International Journal of Mathematics And its Applications
Volume 4, Issue 2–D (2016), 13–22.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Generalized Köthe-Toeplitz Dual of Some Geometric Difference Sequence Spaces

Research Article

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Abstract: Cengin Türkmen and F. Basar [2] have some basic results on the sets of sequences with geometric calculus. K.Boruah, Bipan Hazarika and Mikail Et [1] have generalized geometric difference sequence spaces and its dual. In this paper we have introduce concept of $\eta-$duals and determine $\eta-$duals of some geometric difference sequence spaces.

Keywords: $\eta-$duals, geometric difference sequence spaces $l_{G_\infty}(\Delta_G^m), c^G(\Delta_G^m)$ and $c_0^G(\Delta_G^m)$, geometric complex numbers.

1. Introduction

Geometric calculus is an alternative to the usual calculus of Newton and Leibnitz. It provides differentiation and integration tools based on multiplication instead of addition. Every property in Newtonian calculus has an analog in multiplicative calculus. Generally speaking multiplicative calculus is a methodology that allows one to have a different look at problems which can be investigated via calculus. In some cases, for example for growth related problems, the use of multiplicative calculus is advocated instead of a traditional Newtonian one.

Let l_∞, c and c_0 be the classical sequence spaces of bounded, convergent and null sequences and $\mathbb{C}(G)$ be the set geometric complex numbers [2]. Kizmaz [6] introduced classical difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. Kizmaz , generalized sequence spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ were introduced by Mikail Et and Rifat Colak [4], C.Türkmen and F. Basar [2] defined $\omega(G)$, $l_\infty(G)$, $c(G)$, $c_0(G)$ and $l_p(G)$, the sequence spaces of all, bounded, convergent, null and absolutely p-summable sequences over the geometric complex field $\mathbb{C}(G)$ which correspond to the sets ω, l_∞, c, c_0 and l_p over the complex field $\mathbb{C}$, respectively. That is say that

$$\omega(G) = \{x = (x_k) : x_k \in \mathbb{C}(G) \text{ for all } k \in \mathbb{N}\},$$
$$l_\infty(G) = \{x = (x_k) \in \omega(G) : \sup_{k\in\mathbb{N}} |x_k|_G < \infty\},$$
$$c(G) = \{x = (x_k) \in \omega(G) : {}^G\lim_{k\to\infty} |x_k \ominus l|_G = 1\},$$
$$c_0(G) = \{x = (x_k) \in \omega(G) : {}^G\lim_{k\to\infty} x_k = 1\},$$
$$l_p(G) = \{x = (x_k) \in \omega(G) : G\sum_{k=0}^{\infty} |x_k|_G^{pG} < \infty\}$$

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Note: Here, symbols l_{G_∞}, c^G, c_0^G will be used to denote $l_\infty(G)$, $c(G)$, and $c_0(G)$ respectively.

C.Türkmen and F. Basar [2], Kizmaz [6], defined $l_{G_\infty}(\Delta_G)$, $c^G(\Delta_G)$ and $c_0^G(\Delta_G)$ the geometric difference sequence spaces of bounded, convergent and null sequences $x = (x_k)$ [13] as

$$
\begin{aligned}
l_{G\infty}(\Delta_G) &= \{x = (x_k) \in \omega(G) : \Delta_G x \in l_{G_\infty}\},\\
c^G(\Delta_G) &= \{x = (x_k) \in \omega(G) : \Delta_G x \in c^G\},\\
c_0^G(\Delta_G) &= \{x = (x_k) \in \omega(G) : \Delta_G x \in c_0^G\}.
\end{aligned}
$$

where $\Delta_G x = x_k \ominus x_{k+1}$. Then we have proved that this is a Banach space with the norm.

$$
\| x \|_{\Delta G}^G = |x_1|_G \oplus \| \Delta_G x \|_{G_\infty} \tag{a}
$$

where $\Delta_G x = (\Delta_G x_k) = (x_k \ominus x_{k+1})$. Also it can be proved that $c^G(\Delta_G)$ and $c_0^G(\Delta_G)$ are Banach spaces with the norm (a). Mikail Et and Rifat Colak[4], Khirod and Bipan [13], define the following new sequence spaces

