International Journal of Mathematics And its Applications
Volume 4, Issue 2–D (2016), 35–48.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

An $M/G/1$ Retrial Queue with Multiple Working Vacation

Research Article

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Abstract: We consider an $M/G/1$ retrial queue with general retrial times and multiple working vacation. During the working vacation period, customers can be served at a lower rate. Both service times in a vacation period and in a service period are generally distributed random variables. Using supplementary variable method we obtain the probability generating function for the number of customers and the average number of customers in the orbit. Further more, we carry out the waiting time distribution and some special cases of interest are discussed. Finally, some numerical results are presented.

MSC: 60K25, 60K30.

Keywords: Retrial queues, Working vacation and Supplementary variable method.

1. Introduction

If the server is found to be busy, arriving customers join a trial queue (called orbit), retry for service after some random amount of time. In telephone switching system we do have this type of applications and hence in the last two decades retrial queues have been investigated extensively. Moreover, retrial queues are also used as mathematical models for several computer systems: packet switching networks, shared bus local area networks operating under the carrier-sense multiple access protocol and collision avoidance star local area networks, etc. For more recent references, see the bibliographical overviews in Artalejo [2]. Further, a comprehensive comparison between retrial queues and their standard counterpart with classical waiting line can be found in Artalejo and Falin [4].

A large number of researchers working in various fields have analyzed retrial queues. For a detailed review of literature on retrial queues one may refer Falin [8, 9], Falin and Templeton [10], Yang and Templeton [25], Artalejo and Gomez-corral [3], Choo and Conolly [6], Gomez-corral [11], Renganathan et al. [18] and Kalyanaraman and Srinivasan [13, 14].

Recently, the queueing systems with vacations have been studied extensively, along with a comprehensive and excellent study on the vacation models, including some applications such as production/inventory system, communication systems and computer systems. As we know, there are mainly two vacation policies: classical vacation policy (also called ordinary vacation) and working vacation policy. The characteristic of a working vacation is that the server serves customers at a lower service rate during the vacation period, but in the case of classical vacation, the server stops the service completely

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during the vacation period.

In the literature of queueing systems with vacations has been discussed through a considerable amount of work in the recent past. Doshi [7] has recorded prior work on vacation models and their applications in his survey paper. In recent years few authors who were concentrated on vacation queues are Madan and Gautam Choudhury [15], Kalyanaraman and Pazhani Bala Murugan [12] and Thangaraj and Vanitha [22].

Servi and Finn [21] studied an $M/M/1$ queue with multiple working vacation and obtained the probability generating function for the number of customers in the system and the waiting time distribution. Some other notable works were done by Wu and Takagi [24], Tian et al. [23], Aftab Begum [1] and Santhi and Pazhani Bala Murugan [19, 20].

In this paper we study an Non-Markovian retrial queue with multiple working vacation. The organization of the paper is as follows. In section 2, we describe the model. In section 3, we obtain the steady state probability generating function. Particular cases are discussed in section 4. Some performance measures are obtained in section 5 and in section 6 numerical study is presented.

2. Model Description

We consider an $M/G/1$ queueing system where the primary customers arrive according to a Poisson process with arrival rate $\lambda(>0)$. We assume that there is no waiting space and therefore if an arriving customer (external or repeated) finds the server occupied, he leaves the service area and joins a pool of blocked customers called orbit. We will assume that only the customer at the head of the orbit is allowed to reach the server at a service completion instant. The retrial time follows a general distribution, with distribution functions $B(x)$.Let $b(x)$ and $B^*(\theta)$ denote the probability density function and Laplace Stieltjes Transform of $B(x)$ respectively for regular service period and let $a(x)$,$A^*(\theta)$ denote the probability density function and Laplace Stieltjes Transform of $A(x)$ respectively for working vacation period. Just after the completion of a service, if any customer is in orbit the next customer to gain service is determined by a competition between the primary customer and the orbit customer. The service time is assumed to follow general distribution, with distribution function $S_b(x)$ and density function $s_b(x)$. Let $S_b^*(\theta)$ be the Laplace Stieltjes Transform(LST) of the service time S_b.

Whenever the orbit becomes empty at a service completion instant the server starts a working vacation and the duration of the vacation time follows an exponential distribution with rate η. At a vacation completion instant if there are customers in the system the server will start a new busy period. Otherwise he takes another working vacation. This type of vacation policy is called multiple working vacation. During the working vacation period, the server provides service with service time S_v which follows a general distribution with distribution function $S_v(x)$.Let $s_v(x)$ be the probability density function and $S_v^*(\theta)$ be the Laplace Stieltjes Transform of $S_v(x)$. Further, it is noted that the service interrupted at the end of a vacation is lost and it is restarted with different distribution at the beginning of the following service period. We assume that inter-arrival times, service times, working vacation times and a retrial times are mutually independent.

