beta(x), \mu_i^{1-s} r\right), \\ N'\left(\beta(x), \mu_i^{1-4s} r\right). \end{cases}$$

Hence the inequality (49) holds for the following cases.

Case:1 $L = 2^1$ for $s = 0$ if $i = 0$

$$N\left(f_a(x) - A(x), r\right) \geq N'\left(\beta(x), \frac{L^{1-0}}{1-L} r\right) = N'\left(\epsilon, -2r\right)$$

Case:2 $L = \left(\frac{1}{2}\right)^1$ for $s = 0$ if $i = 1$

$$N\left(f_a(x) - A(x), r\right) \geq N'\left(\beta(x), \frac{L^{1-1}}{1-L} r\right) = N'\left(\epsilon, 2r\right)$$

Case:3 $L = 2^{1-s}$ for $s < 1$ if $i = 0$

$$N\left(f_a(x) - A(x), r\right) \geq N'\left(\beta(x), \frac{L^{1-0}}{1-L} r\right) = N'\left(\epsilon||x||^s \frac{(2^s+1)}{2^s}, \frac{2}{2^s-2} r\right)$$

Case:4 $L = 2^{s-1}$ for $s > 1$ if $i = 1$

$$N\left(f_a(x) - A(x), r\right) \geq N'\left(\beta(x), \frac{L^{1-1}}{1-L}r\right) = N'\left(\epsilon ||x||^s \frac{(2^s+1)}{2^s}, \frac{2}{2-2^s}r\right)$$

Case:5 $L = 2^{1-4s}$ for $s < \frac{1}{4}$ if $i = 0$

$$N\left(f_a(x) - A(x), r\right) \geq N'\left(\beta(x), \frac{L^{1-0}}{1-L}r\right) = N'\left(\epsilon ||x||^{4s} \frac{(2^{4s}+1)}{2^{4s}}, \frac{2}{2^{4s}-2}r\right)$$

Case:6 $L = 2^{4s-1}$ for $s > \frac{1}{4}$ if $i = 1$

$$N\left(f_a(x) - A(x), r\right) \geq N'\left(\beta(x), \frac{L^{1-1}}{1-L}r\right) = N'\left(\epsilon ||x||^{4s} \frac{(2^{4s}+1)}{2^{4s}}, \frac{2}{2-2^{4s}}r\right)$$

Hence the proof is complete. □

The following theorem provide the stability result of (2) for f is even function using fixed point method. The proof of the Theorem 4.5 and Corollary 4.6 is similar to that of Theorem 4.3. Hence the details of the proof is omitted.

Theorem 4.5. *Let $f_q : X \to Y$ be a even mapping for which there exist a function $\vartheta : X^4 \to Z$ with the condition*

$$\lim_{n\to\infty} N'\left(\vartheta\left(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n t\right), \mu_i^{2n} r\right) = 1 \quad \forall\ x, y, z, t \in X, r > 0 \tag{61}$$

and satisfying the functional inequality

$$N\left(D\ f_q(x, y, z, t), r\right) \geq N'\left(\vartheta(x, y, z, t), r\right) \quad \forall\ x, y, z, t \in X, r > 0. \tag{62}$$

If there exists $L = L(i)$ such that the function

$$x \to \beta(x) = \vartheta\left(x, \frac{x}{2}, 0, 0\right),$$

has the property

$$N'\left(L\frac{1}{\mu_i^2}\beta(\mu_i x), r\right) = N'\left(\beta(x), r\right) \quad \forall\ x \in X, r > 0. \tag{63}$$

Then there exists a unique quadratic function $Q : X \to Y$ satisfying the functional equation (2) and

$$N\left(f_q(x) - Q(x), r\right) \geq N'\left(\beta(x), \frac{L^{1-i}}{1-L}r\right) \quad \forall\ x \in X, r > 0. \tag{64}$$