$$
\begin{aligned}
l_\infty^G(\Delta_G^m) &= \{x = (x_k) : \Delta_G^m x \in l_\infty^G\},\\
c^G(\Delta_G^m) &= \{x = (x_k) : \Delta_G^m x \in c^G\},\\
c_0^G(\Delta_G^m) &= \{x = (x_k) : \Delta_G^m x \in c_0^G\}, \quad \text{where } m \in \mathbb{N}\\
\Delta_G^0 x &= (x_k)\\
\Delta_G x &= (\Delta_G x_k) = (x_k \ominus x_{k+1})\\
\Delta_G^2 x &= (\Delta_G^2 x_k) = (\Delta_G x_k \ominus \Delta_G x_{k+1})\\
&= (x_k \ominus x_{k+1} \ominus x_{k+1} \oplus x_{k+2})\\
&= (x_k \ominus e^2 \odot x_{k+1} \oplus x_{k+1})\\
\Delta_G^3 x &= (\Delta_G^3 x_k) = (\Delta_G^2 x_k \ominus \Delta_G^2 x_{k+1})\\
&= (x_k \ominus e^3 \odot x_{k+1} \oplus e^3 \odot x_{k+1} \ominus x_{k+3})\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\Delta_G^m x &= (\Delta_G^m x_k) = (\Delta_G^{m-1} x_k \ominus \Delta_G^{m-1} x_{k+1})\\
&= G\sum_{v=0}^{m} (\ominus e)^{vG} \odot e^{\binom{m}{v}} \odot x_{k+v}), \quad \text{with } (\ominus e)^{0_G} = e.
\end{aligned}
$$

Then it can be easily proved that $l_{G_\infty}(\Delta_G^m)$, $c^G(\Delta_G^m)$ and $c_0^G(\Delta_G^m)$ are normed linear spaces with norm.

$$
\| x \|_{\Delta_G}^G = G\sum_{i=1}^{m} |x_i|_G \oplus \| \Delta_G^m x \|_{G_\infty} .
$$

2. Main Results

Theorem 2.1. *The sequence spaces $l_{G_\infty}(\Delta_G^m)$, $c^G(\Delta_G^m)$ and $c_0^G(\Delta_G^m)$ are Banach spaces with the norm*

$$
\| x \|_{\Delta_G}^G = G\sum_{i=1}^{m} |x_i|_G \oplus \| \Delta_G^m x \|_{G_\infty} .
$$

Proof. Let (x_p) be a Cauchy sequence in $l_{G_\infty}(\Delta_G^m)$, where $x_p = (x_i^{(p)}) = (x_1^{(p)},\ x_2^{(p)},\ x_3^{(p)},\ \ldots)$ for $p \in \mathbb{N}$ and $x_k^{(p)}$ is the k^{th} coordinate of x_p. Then

$$\begin{aligned} \| x_p \ominus x_q \|_{\Delta G}^G &= G\sum_{i=1}^{m} |x_i^{(p)} \ominus x_i^{(q)}|_G \oplus \| \Delta_G^m (x_p \ominus x_q) \|_{G_\infty} \\ &= G\sum_{i=1}^{m} |x_i^{(p)} \ominus x_i^{(q)}|_G \oplus \sup_k |\Delta_G^m (x_p \ominus x_q)|_G \to 1 \ \text{ as } p,\ q \to \infty \end{aligned} \tag{1}$$

Hence we get $|x_p \ominus x_q|_G \to 1$ as $p,\ q \to \infty$ and for each $k \in \mathbb{N}$. Therefore $(x_k^{(p)}) = (x_k^{(1)},\ x_k^{(2)},\ x_k^{(3)}, \ldots)$ is a Cauchy sequence in $\mathbb{C}(G)$. Since $\mathbb{C}(G)$ is complete, therefore $(x_k^{(p)})$ is convergent. Suppose that $\lim_{p\to\infty} x_k^{(p)} = x_k$, for each $k \in \mathbb{N}$. Since (x_p) is a Cauchy sequence, for each $\epsilon > 1$, there exists $N = N(\epsilon)$ such that $\| x_p \ominus x_q \|_{\Delta G}^G < \epsilon$ for all $p,\ q \geq N$. Hence from (1)

$$G\sum_{i=1}^{m} |x_i^{(p)} \ominus x_i^{(q)}|_G < \epsilon \ \text{ and } \ |G\sum_{v=0}^{m} (\ominus e)^{vG} \odot e^{\binom{m}{v}} \odot (x_{k+v}^p \ominus x_{k+v}^q)|_G < \epsilon$$

for all $k \in \mathbb{N}$ and $p,\ q \geq N$. So we have

$$\begin{aligned} &\lim_{q\to\infty} G\sum_{i=1}^{m} |x_i^{(p)} \ominus x_i^{(q)}|_G = G\sum_{i=1}^{m} |x_i^{(p)} \ominus x_i|_G < \epsilon \ \text{ and} \\ &\lim_{q\to\infty} |_G \Delta_G^m (x_k^{(p)} \ominus x_k^{(q)}|_G = |\Delta_G^m (x_k^{(p)} \ominus x_k|_G < \epsilon \ \text{ for each } \ p \geq N. \end{aligned}$$