We define the following random variables

$X(t)$- the orbit size at time t.

$S_b^0(t)$- the remaining service time in regular service period.

$S_v^0(t)$- the remaining service time in WV period.

$A^0(t)$- the remaining retrial time in WV period.

$B^0(t)$- the remaining retrial time in regular service period.

$$Y(t) = \begin{cases} 0 & \text{if the server is on WV period at time } t \text{ but not occupied} \\ 1 & \text{if the server is in regular service period at time } t \text{ but not occupied} \\ 2 & \text{if the server is busy on WV period at time } t \\ 3 & \text{if the server is busy in regular service period at time } t \end{cases}$$

so that the supplementary variables $S_b^0(t), S_v^0(t), A^0(t)$ and $B^0(t)$ are introduced in order to obtain the bivariate Markov Process $\{N(t), \partial(t); t \geq 0\}$, where

$$\partial(t) = \begin{cases} A^0(t) & \text{if } Y(t) = 0 \\ B^0(t) & \text{if } Y(t) = 1 \\ S_v^0(t) & \text{if } Y(t) = 2 \\ S_b^0(t) & \text{if } Y(t) = 3 \end{cases}$$

We define the following limiting probabilities:

$$\begin{aligned} Q_{0,0} &= \lim_{t\to\infty} \Pr\{X(t) = 0, Y(t) = 0\} \ , \\ Q_{0,n} &= \lim_{t\to\infty} \Pr\{X(t) = n, Y(t) = 0, x < A^0(t) \leq x + dx\} \ ; \ n \geq 1 \\ P_{0,n} &= \lim_{t\to\infty} \Pr\{X(t) = n, Y(t) = 1, x < B^0(t) \leq x + dx\} \ ; \ n \geq 1 \\ Q_{1,n} &= \lim_{t\to\infty} \Pr\{X(t) = n, Y(t) = 2, x < S_v^0(t) \leq x + dx\} \ ; \ n \geq 0 \\ P_{1,n} &= \lim_{t\to\infty} \Pr\{X(t) = n, Y(t) = 3, x < S_b^0(t) \leq x + dx\} \ ; \ n \geq 0 \end{aligned}$$

We define the Laplace Stieltjes Transform and the probability generating functions as follows,

$$S_b^*(\theta) = \int_0^\infty e^{-\theta x} s_b(x)dx \ ; \qquad S_v^*(\theta) = \int_0^\infty e^{-\theta x} s_v(x)dx \ ; \qquad A^*(\theta) = \int_0^\infty e^{-\theta x} a(x)dx$$

$$B^*(\theta) = \int_0^\infty e^{-\theta x} b(x)dx \ ; \qquad Q_{0,n}^*(\theta) = \int_0^\infty e^{-\theta x} Q_{0,n}(x)dx \ ; \qquad Q_{0,n}^*(0) = \int_0^\infty Q_{0,n}(x)dx \ ;$$

$$Q_{1,n}^*(\theta) = \int_0^\infty e^{-\theta x} Q_{1,n}(x)dx \ ; \qquad Q_{1,n}^*(0) = \int_0^\infty Q_{1,n}(x)dx \ ; \qquad P_{0,n}^*(\theta) = \int_0^\infty e^{-\theta x} P_{0,n}(x)dx \ ;$$

$$P_{0,n}^*(0) = \int_0^\infty P_{0,n}(x)dx \ ; \qquad Q_0^*(z,\theta) = \sum_{n=1}^\infty Q_{0,n}^*(\theta)z^n \ ; \qquad Q_0^*(z,0) = \sum_{n=1}^\infty Q_{0,n}^*(0)z^n \ ;$$

$$Q_0(z,0) = \sum_{n=1}^\infty Q_{0,n}(0)z^n \ ; \qquad Q_1^*(z,\theta) = \sum_{n=0}^\infty Q_{1,n}^*(\theta)z^n \ ; \qquad Q_1^*(z,0) = \sum_{n=0}^\infty Q_{1,n}^*(0)z^n \ ;$$

$$Q_1(z,0) = \sum_{n=0}^\infty Q_{1,n}(0)z^n \ ; \qquad P_0^*(z,\theta) = \sum_{n=1}^\infty P_{0,n}^*(\theta)z^n \ ; \qquad P_0^*(z,0) = \sum_{n=1}^\infty P_{0,n}^*(0)z^n \ ;$$