Corollary 4.6. *Suppose that a even function $f_q : X \to Y$ satisfies the inequality*

$$N\left(D\ f_q(x, y, z, t), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon\left(||x||^s + ||y||^s + ||z||^s + ||t||^s\right), r\right), \\ N'\left(\epsilon\left\{||x||^s\ ||y||^s\ ||z||^s\ ||t||^s + \left(||x||^{4s} + ||y||^{4s} + ||z||^{4s} + ||t||^{4s}\right)\right\}, r\right), \end{cases} \tag{65}$$

for all $r > 0$ and all $x, y, z, t \in X$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$N\left(f_q(x) - A(x), r\right) \geq \begin{cases} N'\left(\epsilon, \frac{4}{3}r\right), \\ N'\left(\epsilon ||x||^s \frac{(2^{2s}+1)}{2^{2s}}, \frac{2}{|2^{2s}-2|}r\right), & s < 2 \quad or \quad s > 2; \\ N'\left(\epsilon ||x||^{4s} \frac{(2^{4s}+1)}{2^{4s}}, \frac{4}{|2^{4s}-4|}r\right), & s < \frac{1}{2} \quad or \quad \frac{1}{2}; \end{cases} \tag{66}$$

for all $x \in X$ and all $r > 0$.

The following theorem provide the stability result of (2) for mixed case in fixed point method.

Theorem 4.7. *Let $f : X \to Y$ be a mapping for which there exist a function $\vartheta : X^4 \to Z$ with the condition (47) and (61) satisfying the functional inequality*

$$N\left(D\ f(x,y,z,t),r\right) \geq N'\left(\vartheta(x,y,z,t),r\right),\ \forall\ x,y,z,t \in X, r > 0. \tag{67}$$

If there exists $L = L(i)$ such that the function

$$x \to \beta(x) = \vartheta\left(x, \frac{x}{2}, 0, 0\right),$$

has the properties (49) and (63) for all $x \in X$. Then there exists a unique additive function $A : X \to Y$ and a unique quadratic function $Q : X \to Y$ satisfying the functional equation (2) and

$$N\left(f(x) - A(x) - Q(x), r\right) \geq N_3\left(\beta(x), r\right),\ \forall\ x \in X, r > 0. \tag{68}$$

Proof. Let $f_o(x) = \dfrac{f_a(x) - f_a(-x)}{2}$ for all $x \in X$. Then $f_o(0) = 0$ and $f_o(-x) = -f_o(x)$ for all $x \in X$. Hence

$$\begin{aligned} N\left(D\ f_o(x,y,z,t),r\right) &= N\left(D\ f_a(x,y,z,t) - D\ f_a(-x,-y,-z,-t), 2r\right) \\ &\geq \min\left\{N'\left(\vartheta(x,y,z,t),r\right), N'\left(\vartheta(-x,-y,-z,-t),r\right)\right\} \end{aligned} \tag{69}$$

for all $x, y, z, t \in X$ and all $r > 0$. Let

$$N_1(\vartheta(x,y,z,t),r) = \min\left\{N'\left(\vartheta(x,y,z,t),r\right), N'\left(\vartheta(-x,-y,-z,-t),r\right)\right\} \tag{70}$$

for all $x, y, z, t \in X$ and all $r > 0$. By Theorems 4.3 there exists a unique additive mapping $A : X \to Y$ such that

$$N\left(f_o(x) - A(x), r\right) \geq N_1\left(\beta(x), \frac{L^{1-i}}{1-L} r\right) \tag{71}$$

for all $x \in X$ and all $r > 0$.
Also, let $f_e(x) = \dfrac{f_q(x) + f_q(-x)}{2}$ for all $x \in X$. Then $f_e(0) = 0$ and $f_e(-x) = f_e(x)$ for all $x \in X$. Hence

$$\begin{aligned} N\left(D\ f_e(x,y,z,t),r\right) &= N\left(D\ f_q(x,y,z,t) - D\ f_q(-x,-y,-z,-t), 2r\right) \\ &\geq \min\left\{N'\left(\vartheta(x,y,z,t),r\right), N'\left(\vartheta(-x,-y,-z,-t),r\right)\right\} \end{aligned} \tag{72}$$