This implies $\| x_p \ominus x \|_{\Delta G}^G < \epsilon^2$ for each $p \geq N$, that is ${}^G\lim_{p\to\infty} x_p = x$, where $x = (x_k)$. Now we have to show that $x \in l_{G_\infty}(\Delta_G^m)$. We have

$$\begin{aligned} |\Delta_G^m x_k|_G &= |G\sum_{v=0}^{m} (\ominus e)^{vG} \odot e^{\binom{m}{v}} \odot x_{k+v}|_G \\ &= |G\sum_{v=0}^{m} (\ominus e)^{vG} \odot e^{\binom{m}{v}} \odot (x_{k+v} \ominus x_{k+v}^N \oplus x_{k+v}^N)|_G \\ &\leq |G\sum_{v=0}^{m} (\ominus e)^{vG} \odot e^{\binom{m}{v}} \odot (x_{k+v}^N \ominus x_{k+v})|_G \oplus |G\sum_{v=0}^{m} (\ominus e)^{vG} \odot e^{\binom{m}{v}} \odot x_{k+v}^N|_G \\ &\leq \| x^N \ominus x \|_{\Delta G}^G \oplus |\Delta_G^m x_k^N|_G = O(e). \end{aligned}$$

Therefore we get $x \in l_{G_\infty}(\Delta_G^m)$. Hence $l_{G_\infty}(\Delta_G^m)$ is a Banach space. It can be shown that $c^G(\Delta_G^m)$ and $c_0^G(\Delta_G^m)$ are closed subspace of $l_{G_\infty}(\Delta_G^m)$. Therefore these sequence spaces are Banach spaces with the same norm defined for $l_{G_\infty}(\Delta_G^m)$, above. □

Lemma 2.2.

(i). $c_0^G(\Delta_G^m) \subsetneq c_0^G(\Delta_G^{m+1})$,

(ii). $c^G(\Delta_G^m) \subsetneq c^G(\Delta_G^{m+1})$,

(iii). $l_{G_\infty}(\Delta_G^m) \subsetneq l_{G_\infty}(\Delta_G^{m+1})$.

Proof. (i) Let $x \in c_0^G(\Delta_G^m)$. Since

$$\begin{aligned} |\Delta_G^{m+1} x_k|_G &= |\Delta_G^m x_k \ominus \Delta_G^m x_{k+1}|_G \\ &\leq |\Delta_G^m x_k|_G \oplus |\Delta_G^m x_{k+1}|_G \to 1 \ \text{ as } \ k \to \infty. \end{aligned}$$

Therefore we obtain $x \in c_0^G(\Delta_G^{m+1})$. Thus $c_0^G(\Delta_G^m) \subset c_0^G(\Delta_G^{m+1})$. This inclusion is strict. For let

$$x = (e^{k^m}) = (e,\ e^{2^m},\ e^{3^m},\ e^{4^m}, \dots,\ e^{k^m}, \dots).$$

Then $x \in c_0^G(\Delta_G^{m+1})$ as $(m+1)^{th}$ geometric difference of e^{k^m} is 1 (geometric zero). But $x \notin c_0^G(\Delta_G^m)$ as m^{th} geometric difference of e^{k^m} is a constant. Hence the inclusion is strict.

The proofs of (ii) and (iii) are similar to that of (i). □

Lemma 2.3.

(i). $c_0^G(\Delta_G^m) \subsetneq c^G(\Delta_G^m)$,

(ii). $c^G(\Delta_G^m) \subsetneq l_{G_\infty}(\Delta_G^m)$.

Proof. Proofs are similar to that of Lemma 2.2. Further more , since the sequence spaces $l_{G_\infty}(\Delta_G^m)$, $c^G(\Delta_G^m)$ and $c_0^G(\Delta_G^m)$ are Banach spaces with continuous coordinates, that is,

$$\| x_p \ominus x \|_{\Delta G}^G \rightarrow 1 \ \text{ implies } \ |x_k^{(p)} \ominus x_k| \rightarrow 1$$

for each $k \in \mathbb{N}$ as $p \rightarrow \infty$, these are also BK-spaces. □

Remark 2.4. *It can be easily proved that c_0^G is a sequence algebra. But in general, $l_{G_\infty}(\Delta_G^m)$, $c^G(\Delta_G^m)$ and $c_0^G(\Delta_G^m)$ are not sequence algebra. For let $x = (e^k)$, $y = (e^{k^{m-1}})$. Clearly $x,\ y \in c_0^G(\Delta_G^m)$. But $x \odot y = (e^k \odot e^{k^{m-1}}) = (e^{k^m}) \notin c_0^G(\Delta_G^m)$ for $m \geq 2$, since m^{th} geometric difference of e^{k^m} is constant.*