$$P_0(z,0) = \sum_{n=1}^\infty P_{0,n}(0)z^n \ ; \qquad P_1^*(z,\theta) = \sum_{n=0}^\infty P_{1,n}^*(\theta)z^n \ ; \qquad P_1^*(z,0) = \sum_{n=0}^\infty P_{1,n}^*(0)z^n \ ;$$

$$P_1(z,0) = \sum_{n=0}^\infty P_{1,n}(0)z^n \ .$$

3. The Orbit Size Distribution

By assuming that the system is in steady state condition, the differential difference equations governing the system are as follows:

$$\lambda Q_{0,0} = P_{1,0}(0) + Q_{1,0}(0) \tag{1}$$

$$-\frac{d}{dx}Q_{0,n}(x) = -(\lambda+\eta)Q_{0,n}(x) + Q_{1,n}(0)a(x) \; ; \quad n \geq 1 \tag{2}$$

$$-\frac{d}{dx}Q_{1,0}(x) = -(\lambda+\eta)Q_{1,0}(x) + Q_{0,1}(0)s_v(x) + \lambda Q_{0,0}s_v(x) \tag{3}$$

$$-\frac{d}{dx}Q_{1,n}(x) = -(\lambda+\eta)Q_{1,n}(x) + \lambda Q_{1,n-1}(x) + Q_{0,n+1}(0)s_v(x) + \lambda s_v(x)\int_0^\infty Q_{0,n}(x)dx \; ; \quad n \geq 1 \tag{4}$$

$$-\frac{d}{dx}P_{0,n}(x) = -\lambda P_{0,n}(x) + P_{1,n}(0)b(x) + \eta b(x)\int_0^\infty Q_{0,n}(x)dx \; ; \quad n \geq 1 \tag{5}$$

$$-\frac{d}{dx}P_{1,0}(x) = -\lambda P_{1,0}(x) + P_{0,1}(0)s_b(x) + \eta s_b(x)\int_0^\infty Q_{1,0}(x)dx \tag{6}$$

$$-\frac{d}{dx}P_{1,n}(x) = -\lambda P_{1,n}(x) + \lambda P_{1,n-1}(x) + P_{0,n+1}(0)s_b(x) + \eta s_b(x)\int_0^\infty Q_{1,n}(x)dx + \lambda s_b(x)\int_0^\infty P_{0,n}(x)dx \; ; \quad n \geq 1 \tag{7}$$

Taking the LST on both sides of the equations from (2) to (7), we get

$$-\int_0^\infty e^{-\theta x}dQ_{0,n}(x) = -(\lambda+\eta)\int_0^\infty e^{-\theta x}Q_{0,n}(x)dx + Q_{1,n}(0)\int_0^\infty e^{-\theta x}a(x)dx$$

$$\theta Q^*_{0,n}(\theta) - Q_{0,n}(0) = (\lambda+\eta)Q^*_{0,n}(\theta) - Q_{1,n}(0)A^*(\theta) \; ; \quad n \geq 1 \tag{8}$$

$$-\int_0^\infty e^{-\theta x}dQ_{1,0}(x) = -(\lambda+\eta)\int_0^\infty e^{-\theta x}Q_{1,0}(x)dx + Q_{0,1}(0)\int_0^\infty e^{-\theta x}s_v(x)dx + \lambda Q_{0,0}\int_0^\infty e^{-\theta x}s_v(x)dx$$

$$\theta Q^*_{1,0}(\theta) - Q_{1,0}(0) = (\lambda+\eta)Q^*_{1,0}(\theta) - Q_{0,1}(0)S^*_v(\theta) - \lambda Q_{0,0}S^*_v(\theta) \tag{9}$$

$$-\int_0^\infty e^{-\theta x}dQ_{1,n}(x) = -(\lambda+\eta)\int_0^\infty e^{-\theta x}Q_{1,n}(x)dx + \lambda\int_0^\infty e^{-\theta x}Q_{1,n-1}(x)dx$$
$$+Q_{0,n+1}(0)\int_0^\infty e^{-\theta x}s_v(x)dx + \lambda\int_0^\infty Q_{0,n}(x)dx\int_0^\infty e^{-\theta x}s_v(x)dx$$

$$\theta Q^*_{1,n}(\theta) - Q_{1,n}(0) = (\lambda+\eta)Q^*_{1,n}(\theta) - \lambda Q^*_{1,n-1}(\theta) - Q_{0,n+1}(0)S^*_v(\theta) - \lambda Q^*_{0,n}(0)S^*_v(\theta) \; ; \quad n \geq 1 \tag{10}$$