for all $x, y, z, t \in X$ and all $r > 0$. Let

$$N_2(\vartheta(x,y,z,t),r) = \min\left\{N'\left(\vartheta(x,y,z,t),r\right), N'\left(\vartheta(-x,-y,-z,-t),r\right)\right\} \tag{73}$$

for all $x, y, z, t \in X$ and all $r > 0$. By Theorem 4.5, there exists unique quadratic mapping $Q : X \to Y$ such that

$$N\left(f_e(x) - Q(x), r\right) \geq N_2\left(\beta(x), \frac{L^{1-i}}{1-L} r\right) \tag{74}$$

for all $x \in X$ and all $r > 0$. Define

$$f(x) = f_e(x) + f_o(x) \tag{75}$$

for all $x \in x$. From (68),(70) and (71), we arrive

$$\begin{aligned} N\left(f(x)-A(x)-Q(x),r\right) &= N\left(f_e(x)+f_o(x)-A(x)-Q(x),r\right) \\ &\geq \min\left\{N\left(f_o(x)-A(x),\frac{r}{2}\right),N\left(f_e(x)-Q(x),\frac{r}{2}\right)\right\} \\ &\geq \min\left\{N_1\left(\beta(x),\frac{L^{1-i}}{1-L}r\right),N_2\left(\beta(x),\frac{L^{1-i}}{1-L}r\right)\right\} \\ &= N_3\left(\beta(x),r\right) \end{aligned}$$

where

$$N_3\left(\beta(x),r\right) = \min\left\{N_1\left(\beta(x),\frac{L^{1-i}}{1-L}r\right),N_2\left(\beta(x),\frac{L^{1-i}}{1-L}r\right)\right\} \tag{76}$$

for all $x \in X$ and all $r > 0$. Hence the theorem is proved. □

The following corollary is the immediate consequence of Corollaries 4.4, 4.6 and Theorem 4.7 concerning the stability for the functional equation (2) in fixed point method.

Corollary 4.8. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(D\ f(x,y,z,t),r\right) \geq \begin{cases} N'\left(\epsilon,r\right), \\ N'\left(\epsilon\left(||x||^s+||y||^s+||z||^s+||t||^s\right),r\right), \\ N'\left(\epsilon\left\{||x||^s\ ||y||^s\ ||z||^s\ ||t||^s+\left(||x||^{4s}+||y||^{4s}+||z||^{4s}+||t||^{4s}\right)\right\},r\right), \end{cases} \tag{77}$$

for all $r > 0$ and all $x,y,z,t \in X$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x)-A(x)-Q(x),r\right) \geq \begin{cases} N_3\left(\epsilon,r\right), \\ N_3\left(\epsilon||x||^s\frac{(2^{2s}+1)}{2^{2s}},r\right), & s \neq 1,2; \\ N_3\left(\epsilon||x||^{4s}\frac{(2^{4s}+1)}{2^{4s}},r\right), & s \neq \frac{1}{4},\frac{1}{2}; \end{cases} \tag{78}$$

for all $x \in X$ and all $r > 0$.

References

[1] J.Aczel and J.Dhombres, *Functional Equations in Several Variables*, Cambridge Univ, Press, (1989).

[2] T.Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, 2(1950), 64-66.

[3] L.M.Arriola and W.A.Beyer, *Stability of the Cauchy functional equation over p−adic fields*, Real Analysis Exchange, 31(125)(2005).

[4] M.Arunkumar, G.Ganapathy, S.Murthy and S.Karthikeyan, *Stability of the Generalized Arun-additive functional equation in Instutionistic fuzzy normed spaces*, International Journal Mathematical Sciences and Engineering Applications, 4(V)(2010), 135-146.

