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# Distinctive Properties of Strongly Magic Square

**Research Article**

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**Abstract:** Magic squares have turned up throughout history, some in a mathematical context and others in philosophical or religious contexts.A magic square is a square array of numbers where the rows, columns, diagonals and co-diagonals add up to the same number. The paper discuss about a well-known class of magic squares; the strongly magic square. The strongly magic square is a magic square with a stronger property that the sum of the entries of the sub-squares taken without any gaps between the rows or columns is also the magic constant. In this paper a generic definition for Strongly Magic Squares is given.Some advanced mathematical properties of strongly magic squares is also discussed.

**Keywords:** Magic Square, Magic Constant, Strongly Magic Square.



## 1. Introduction

A magic square is a square array of numbers where the rows and column add up to the same number. The constant sum is called magic constant or magic number. There are many recreational aspects of magic squares. But, apart from the usual recreational aspects, it is found that these squares possess mathematical properties.

## 2. Notations and Mathematical Preliminaries

(A). *Magic Square:* A magic square of order n is an $n^{th}$ order matrix $[a_{ij}]$ such that

$$\sum_{j=1}^{n} a_{ij} = \rho \quad \text{for} \quad i = 1, 2, \ldots, n \tag{1}$$

$$\sum_{j=1}^{n} a_{ji} = \rho \quad \text{for} \quad i = 1, 2, \ldots, n \tag{2}$$

$$\sum_{i=1}^{n} a_{ii} = \rho, \quad \sum_{i=1}^{n} a_{i,\ n-i+1} = \rho \tag{3}$$

Equation (1) represents the row sum, equation (2) represents the column sum, equation (3) represents the diagonal and co-diagonal sum and symbol$\rho$ represents the magic constant [2].

(B). *Magic Constant:* The constant $\rho$ in the above definition is known as the magic constant or magic number. The magic constant of the magic square A is denoted as $\rho(A)$. For example Fig 1 is a well-known 10th-century $4 \times 4$ magic square on

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display in the Parshvanath Jain temple in Khajuraho, India [3] and fig 2 [4] refers to well known Sri Rama Chakra; both having magic constant 34.

| | | | |
|---|---|---|---|
| 7 | 12 | 1 | 14 |
| 2 | 13 | 8 | 11 |
| 16 | 3 | 10 | 5 |
| 9 | 6 | 15 | 4 |

Fig.1: The Chautisa Yantra

| | | | |
|---|---|---|---|
| 9 | 16 | 5 | 4 |
| 7 | 2 | 11 | 14 |
| 12 | 13 | 8 | 1 |
| 6 | 3 | 10 | 15 |

Fig.2: The Sri Rama Chakra

(C). *Semi Magic Square:* Semi-magic square is a square that fails to be a magic square only because one or both of the main diagonal sums do not equal the magic constant [6].

(D). *Strongly magic square (SMS):* Let $A = [a_{ij}]$ be a matrix of order $n^2 \times n^2$, such that

$$\sum_{j=1}^{n^2} a_{ij} = \rho \quad \text{for} \quad i = 1, 2, \ldots, n^2 \tag{4}$$

$$\sum_{j=1}^{n^2} a_{ji} = \rho \quad \text{for} \quad i = 1, 2, \ldots, n^2 \tag{5}$$

$$\sum_{i=1}^{n^2} a_{ii} = \rho, \qquad \sum_{i=1}^{n^2} a_{i,\ n^2-i+1} = \rho \tag{6}$$

$$\sum_{l=0}^{n-1} \sum_{k=0}^{n-1} a_{i+k,\ j+l} = \rho \quad \text{for} \quad i\ , j = 1, 2, \ldots, n^2 \tag{7}$$

where the subscripts are congruent modulo $n^2$. Equation (4) represents the row sum, equation (5) represents the column sum, equation (6) represents the diagonal and co-diagonal sum, equation (7) represents the $n \times n$ sub-square sum with no gaps in between the elements of rows or columns and is denoted as $M_{0C}^{(n)}$ or $M_{0R}^{(n)}$ and $\rho$ is the magic constant.

*Note:* The $n^{th}$ order subsquare sum with k column gaps or k row gaps is generally denoted as $M_{kC}^{(n)}$ or $M_{kR}^{(n)}$ respectively.

