



Perfect domination edge subdivision critical and stable Graphs

Research Article

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Abstract: Let G be a graph. A subset S of vertices in a graph G is a perfect dominating set if every vertex in $V \setminus S$ is adjacent to exactly one vertex in S . A graph is perfect domination edge subdivision critical if the subdivision of an arbitrary edge increases the perfect domination number. On the other hand, a graph is perfect domination edge subdivision stable if the subdivision of an arbitrary edge leaves the perfect domination number unchanged. In this paper, we initiate the study of perfect domination critical and stable graphs upon edge subdivision. We discuss some graphs which are perfect domination critical and stable.

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1. Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E , such that $|V| = n$ and $|E| = m$. A dominating set S is a subset of V such that every vertex in $V \setminus S$ is adjacent to at least one vertex in S . Further, S is a perfect dominating set if every vertex of $V \setminus S$ is adjacent to exactly one vertex in S . The perfect domination number denoted $\gamma_p(G)$ is the minimum cardinality of a perfect dominating set in G . Any perfect dominating set of cardinality $\gamma_p(G)$ is called a $\gamma_p(G)$ -set.

When we study a graphical parameter, it is equally important to study its behavior of the parameter when the graph is modified by applying the graph operations. For instance, the effects of the operations like removing or adding an edge or a vertex have been considered on the parameter domination number. The graphs whose domination number increase or decreases by such operations are named as domination critical graphs. Sumner et.al.[10] initiated the study of domination critical graphs. Favaron et. al.[6] have studied the effect of domination criticalness on the diameter of a graph. Removal of a vertex can increase the domination number by more than one, but can decrease it by at most one. Motivated by this Brigham[1] defined the concept of vertex domination critical graph. Further properties of these graphs were explored in [7]. Van der Merwe [11] initiated the study of those graphs where the total domination number decreases upon the addition of any edge. Chellali et.al. [2] studied double domination stable graphs upon edge removal. Chen et. al. [3] studied connected domination critical graphs i.e. graphs whose connected domination number decreases when an edge in the complement of

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the graph is added. Lemanska et. al. [9] have discussed the weakly connected domination stable trees. Motivated by these concepts, we study the effects of edge subdivision on the perfect domination number.

An edge $e = \{u, v\}$ is said to be subdivided if the edge uv is deleted and two new edges ux and xv are added. x is a new vertex and is called the subdivision vertex. Subdivision of an edge in a graph can cause its perfect domination number to increase, to decrease, or to remain the same. Let G_e^* denote the graph obtained on subdividing an arbitrary edge e of G . Further, the subdivision of an edge increases the number of vertices and number of edges of the graph G by one each.

We assume that, in this paper by a graph, we mean a undirected, finite graph having no loops and multiple edges. As usual, for a real number x , the symbols $\lfloor x \rfloor$, $\lceil x \rceil$ respectively denotes the largest integer not greater than x and the smallest integer not less than x .

2. Effect of Edge Subdivision

In this section, we consider the effects that subdivision of an edge in a graph has on the perfect domination number. We begin with the remark that the perfect domination number of a path P_n and a cycle C_n on n vertices is easy to compute.

Observation 2.1. For an integer $n \geq 1$, $\gamma_p(P_n) = \lceil \frac{n}{3} \rceil$.

Observation 2.2. For an integer $n \geq 1$, we have

$$\gamma_p(C_n) = \begin{cases} \frac{n}{3}, & \text{if } n \equiv 0 \pmod{3}; \\ \lfloor \frac{n}{3} \rfloor + 1, & \text{if } n \equiv 1 \pmod{3}; \\ \lfloor \frac{n}{3} \rfloor + 2, & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

Subdivision of an edge in a graph can cause its perfect domination number to increase, to decrease, or to remain the same. This can be illustrated with the following example. First, let G be a star with n vertices. Then $\gamma_p(G) = 1$ and $\gamma_p(G_e^*) = 2$, for any edge e in G . To show the decrease in γ_p , consider a Cycle graph G with 5 vertices, then $\gamma_p(G) = 3$ but $\gamma_p(G_e^*) = 2$. Finally, let G be a path with 4 vertices. Then $\gamma_p(G) = 2$ and $\gamma_p(G_e^*) = 2$, for any edge e in G . Therefore, for any graph G , we may define the following weak partition of its edge set $E(G)$, where by a weak partition of a set we mean a partition of the set in which some of the subsets may be empty.

Definition 2.3. For a graph G , we define a weak partition $E(G) = E^0(G) \cup E^+(G) \cup E^-(G)$, where

$$E^0(G) = \{e \in E(G) | \gamma_p(G_e^*) = \gamma_p(G)\},$$

$$E^+(G) = \{e \in E(G) | \gamma_p(G_e^*) > \gamma_p(G)\},$$

$$E^-(G) = \{e \in E(G) | \gamma_p(G_e^*) < \gamma_p(G)\}.$$

Definition 2.4. A graph G is defined to be perfect domination edge subdivision critical or γ_p -critical for short, if $\gamma_p(G_e^*) > \gamma_p(G)$ for every edge $e \in E(G)$.