Let us define the operator $D : l_{G_\infty}(\Delta_G^m) \rightarrow l_{G_\infty}(\Delta_G^m)$ as $Dx = (1,\ 1,\ 1, \dots,\ 1,\ x_{m+1},\ x_{m+2}, \dots)$, where $x = (x_1,\ x_2,\ x_3, \dots,\ x_m,\ x_{m+1} \dots) \in l_{G_\infty}(\Delta_G^m)$. It is trivial that D is bounded linear operator on $l_{G_\infty}(\Delta_G^m)$. Further more, the set

$$D[l_{G_\infty}(\Delta_G^m)] = Dl_{G_\infty}(\Delta_G^m) = \{x = (x_k) : x \in l_{G_\infty}(\Delta_G^m), x_1 = x_2 = \cdots = x_m = 1\}$$

is a subspace of $l_{G_\infty}(\Delta_G^m)$ and

$$\begin{aligned} \| x \|_{\Delta G}^G &= |x_1|_G \oplus |x_2|_G \oplus \cdots \oplus |x_m|_G \oplus \| \Delta_G^m x \|_{G_\infty} \\ &= 1 \oplus 1 \oplus \ldots \oplus 1 \oplus \| \Delta_G^m x \|_{G_\infty} \\ &= \| \Delta_G^m x \|_{G_\infty} \end{aligned}$$
$$\text{Therefore} \quad \| x \|_{\Delta G}^G = \| \Delta_G^m x \|_{G_\infty} \quad \text{in} \ \ Dl_{G_\infty}(\Delta_G^m).$$

Now let us define $\Delta^m : Dl_{G_\infty}(\Delta_G^m) \rightarrow l_{G_\infty}$; $\Delta_G^m x = y = (\Delta_G^{m-1} x_k \ominus \Delta_G^{m-1} x_{k+1})$.

Δ_G^m **is a linear homomorphism:** Let $x, y \in Dl_{G_\infty}(\Delta_G^m)$. Then

$$\begin{aligned} \Delta_G^m(x_k \oplus y_k) &= G\sum_{v=0}^{m} (\ominus e)^{vG} \odot e^{\binom{m}{v}} \odot (x_k \oplus y_k) \\ &= G\sum_{v=0}^{m} (\ominus e)^{vG} \odot e^{\binom{m}{v}} \odot x_k \oplus G\sum_{v=0}^{m} (\ominus e)^{vG} \odot e^{\binom{m}{v}} \odot y_k \\ &= \Delta_G^m x_k \oplus \Delta_G^m y_k \end{aligned}$$

$$\begin{aligned}\text{Therefore}\ \ \Delta_G^m(x \oplus y) &= \Delta_G^m x \oplus \Delta_G^m y. \text{ For } \alpha \in \mathbb{C}(G)\\ \Delta_G^m(\alpha \odot x) &= (\Delta_G^m \alpha \odot x_k)\\ &= (G\sum_{v=0}^{m}(\ominus e)^{vG} \odot e^{\binom{m}{v}} \odot \alpha \odot x_k)\\ &= (\alpha \odot G\sum_{v=0}^{m}(\ominus e)^{vG} \odot e^{\binom{m}{v}} \odot x_k)\\ &= \alpha \odot \Delta_G^m \odot x.\end{aligned}$$

This implies that Δ_G^m is a linear homomorphism. Hence $Dl_{G_\infty}(\Delta_G^m)$ and l_{G_∞} are equivalent as topological spaces [9]. Δ_G^m and $(\Delta_G^m)^{-1}$ are norm preserving and $\| \Delta_G^m \|_{G_\infty} = \| (\Delta_G^m)^{-1} \|_{G_\infty} = e$. Let $[l_{G_\infty}]'$ and $[Dl_{G_\infty}(\Delta_G^m)]'$ denote the continuous duals of l_{G_∞} and $Dl_{G_\infty}(\Delta_G^m)$, respectively. It can be shown that $s : [Dl_{G_\infty}(\Delta_G^m)]' \to [l_{G_\infty}]'$; $f_\Delta \to f_\Delta \circ (\Delta_G^m)^{-1} = f$ is a linear isometry. So $[Dl_{G_\infty}(\Delta_G^m)]'$ is equivalent to $[l_{G_\infty}]'$. In the same way, it can be shown that $Dc^G(\Delta_G^m)$ and $Dc_0^G(\Delta_G^m)$ are equivalent as topological spaces to c^G and c_0^G, respectively. Also

$$[Dc^G(\Delta_G^m)]' \cong [Dc_0^G(\Delta_G^m)]' \cong l_1^G$$

where $l_1^G = \{x = (x_k) : G\sum\limits_{k=1}^{\infty} |x_k|_G < \infty\}$.