(E). *Examples of Strongly Magic Square*

$$A = \begin{bmatrix} 16 & 5 & 4 & 9 \\ 2 & 11 & 14 & 7 \\ 13 & 3 & 8 & 10 \\ 1 & 12 & 15 & 6 \end{bmatrix} \quad \text{is of order 4.}$$

In $A$ all sixteen $2 \times 2$ subsquares (allowing wrap-around) have entries adding to 34.

$$B = \begin{bmatrix} 26 & 64 & 78 & 44 & 1 & 15 & 62 & 46 & 33 \\ 39 & 5 & 16 & 57 & 50 & 34 & 21 & 68 & 79 \\ 58 & 54 & 29 & 22 & 72 & 74 & 40 & 9 & 11 \\ 24 & 71 & 73 & 42 & 8 & 10 & 60 & 53 & 28 \\ 43 & 3 & 14 & 61 & 48 & 32 & 25 & 66 & 77 \\ 56 & 49 & 36 & 20 & 67 & 81 & 38 & 4 & 18 \\ 19 & 69 & 80 & 37 & 6 & 17 & 55 & 51 & 35 \\ 41 & 7 & 12 & 59 & 52 & 30 & 23 & 70 & 75 \\ 63 & 47 & 31 & 27 & 65 & 76 & 45 & 2 & 13 \end{bmatrix} \quad textisoforder9.$$

In $B$ all $3 \times 3$ sub-square with no gaps in between the elements of rows or columns add to the magic constant 369.

## 3. Propositions

**Proposition 3.1.** *Let $A = [a_{ij}]$ be a square matrix of order $n^2 \times n^2$ such that $a_{ij} = a$ for every $i,\ j = 1, 2, \ldots, n^2$, then $A$ denoted as $[a]$ is a Strongly magic square with $\rho(A) = n^2 a$.*

*Proof.* The sum of the $i^{th}$ row elements are given by

$$\sum_{j=1}^{n^2} a_{ij} = \sum_{j=1}^{n^2} a = n^2 a.$$

The sum of the $j^{th}$ column elements are given by

$$\sum_{j=1}^{n^2} a_{ji} = \sum_{j=1}^{n^2} a = n^2 a.$$

For diagonal sum we have,

$$\sum_{i=1}^{n^2} a_{ii} = \sum_{i=1}^{n^2} a = n^2 a.$$

For co-diagonal sum we have,

$$\sum_{i=1}^{n^2} a_{i,n^2-i+1} = \sum_{i=1}^{n^2} a = n^2 a.$$

For the $n \times n$ sub Square sum ${M_{0C}}^{(n)}$ we have,

$$\sum_{l=0}^{n-1}\sum_{k=0}^{n-1} a_{i+k,j+l} = \sum_{l=0}^{n-1}\sum_{k=0}^{n-1} a = n^2 a.$$

□

**Proposition 3.2.** *Let $A = [a_{ij}]_{n^2 \times n^2}$ be a SMS with $\rho(A) = a$ and $B = [b_{ij}]_{n^2 \times n^2}$ be another SMS with $\rho(B) = b$, then $AB$ is a semi magic square with $\rho(AB) = ab = \rho(A) \times \rho(B)$.*

*Proof.* Let $AB = C = [c_{ij}]_{n^2 \times n^2}$ where $c_{ij} = \sum_{k=1}^{n^2} a_{ik} b_{kj}$. The sum of the $i^{th}$ row elements are given by

$$\begin{aligned}\sum_{j=1}^{n^2} c_{ij} &= \sum_{j=1}^{n^2}\sum_{k=1}^{n^2} a_{ik} b_{kj} \quad \text{for} \quad i = 1, 2, \ldots, n^2 \\ &= \sum_{k=1}^{n^2} a_{ik} b_{k1} + \sum_{k=1}^{n^2} a_{ik} b_{k2} + \cdots + \sum_{k=1}^{n^2} a_{ik} b_{kn^2} \\ &= \sum_{k=1}^{n^2} a_{ik}(b_{k1} + b_{k2} + \cdots + b_{kn^2}) = \sum_{k=1}^{n^2} a_{ik}(b) \\ &= ab\end{aligned}$$

Similarly we can calculate the sum of the column elements and can be found to be $ab$. The diagonal sum;

$$\sum_{i=1}^{n^2} c_{ii} = \sum_{i=1}^{n^2}\sum_{k=1}^{n^2} a_{ik} b_{ki} = \sum_{k=1}^{n^2} a_{ik} b_{k1} + \sum_{k=1}^{n^2} a_{ik} b_{k2} + \cdots \sum_{k=1}^{n^2} a_{ik} b_{kn^2}.$$

This sum need not be equal to $ab$. We will give an example of a $4 \times 4$ SMS to illustrate the above fact. Let $A = \begin{bmatrix} 16 & 5 & 4 & 9 \\ 2 & 11 & 14 & 7 \\ 13 & 8 & 1 & 12 \\ 3 & 10 & 15 & 6 \end{bmatrix}$ with $\rho(A) = 34$ and $B = \begin{bmatrix} 3 & 13 & 2 & 16 \\ 10 & 8 & 11 & 5 \\ 15 & 1 & 14 & 4 \\ 6 & 12 & 7 & 9 \end{bmatrix}$ with $\rho(B) = 34$. Then, $AB = \begin{bmatrix} 212 & 360 & 206 & 378 \\ 368 & 212 & 370 & 206 \\ 206 & 378 & 212 & 360 \\ 370 & 206 & 368 & 212 \end{bmatrix}$ is a semi Magic Square with $\rho(AB) = 34^2$. □