In symbols, G is γ_p -critical if $E(G) = E^+(G)$. We too consider those graphs whose perfect domination number changes upon the subdivision of an edge. However, we consider both the types of changes, that is, increase and decrease in perfect domination number on subdividing an edge in G . We are now in a position to define our two main concepts in this paper.

Definition 2.5. A graph G is γ_p -changing if $\gamma_p(G_e^*) \neq \gamma_p(G)$ for every edge $e \in E(G)$, while a graph G is γ_p -stable if $\gamma_p(G_e^*) = \gamma_p(G)$ for every edge $e \in E(G)$.

Thus a graph G is γ_p -changing if the subdivision of any edge from G either increases or decreases the perfect domination number. That is, $E(G) = E^-(G) \cup E^+(G)$. A graph G is γ_p -stable if $E(G) = E^0(G)$. It follows that γ_p -critical graphs are a subset of γ_p -changing graphs.

Definition 2.6. *The multi-star graph $K_m(a_1, a_2, \dots, a_m)$ is a graph of order $a_1 + a_2 + \dots + a_m + m$ formed by joining a_1, a_2, \dots, a_m end-edges to m vertices of K_m .*

Example 2.7.

- (1). *Let G be a complete graph. Then we have $\gamma_p(G) = 1$. Clearly, subdividing any arbitrary edge of G , we observe that $\gamma_p(G_e^*) = 2$. Therefore, for any positive integer n , the complete graph K_n is a γ_p -changing graph.*
- (2). *Let G be a star with n vertices. Then $\gamma_p(G) = 1$. But $\gamma_p(G_e^*) = 2$, for any edge $e \in E(G)$. In particular, the multi-star graph $K(a_1, a_2, \dots, a_k)$ is γ_p -changing graph.*
- (3). *Let $G \cong K_{m,n}$ be a complete bipartite graph. Then G is γ_p -changing.*

The following observation follows from the definition of the edge subdivision.

Observation 2.8. *If G is connected then G_e^* is also connected. Further, if G is not a connected graph then both G and G_e^* have same number of components.*

3. Main Results

Here, we discuss some of the graphs which are γ_p -changing and γ_p -stable.

Lemma 3.1. *For $n \geq 3$, the Cycle graph C_n is γ_p -changing.*

Proof. Let C_n be a cycle graph and let $V = \{v_1, v_2, \dots, v_n\}$ denotes the vertex set of the Cycle with $n \geq 3$. We first note that, the graph obtained by subdividing an arbitrary edge of C_n is the cycle graph C_{n+1} . Now, we may consider the following possible cases here:

Case 1: Suppose $n \equiv 0(mod 3)$. Then, we have $\gamma_p(C_n) = \frac{n}{3}$ and the set $S = \{v_1, v_4, \dots, v_{n-2}\}$ will be the γ_p -set of C_n . On subdividing an arbitrary edge of C_n we obtain the cycle graph C_{n+1} . Further, we have $\gamma_p(C_{n+1}) = \lfloor \frac{n}{3} \rfloor + 1$. In fact γ_p -set of C_{n+1} is obtained by adding one more vertex to the γ_p -set of C_n . Thus, the subdivision of an edge, here, in this case increases the perfect domination number of the graph.

Case 2: Suppose $n \equiv 1(mod 3)$. This case is similar to the above.

Case 3: Suppose $n \equiv 2(mod 3)$. By Observation 2, we have $\gamma_p(C_n) = \lfloor \frac{n}{3} \rfloor + 2$. Clearly, the graph obtained by subdividing an arbitrary edge of C_n is C_{n+1} . The γ_p -set of C_{n+1} is obtained by removing exactly one vertex from the γ_p -set of C_n . Therefore, in this case, the subdivision of an arbitrary edge in the graph decreases the perfect domination number of the graph. □

Lemma 3.2. *For $n \geq 1$, the Path graph P_n is γ_p -stable if and only if $n \equiv 1$ or $2(mod 3)$.*

Proof. Let P_n be a cycle graph and $V = \{v_1, v_2, \dots, v_n\}$ denotes the vertex set of the Path with n vertices. Here also, We note that, the graph obtained by subdividing an arbitrary edge of P_n is the path graph P_{n+1} . We consider two cases here:

Case 1: Suppose $n = 3k + 1$, for some integer $k \geq 1$. from the observation 2.1, we have $\gamma_p(P_n) = k + 1$. Since, the edge subdivision graph of P_n is the graph P_{n+1} , we have again by the same observation that $\gamma_p(P_{n+1}) = k + 1$. Hence P_n is edge stable whenever $n = 3k + 1$.