[5] M.Arunkumar and S.Karthikeyan, *Solution and Stability Of n-Dimensional Mixed Type Additive and Quadratic Functional Equation*, Far East Journal of Applied Mathematics, 54(1)(2011), 47-64.

[6] M.Arunkumar and John M.Rassias, *On the generalized Ulam-Hyers stability of an AQ-mixed type functional equation with counter examples*, Far East Journal of Applied Mathematics, 71(2)(2012), 279-305.

[7] M.Arunkumar, *Solution and stability of modified additive and quadratic functional equation in generalized 2-normed spaces*, International Journal Mathematical Sciences and Engineering Applications, 7(I)(2013), 383-391.

[8] M.Arunkumar, *Generalized Ulam - Hyers stability of derivations of a AQ - functional equation*, Cubo A Mathematical Journal dedicated to Professor Gaston M. N'Gurkata on the occasion of his 60th Birthday, 15(1)(2013), 159-169.

[9] M.Arunkumar and P.Agilan, *Additive Quadratic functional equation are Stable in Banach space: A Fixed Point Approach*, International Journal of pure and Applied Mathematics, 86(6)(2013), 951-963.

[10] M.Arunkumar and P.Agilan, *Additive Quadratic Functional Equation are Stable in Banach space: A Direct Method*, Far East Journal of Applied Mathematics, 80(1)(2013), 105-121.

[11] M.Arunkumar, G.Shobana and S.Hemalatha, *Ulam - Hyers, Ulam - TRassias, Ulam-GRassias, Ulam - JRassias Stabilities of A Additive - Quadratic Mixed Type Functional Equation In Banach Spaces*, International Conference on Mathematics and Computing, ICMCE (Accepted).

[12] M.Arunkumar, P.Agilan and C.Devi Shyamala Mary, *Permanence of A Generalized AQ Functional Equation In Quasi-Beta Normed Spaces*, International Conference on Mathematics and Computing, ICMCE (Accepted).

[13] M.Arunkumar, *Perturbation of n Dimensional AQ - mixed type Functional Equation via Banach Spaces and Banach Algebra : Hyers Direct and Alternative Fixed Point Methods*, International Journal of Advanced Mathematical Sciences (IJAMS) (Submitted).

[14] S.Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, (2002).

[15] M.Eshaghi Gordji, *Stability of an Additive-Quadratic Functional Equation of Two Variables in F-Spaces*, J. Nonlinear Sci. Appl., 2(4)(2009), 251-259

[16] M.Eshaghi Gordji, N.Ghobadipour and J.M.Rassias, *Fuzzy Stability of Additive-Quadratic Functional Equations*, arxiv:0903.0842v1 [math.fa], (2009).

[17] M.Eshaghi Gordji and M.Bavand Savadkouhi, *Stability of cubic and quartic functional equations in non-Archimedean spaces*, Acta Appl. Math., 110(2009).

[18] P.Gavruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., 184 (1994), 431-436.

[19] D.H.Hyers, *On the stability of the linear functional equation*, Proc.Nat. Acad.Sci.,U.S.A., 27(1941), 222-224.

[20] D.H.Hyers, G.Isac and Th.M.Rassias, *Stability of functional equations in several variables*, Birkhauser, Basel, (1998).

[21] S.M.Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, (2001).

[22] Pl.Kannappan, *Functional Equations and Inequalities with Applications*, Springer Monographs in Mathematic.

[23] A.Khrennikov, *Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models*, Kluwer Academic, Dordrecht, (1997).

[24] B.Margoils and J.B.Diaz, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc., 126(74)(1968), 305-309.

[25] A.K. Mirmostafaee and M.S.Moslehian, *Fuzzy versions of Hyers-Ulam-Rassias theorem*, Fuzzy Sets and Systems, 159(6)(2008), 720729.

[26] A.K.Mirmostafaee, M.Mirzavaziri and M.S.Moslehian, *Fuzzy stability of the Jensen functional equation*, Fuzzy Sets and Systems, 159(6)(2008), 730-738.