**Remark 3.3.** *Similar result holds for matrix BA also.*

**Proposition 3.4.** *If* $A = [a_{ij}]$ *be a square matrix of order* $n^2 \times n^2$ *such that* $a_{ij} = a$ *for every* $i,\ j = 1, 2, \ldots, n^2$, *with* $\rho(A) = n^2 a$ *and* $B = [b_{ij}]$ *be a square matrix of order* $n^2 \times n^2$ *such that* $b_{ij} = b$ *for every* $i,\ j = 1, 2, \ldots, n^2$ *with* $\rho(B) = n^2 b$, *then AB will be a Strongly Magic Square with* $\rho(AB) = n^4 ab = \rho(A) \times \rho(B)$.

*Proof.* Let $AB = C = [c_{ij}]_{n^2 \times n^2}$, where $C = [c]$ and $c = n^2 ab$. Then by Proposition 3.1, $C$ will be an SMS with $\rho(C) = n^2(n^2 ab) = n^4 ab$. □

**Proposition 3.5.** *Let* $A = [a_{ij}]_{n^2 \times n^2}$ *be a SMS with* $\rho(A) = a$ *and* $B = [b_{ij})]_{n^2 \times n^2}$ *be another SMS with* $\rho(B) = b$, *then* $AB - BA$ *is a semi magic square with magic constant 0.*

*Proof.* By Proposition 3.2, AB and BA are clearly semi magic squares. Therefore $AB - BA$ is a semi magic square. Since if square matrices A,B are Semi magic squares, then $A - B$ is also a semi magic square with

$$\rho(A - B) = \rho(A) - \rho(B) \quad [5]$$

$$\begin{aligned}\rho(AB - BA) &= \rho(AB) - \rho(BA)\\ &= [\rho(A) \times \rho(B)] - [\rho(B) \times \rho(A)] \quad \text{(By Proposition 3.2)}\\ &= ab - ba = 0.\end{aligned}$$

□

**Proposition 3.6.** *Let A be a SMS of order n with* $A = a$, *then* $B = \left[A - \frac{a}{n} U\right]$ *is also a SMS with* $\rho(B) = 0$ *where* $U = \{1,$ *for every* $i,\ j = 1, 2, \ldots, n\}$.

*Proof.* Let $A = [a_{ij}]$ and $B = [b_{ij}] = \left[a_{ij} - \frac{a}{n}\right]$. The sum of the $i^{th}$ row elements are given by

$$\begin{aligned}\sum_{j=1}^{n} b_{ij} &= \sum_{j=1}^{n} \left[a_{ij} - \frac{a}{n}\right]\\ &= \sum_{j=1}^{n} a_{ij} - a\\ &= a - a = 0 \quad \left(\text{Since } \sum_{j=1}^{n} a_{ij} = \rho(A) = a\right).\end{aligned}$$

Similarly we can calculate the sum of the column elements. For the sum of the diagonal elements;

$$\begin{aligned}\sum_{i=1}^{n} b_{ii} &= \sum_{i=1}^{n} \left[a_{ii} - \frac{a}{n}\right]\\ &= \sum_{i=1}^{n} a_{ii} - a\\ &= a - a = 0\end{aligned}$$

For the sum of the co-diagonal elements;

$$\begin{aligned}\sum_{i=1}^{n} b_{i,n-i+1} &= \sum_{i=1}^{n} \left[a_{i,n-i+1} - \frac{a}{n}\right]\\ &= \sum_{i=1}^{n} a_{i,n-i+1} - a\\ &= a - a = 0.\end{aligned}$$

For the sum of the $n \times n$ sub square elements ${M_{0C}}^{(n)}$;

$$\sum_{l=0}^{\sqrt{n}-1}\sum_{k=0}^{\sqrt{n}-1} b_{i+k,j+l} = \sum_{l=0}^{\sqrt{n}-1}\sum_{k=0}^{\sqrt{n}-1}\left[a_{i+k,j+l} - \frac{a}{n}\right]$$
$$= \sum_{l=0}^{\sqrt{n}-1}\sum_{k=0}^{\sqrt{n}-1}\left[a_{i+k,j+l}\right] - a = 0.$$

□

**Proposition 3.7.** *Let A be a SMS of order n with* $(A) = a$, *then* $B = \left[A - \frac{a}{\sqrt{n}}U\right]$ *is also a SMS with* $\rho(B) = -(\sqrt{n}-1)\rho(A)$, *where* $U = \{1,\ \textit{for every}\ i,\ \ j = 1, 2, \ldots, n\}$.

*Proof.* Proceeding similarly as in Proposition 5 we will get the required result. □

## 4. Conclusion

While magic squares are recreational in grade school, they may be treated somewhat more seriously in different mathematical courses. The study of strongly magic squares is an emerging innovative area in which mathematical analysis can be done. Here some advanced properties regarding strongly magic squares are described.

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