Case 2: Suppose $n = 3k + 2$, for some integer $k \geq 1$. Then, the graph obtained by subdividing an arbitrary edge of P_n is again a path graph with $3(k + 1)$ vertices. Thus by observation 2.1 we have, $\gamma_p(P_n) = \gamma_p(P_{n+1}) = k + 1$. Hence P_n is edge stable whenever $n \equiv 1 \text{ or } 2 \pmod{3}$.

Conversely, suppose $n \equiv 0 \pmod{3}$. Then, $n = 3k$, for some integer $k \geq 1$. From the observation 2.1, we have $\gamma_p(P_n) = k$. But, the graph obtained by subdividing an arbitrary edge of P_n is again a path graph with $3k + 1$ vertices. Hence, we have $\gamma_p(P_{n+1}) = k + 1 > k = \gamma_p(P_n)$. \square

Corollary 3.3. *For $n \geq 1$, the Path graph P_n is γ_p -changing if and only if $n \equiv 0 \pmod{3}$.*

Proposition 3.4. *Let G be any graph with $\gamma_p(G) = 1$. Then G is γ_p -changing.*

Proof. Let G be a graph and let $S = \{v\}$ be a $\gamma_p(G)$ -set of G . Clearly, the degree of the vertex v is $n - 1$, where $n = |V|$. Then, subdividing an arbitrary edge $e = uv$ of G and inserting the vertex x between u and v it follows that the vertex u will be not be dominated by v . Therefore, at least two vertices are required to obtain the perfect dominating set of G and so we get, $\gamma_p \geq 2$. Thus G is a γ_p -changing graph. \square

Proposition 3.5. *Let G be any graph having an isolated vertex. Then \overline{G} is a γ_p -changing graph.*

Proof. Let G be any graph with an isolated vertex v . Then $S = \{v\}$ is a perfect dominating set of \overline{G} . Hence $\gamma_p(\overline{G}) = 1$ and so from Proposition 1, it follows that \overline{G} is a γ_p -changing graph. \square

Proposition 3.6. *Let G be any graph with n vertices. Then $2 \leq \gamma_p(G_e^*) \leq n$.*

Proof. The first inequality follows from the proposition 3.1 and the second inequality follows trivially. \square

The bounds established in proposition 3 are sharp. This may be seen through the following example. The case of the lower bound follows by proposition 2.1. The sharpness of the upper bound is achieved may be seen by taking a wheel $G \cong W_n$ with $n \geq 4$ vertices. Then, for any value of n , $\gamma_p(G) = 1$ but if $e = uv$ is incident to vertex at the center of G then $\gamma_p(G_e^*) = n$. Otherwise, $\gamma_p(G_e^*) = 2$.

As a consequence of above proposition, the graphs K_n , K_{n-1} , W_n are γ_p -changing graphs.

Proposition 3.7. *Let G be a graph. Then G is γ_p -changing if the end vertices of a subdividing edge lies in the γ_p -set of G .*

Proof. Suppose S is a γ_p -set of G and $e = uv$ is a subdividing edge of G . On edge subdivision, the vertex x is added to the vertex set of G . Clearly S will not be a γ_p -set of G_e^* , since x is adjacent to both u and v . Hence, $\gamma_p(G_e^*) \geq |S| + 1$. On the other hand, $S \cup \{x\}$ is a perfect dominating set of G_e^* . Hence $\gamma_p(G) < \gamma_p(G_e^*)$. \square

References

- [1] R.C.Brigham, P.Z.Chinn and R.D.Dutton, *Vertex domination critical graphs*, Networks, 18(1988), 173-179.
- [2] M.Chellali and T.W.Haynes, *Double domination stable graphs upon edge removal*, Aus. J. Comb., 47(2010), 157-164.
- [3] X.Chen, L.Sun and D.Ma, *Connected domination critical graphs*, Applied Mathematics Letters, 17(2004), 503-507.
- [4] X.Chen and M.Y.Sohn, *A note on the total domination vertex critical graphs*, Ars Comb., 88(2008), 289-294.
- [5] W.J.Desormeaux, T.W.Haynes and M.A.Henning, *Total domination changing and stable graphs upon vertex removal*, Dis. Appl. Math., 159(2011), 1548-1554.
- [6] O.Favaron, D.Sumner and E.Wojcicka, *The diameter of domination critical graphs*, J. Graph Theory, 18(1994), 723-734.

- [7] J.Fulman, D.Hanson and G.MacGillivray, *Vertex domination-critical graphs*, Networks, 25(2)(1995), 41-43.
- [8] T.W.Haynes, C.M.Mynhardt and L.C.Van der Merwe, *Total domination edge critical graphs*, Utilitas Math., 54(1988), 229-240.
- [9] M.Lemanska, J.Raczek and Gdansk, *Weakly connected domination stable trees*, Czechoslovak Math. J., 59(134)(2009), 95-100.
- [10] D.P.Sumner and P.Blitich, *Domination critical graphs*, J. Combin. Theory Ser., B34(1983), 65-76.
- [11] L.C.Van der Merwe, *Total domination edge critical graphs*, Ph.D. Thesis, University of South Africa, (1999).