3. Dual Spaces of $l_{G_\infty}(\Delta_G^m)$ and $c^G(\Delta_G^m)$

In this section we construct the $\eta-$dual spaces of $l_{G_\infty}(\Delta_G^m)$ and $c^G(\Delta_G^m)$. Also we show that these spaces are not perfect space relative to $\eta-$dual.

Lemma 3.1. *$\sup_k |x_k \ominus x_{k+1}|_G < \infty$ i.e. $\sup_k |\Delta x_k|_G < \infty$ if and only if $\sup_k e^{k^{-1}} \odot |x_k|_G < \infty$ and $\sup_k |x_k \ominus e^{k(k+1)^{-1}} \odot x_{k+1}|_G < \infty$.*

Corollary 3.2. *$\sup_k |\Delta_G^{m-1} x_k \ominus \Delta_G^{m-1} x_{k+1}|_G < \infty$ if and only if $\sup_k e^{k^{-1}} \odot |\Delta_G^{m-1} x_k|_G < \infty$ and $\sup_k |\Delta_G^{m-1} x_k \ominus e^{k(k+1)^{-1}} \odot \Delta_G^{m-1} x_{k+1}|_G < \infty$.*

Lemma 3.3. *$\sup_k e^{k^{-i}} \odot |\Delta_G x_k|_G < \infty$ implies $\sup_k e^{-(i+1)} \odot |x_k|_G < \infty$ for each $i \in \mathbb{N}$.*

Lemma 3.4. *$\sup_k e^{k^{-i}} \odot |\Delta_G^{m-1} x_k|_G < \infty$ implies $\sup_k e^{-(i+1)} \odot |\Delta_G^{m-(i+1)} x_k|_G < \infty$ for all $i,\ m \in \mathbb{N}$ and $1 \le i < m$.*

Corollary 3.5. *$\sup_k e^{k^{-1}} \odot |\Delta_G^{m-1} x_k| < \infty$ implies $\sup_k e^{k^{-m}} \odot |x_k| < \infty$.*

Corollary 3.6. *If $x \in l_{G_\infty}(\Delta_G^m)$ then $\sup_k e^{k^{-m}} |x_k|_G < \infty$.*

Proof.

$$\begin{aligned}\text{Let}\ \ x \in l_{G_\infty}(\Delta_G^m) &\Rightarrow \Delta_G^m x \in l_{G_\infty}\\ &\Rightarrow \sup_k |\Delta_G^m x_k|_G < \infty\\ &\Rightarrow \sup_k |\Delta_G^{m-1} x_k \ominus \Delta_G^{m-1} x_{k+1}|_G < \infty\\ &\Rightarrow \sup_k e^{k^{-1}} \odot |\Delta_G^m x_k|_G < \infty \ \ \text{by Corollary 3.2}\\ &\Rightarrow \sup_k e^{k^{-m}} \odot |x_k|_G < \infty \ \ \text{by Corollary 3.5}\end{aligned}$$

□

Definition 3.7. *If X is sequence space, it is defined as*

$$X^{\alpha} = \left\{ a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \ \textit{for each} \ x \in X \right\},$$
$$X^{\beta} = \left\{ a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \ \textit{ is convergent, for each} \ x \in X \right\},$$
$$X^{\gamma} = \left\{ a = (a_k) : \sup_n |\sum_{k=1}^{n} a_k x_k| < \infty, \ \textit{for each} \ x \in X \right\}.$$

Then X^{α}, X^{β} and X^{γ} are called α-dual (or Köthe-Toeplitz dual), β-dual (or generalized Köthe-Toeplitz dual) and γ-dual spaces of X, respectively.