[27] A.K.Mirmostafaee and M.S.Moslehian, *Stability of additive mappings in non-Archimedean fuzzy normed spaces*, Fuzzy Sets Syst., 160(2009).

[28] S.A.Mohiuddine, *Stability of Jensen functional equation in intuitionistic fuzzy normed space*, Chaos, Solitons Fractals,

42(2009).

[29] M.S.Moslehian and Th.M. Rassias, *Stability of functional equations in non-Archimedean spaces*, Applicable Analysis and Discrete Mathematics, 1(2007).

[30] M.S.E.I.Naschie, *A review of applications and results of E-infinity theory*, Int. J. Nonlinear Sci. Numer. Simul., 8(11)(2007).

[31] M.S.E.I.Naschie, *On a fuzzy Khler-like manifold which is consistent with two-slit experiment*, Int. J. Nonlinear Sci. Numer. Simul., 6(95)(2005).

[32] M.S.E.I.Naschie, *The idealized quantum two-slit Gedanken experiment revisited-criticism and reinterpretation*, Chaos, Solitons Fractals, 27(9)(2006).

[33] M.S.E.I.Naschie, *On two new fuzzy Kahler manifolds*, Klein modular space and Hooft holographic principles, Chaos, Solitons Fractals 29(2006).

[34] M.S.E.I.Naschie, *Fuzzy dodecahedron topology and E-infinity space-time as a model for quantum physics*, Chaos, Solitons Fractals, 30(2006).

[35] Matina J.Rassias, M.Arunkumar and S.Ramamoorthi, *Stability of the Leibniz additive-quadratic functional equation in Quasi-Beta normed space: Direct and fixed point methods*, Journal Of Concrete And Applicable Mathematics (JCAAM), 14(1-2)(2014), 22-46.

[36] C.Park, *Orthogonal Stability of an Additive-Quadratic Functional Equation*, Fixed Point Theory and Applications, doi:10.1186/1687-1812-2011-66.

[37] J.M.Rassias, *On approximately of approximately linear mappings by linear mappings*, J. Funct. Anal. USA, 46(1982), 126-130.

[38] J.M.Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., 72(1978), 297-300.

[39] J.M.Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Acedamic Publishers, Dordrecht, Bostan London, (2003).

[40] K.Ravi, M.Arunkumar and J.M.Rassias, *On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation*, International Journal of Mathematical Sciences, 3(08)(2008), 36-47.

[41] A.M. Robert, *A Course in $p-$adic Analysis*, Springer-Verlag, New York, (2000).

[42] R.Saadati, *A note on Some results on the IF-normed spaces*, Chaos, Solitons Fractals, 41(2009).

[43] Sun Sook Jin and Yang Hi Lee, *A Fixed Point Approach to The Stability of the Cauchy Additive and Quadratic Type Functional Equation*, Journal of Applied Mathematics 16 pages, doi:10.1155/2011/817079.

[44] Sun Sook Jin, Yang Hi Lee, *Fuzzy Stability of a Quadratic-Additive Functional Equation*, International Journal of Mathematics and Mathematical Sciences 6 pages, doi:10.1155/2011/504802

[45] V.S.Vladimirov, I.V.Volovich and E.I.Zelenov, *$p-$adic Analysis and Mathematical Physics*, World Scientific, Singapore, (1994).

[46] S.M.Ulam, *Problems in Modern Mathematics,* Science Editions, Wiley, NewYork, (1964).

[47] T.Z.Xu, J.M.Rassias and W.X.Xu, *Stability of a general mixed additive-cubic functional equation in non-Archimedean fuzzy normed spaces*, J. Math. Phy. 51(2010).

[48] G.Zamani Eskandani, Hamid Vaezi and Y.N.Dehghan, *Stability of a Mixed Additive and Quadratic Functional Equation in Non-Archimedean Banach Modules*, Taiwanese Journal of Mathematics, 14(4)(2010), 1309-1324.