If X is sequence space, then η-dual of X is define as

$$X^{\eta} = \left\{ a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k|^r < \infty \ \text{ for all } \ x \in X \right\} \ \text{ and } \ r \geq 1$$

X is set to be perfect or $\eta-$ reflexive if $X^{\eta\eta} = X$, taking $r = 1$ in above definition we get α-dual of X. We define and have prove that [13].

$$[sl_{G_\infty}(\Delta_G)]^{\alpha} = \left\{ a = (a_k) : G\sum_{k=1}^{\infty} e^k \odot |a_k|_G < \infty \right\}.$$
$$[sl_{G_\infty}(\Delta_G)]^{\beta} = \left\{ a = (a_k) : G\sum_{k=1}^{\infty} e^k \odot a_k \ \text{ is convergent with } \ G\sum_{k=1}^{\infty} |R_k|_G < \infty \right\}.$$
$$[sl_{G_\infty}(\Delta_G)]^{\gamma} = \left\{ a = (a_k) : \sup_n |G\sum_{k=1}^{n} e^k|_G < \infty, \ \ G\sum_{k=1}^{\infty} |R_k|_G < \infty \right\}$$
$$[sl_{G_\infty}(\Delta_G)]^{\eta} = \left\{ a = (a_k) : G\sum_{k=1}^{\infty} e^{kr} \odot |a_k|_G^r < \infty, \ \ \text{for all } \ x \in X \text{ and } r \geq 1 \right\}$$

where $R_k = G \sum\limits_{n=k+1}^{\infty} a_n$ and $s : l_{G_\infty}(\Delta_G) \to l_{G_\infty}(\Delta_G)$, $x \to sx = y = (1, x_2, x_3, \dots)$.

Theorem 3.8. *Let $U_1 = \left\{ a = (a_k) : G \sum\limits_{k=1}^{\infty} e^{k^{mr}} \odot |a_k|_G^r < \infty \right\}$. Then $[Dl_{G_\infty}(\Delta_G^m)]^{\eta} = U_1$.*

Proof. Let $a \in U_1$, then using Corollary 3.2 for $x \in Dl_{G_\infty}(\Delta_G^m)$ we have

$$G\sum_{k=1}^{\infty} |a_k \odot x_k|_G^r = G\sum_{k=1}^{\infty} \{ e^{k^{mr}} \odot |a_k|_G^r \odot e^{k^{-mr}} \odot |x_k|_G^r \} < \infty \ \text{ by Corollary 3.5.}$$

This implies that $a \in [Dl_{G_\infty}(\Delta_G^m)]^{\eta}$. Therefore

$$U_1 \subseteq [Dl_{G_\infty}(\Delta_G^m)]^{\eta} \tag{2}$$

Conversely, let $a \in [Dl_{G_\infty}(\Delta_G^m)]^{\eta}$. Then $G \sum\limits_{k=1}^{\infty} |a_k \odot x_k|_G^r < \infty$ by definition of η-dual, for $x \in Dl_{G_\infty}(\Delta_G^m)$. So we take.

$$x_k = \begin{cases} 1, & \text{if } k \leq m \\ e^{k^m}, & \text{if } k > m \end{cases} \tag{3}$$

Then $x = \left(1, 1, 1, \dots, 1, e^{(m+1)^m}, e^{(m+2)^m}, \dots \right) \in Dl_{G_\infty}(\Delta_G^m)$. Therefore

$$\begin{aligned} G\sum_{k=1}^{\infty} e^{k^{mr}} \odot |a_k|_G^r &= G\sum_{k=1}^{m} e^{k^{mr}} \odot |a_k|_G^r \oplus G \sum_{k=m+1}^{\infty} e^{k^{mr}} \odot |a_k|_G^r \\ &= G\sum_{k=1}^{m} e^{k^{mr}} \odot |a_k|_G^r \oplus G\sum_{k=1}^{\infty} |a_k \odot x_k|_G^r < \infty \end{aligned}$$

Since $a_k \odot x_k = 1$ (the geometric zero)for $k = 1,\ 2\ldots, m$. Therefore $a \in U_1$. This implies

$$[Dl_{G_\infty}(\Delta_G^m)]^\eta \subseteq U_1. \tag{4}$$

Then from (2) and (4), we get $[Dl_{G_\infty}(\Delta_G^m)]^\eta = U_1$. □

Theorem 3.9. $[Dl_{G_\infty}(\Delta_G^m)]^\eta = [Dc^G(\Delta_G^m)]^\eta$.

Proof. Since $Dc^G(\Delta_G^m) \subseteq Dl_{G_\infty}(\Delta_G^m)$, hence

$$[Dl_{G_\infty}(\Delta_G^m)]^\eta \subseteq [Dc^G(\Delta_G^m)]^\eta. \tag{5}$$

Again let $a \in [Dc^G(\Delta_G^m)]^\eta$. Then $G\sum_{k=1}^{\infty} |a_k \odot x_k|_G^r < \infty$ for each $x \in Dc^G(\Delta_G^m)$. If we take $x = (x_k)$ which is defined in (3), we have

$$G\sum_{k=1}^{\infty} e^{k^{mr}} \odot |a_k|_G^r = G\sum_{k=1}^{m} e^{k^{mr}} \odot |a_k|_G^r \oplus G\sum_{k=1}^{\infty} |a_k \odot x_k|_G^r < \infty.$$

This implies that $a \in [Dl_{G_\infty}(\Delta_G^m)]^\eta$. Thus

$$[Dc^G(\Delta_G^m)]^\eta \subseteq [Dl_{G_\infty}(\Delta_G^m)]^\eta. \tag{6}$$

From (5) and (6)

$$[Dl_{G_\infty}(\Delta_G^m)]^\eta = [Dc^G(\Delta_G^m)]^\eta.$$

□

Theorem 3.10.

(i). $[l_{G_\infty}(\Delta_G^m)]^\eta = [Dl_{G_\infty}(\Delta_G^m)]^\eta$.

(ii). $[c^G(\Delta_G^m)]^\eta = [Dc^G(\Delta_G^m)]^\eta$.

Proof.

(i). Since $Dl_{G_\infty}(\Delta_G^m) \subseteq l_{G_\infty}(\Delta_G^m)$, so

$$[l_{G_\infty}(\Delta_G^m)]^\eta \subseteq [Dl_{G_\infty}(\Delta_G^m)]^\eta. \tag{7}$$

Let $a \in [Dl_{G_\infty}(\Delta_G^m)]^\eta$ and $x \in l_{G_\infty}(\Delta_G^m)$. From Corollary 3.6, we have

$$G\sum_{k=1}^{\infty} |a_k \odot x_k|_G^r = G\sum_{k=1}^{\infty} e^{k^{mr}} \odot |a_k|_G^r \odot (e^{k^{-mr}} \odot |x_k|_G^r) < \infty.$$

Hence $a \in [l_{G_\infty}(\Delta_G^m)]^\eta$. Therefore

$$[Dl_{G_\infty}(\Delta_G^m)]^\eta \subseteq [l_{G_\infty}(\Delta_G^m)]^\eta. \tag{8}$$

From (7) and (8) we have

$$[l_{G_\infty}(\Delta_G^m)]^\eta = [Dl_{G_\infty}(\Delta_G^m)]^\eta$$

(ii). Since $Dc^G(\Delta_G^m) \subseteq c^G(\Delta_G^m)$ implies

$$[c^G(\Delta_G^m)]^\eta \subseteq [Dc^G(\Delta_G^m)]^\eta. \tag{9}$$

Let $a \in [Dc^G(\Delta_G^m)]^\eta$ and $x \in c^G(\Delta_G^m)$. From Corollary 3.6 we get

$$G\sum_{k=1}^{\infty} |a_k \odot x_k|_G^r = G\sum_{k=1}^{\infty} e^{k^{mr}} \odot |a_k|_G^r \odot (e^{k^{-mr}} \odot |x_k|_G^r) < \infty.$$

for $x \in c^G(\Delta_G^m) \subseteq l_{G_\infty}(\Delta_G^m)$. Hence $a \in [c^G(\Delta_G^m)]^\eta$. Therefore

$$[Dc^G(\Delta_G^m)]^\eta \subseteq [c^G(\Delta_G^m)]^\eta. \tag{10}$$

From (9) and (10)

$$[c^G(\Delta_G^m)]^\eta = [Dc^G(\Delta_G^m)]^\eta.$$

□

Theorem 3.11. *Let X stand for l_{G_∞} or c^G. Then*

$$[X(\Delta_G^m)]^\eta = \{a = (a_k) : G\sum_{k=1}^{\infty} e^{k^{mr}} \odot |a_k|_G^r < \infty\}.$$

Proof.

$$\begin{aligned}
[l_{G_\infty}(\Delta_G^m)]^\eta &= [Dl_{G_\infty}(\Delta_G^m)]^\eta \text{ by Theorem 3.10} \\
&= \{a = (a_k) : G\sum_{k=1}^{\infty} e^{k^{mr}} \odot |a_k|_G^r < \infty\} \text{ by Theorem 3.8}
\end{aligned}$$

Again

$$\begin{aligned}
[c^G(\Delta_G^m)]^\eta &= [Dc^G(\Delta_G^m)]^\eta \text{ by Theorem 3.10} \\
&= [Dl_{G_\infty}(\Delta_G^m)]^\eta \text{ by Theorem 3.9} \\
&= \{a = (a_k) : G\sum_{k=1}^{\infty} e^{k^{mr}} \odot |a_k|_G^r < \infty\} \text{ by Theorem 3.8} \\
[l_{G_\infty}(\Delta_G^m)]^\eta &= [c^G(\Delta_G^m)]^\eta
\end{aligned}$$

□

Corollary 3.12. *For $X = l_{G_\infty}$ or c^G, we get*

$$[X(\Delta_G)]^\eta = \{a = (a_k) : G\sum_{k=1}^{\infty} e^{k^r} \odot |a_k|_G^r < \infty\}, \text{ and}$$

$$[X(\Delta_G^2)]^\eta = \{a = (a_k) : G\sum_{k=1}^{\infty} e^{k^{2r}} \odot |a_k|_G^r < \infty\}.$$

Proof. Putting $m = 1$ and $m = 2$ in Theorem 3.11 the results follows. Hence $[l_{G_\infty}(\Delta_G)]^\eta = [c^G(\Delta_G)]^\eta$ and $[l_{G_\infty}(\Delta_G^2)]^\eta = [c^G(\Delta_G^2)]^\eta$. □

Theorem 3.13. *Let X stand for l_{G_∞} or c^G and $U_2 = \{a = (a_k) : \sup_k e^{k^{-mr}} \odot |a_k|_G^r < \infty\}$. Then $[X(\Delta_G^m)]^{\eta\eta} = U_2$.*

Proof. Let $a \in U_2$ and $x \in [X(\Delta_G^m)]^\eta$. Then by definition of U_2 and by Theorem 3.8, we have

$$G\sum_{k=1}^{\infty}|a_k \odot x_k|_G^r = G\sum_{k=1}^{\infty} e^{k^{mr}} \odot |x_k|_G^r \odot e^{k^{-mr}} \odot |a_k|_G^r$$
$$\leq G\sum_{k=1}^{\infty} e^{k^{mr}} \odot |x_k|_G^r \odot \sup_k e^{k^{-mr}} \odot |a_k|_G^r < \infty.$$

Hence $a \in [X(\Delta_G^m)]^{\eta\eta}$.

Conversely, let $a \in [X(\Delta_G^m)]^{\eta\eta}$ and $a \notin U_2$. Then we have

$$\sup_k e^{k^{-mr}} \odot |a_k|_G^r = \infty.$$

Hence there exists a strictly increasing sequence $(e^{k(i)})$ of geometric integers [2], where $k(i)$ is a strictly increasing sequence of positive integers such that

$$e^{[k(i)]^{-mr}} \odot |a_{k(i)}|_G^r > e^{i^{mr}}.$$

Let us define the sequence x by

$$x_k = \begin{cases} \left(|a_{k(i)}|_G\right)^{-1_G}, & k = k(i) \\ 1, & k \neq k(i) \end{cases}$$

where $\left(|a_{k(i)}|_G\right)^{-1_G}$ is the geometric inverse of $|a_{k(i)}|_G$ so that

$$|a_{k(i)}|_G \odot \left(|a_{k(i)}|_G\right)^{-1_G} = e.$$

Then we get

$$G\sum_{k=1}^{\infty} e^{k^{mr}} \odot |x_k|_G^r = G\sum e^{[k(i)]^{mr}} \odot \left[|a_{k(i)}|_G^r\right]^{-1_G} \leq e^{i^{-mr}} < \infty.$$

Hence $x \in [X(\Delta_G^m)]^\eta$ and $G\sum_{k=1}^{\infty}|a_k \odot x_k|_G^r = \sum e = \infty$. This is a contradiction as $a \in [X(\Delta_G^m)]^{\eta\eta}$. Hence $a \in U_2$.

$$\therefore [l_{G_\infty}(\Delta_G^m)]^{\eta\eta} = [c^G(\Delta_G^m)]^{\eta\eta}.$$

□

Corollary 3.14. *For $X = l_{G_\infty}$ or c^G, we get*

$$[X(\Delta_G^2)]^{\eta\eta} = \{a = (a_k) : G\sum_{k=1}^{\infty} e^{k^{-2r}} \odot |a_k|_G^r < \infty\}.$$

Proof. In Theorem 3.13 putting $m = 2$ we obtain the result.

$$[l_{G_\infty}(\Delta_G^2)]^{\eta\eta} = [c^G(\Delta_G^2)]^{\eta\eta}.$$

□

Corollary 3.15. *The sequence spaces $l_{G_\infty}(\Delta_G^m)$ and $c^G(\Delta_G^m)$ are not perfect space relative to η-dual.*

Proof. Proof is trivial as $X^{\eta\eta} \neq X$ for $X = l_{G_\infty}(\Delta_G^m)$ or $c^G(\Delta_G^m)$. □

Acknowledgement

The authors would like to record their gratitude to the reviewer for his careful reading and making some useful corrections which improved the presentation of the paper.

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