$$-\int_0^\infty e^{-\theta x}dP_{0,n}(x) = -\lambda\int_0^\infty e^{-\theta x}P_{0,n}(x)dx + P_{1,n}(0)\int_0^\infty e^{-\theta x}b(x)dx + \eta\int_0^\infty Q_{0,n}(x)dx\int_0^\infty e^{-\theta x}b(x)dx$$

$$\theta P^*_{0,n}(\theta) - P_{0,n}(0) = \lambda P^*_{0,n}(\theta) - P_{1,n}(0)B^*(\theta) - \eta B^*(\theta)Q^*_{0,n}(0) \; ; \quad n \geq 1 \tag{11}$$

$$-\int_0^\infty e^{-\theta x}dP_{1,0}(x) = -\lambda\int_0^\infty e^{-\theta x}P_{1,0}(x)dx + P_{0,1}(0)\int_0^\infty e^{-\theta x}s_b(x)dx + \eta\int_0^\infty e^{-\theta x}s_b(x)dx\int_0^\infty Q_{1,0}(x)dx$$

$$\theta P^*_{1,0}(\theta) - P_{1,0}(0) = \lambda P^*_{1,0}(\theta) - P_{0,1}(0)S^*_b(\theta) - \eta S^*_b(\theta)Q^*_{1,0}(0) \tag{12}$$

$$-\int_0^\infty e^{-\theta x}dP_{1,n}(x) = -\lambda\int_0^\infty e^{-\theta x}P_{1,n}(x)dx + \lambda\int_0^\infty e^{-\theta x}P_{1,n-1}(x)dx + P_{0,n+1}(0)\int_0^\infty e^{-\theta x}s_b(x)dx$$
$$+\eta\int_0^\infty e^{-\theta x}s_b(x)dx\int_0^\infty Q_{1,n}(x)dx + \lambda\int_0^\infty e^{-\theta x}s_b(x)dx\int_0^\infty P_{0,n}(x)dx$$

$$\theta P^*_{1,n}(\theta) - P_{1,n}(0) = \lambda P^*_{1,n}(\theta) - \lambda P^*_{1,n-1}(\theta) - P_{0,n+1}(0)S^*_b(\theta) - \eta S^*_b(\theta)Q^*_{1,n}(0) - \lambda S^*_b(\theta)P^*_{0,n}(0) \; ; \quad n \geq 1 \tag{13}$$

Multiplying (8) with z^n and summed over n from 1 to ∞, we get

$$\begin{aligned}\theta\sum_{n=1}^{\infty}Q^*_{0,n}(\theta)z^n-\sum_{n=1}^{\infty}Q_{0,n}(0)z^n &= (\lambda+\eta)\sum_{n=1}^{\infty}Q^*_{0,n}(\theta)z^n-A^*(\theta)\sum_{n=1}^{\infty}Q_{1,n}(0)z^n\\ [\theta-(\lambda+\eta)]Q^*_0(z,\theta) &= Q_0(z,0)-A^*(\theta)Q_1(z,0)+A^*(\theta)Q_{1,0}(0)\end{aligned}\tag{14}$$

z^n times (10) summed over n from 1 to ∞ and added up with (9) gives

$$\begin{aligned}\theta\sum_{n=0}^{\infty}Q^*_{1,n}(\theta)z^n-\sum_{n=0}^{\infty}Q_{1,n}(0)z^n &= (\lambda+\eta)\sum_{n=0}^{\infty}Q^*_{1,n}(\theta)z^n-\lambda z\sum_{n=1}^{\infty}Q^*_{1,n-1}(\theta)z^{n-1}\\ &\quad -S^*_v(\theta)\sum_{n=0}^{\infty}Q_{0,n+1}(0)z^n-\lambda Q_{0,0}S^*_v(\theta)-\lambda S^*_v(\theta)\sum_{n=1}^{\infty}Q^*_{0,n}(0)z^n\end{aligned}$$

$$[\theta-(\lambda-\lambda z+\eta)]Q^*_1(z,\theta)=Q_1(z,0)-\frac{S^*_v(\theta)}{z}Q_0(z,0)-\lambda Q_{0,0}S^*_v(\theta)-\lambda S^*_v(\theta)Q^*_0(z,0)\tag{15}$$

Inserting $\theta=(\lambda+\eta)$ in (14), we get

$$Q_0(z,0)=A^*(\lambda+\eta)[Q_1(z,0)-Q_{1,0}(0)]\tag{16}$$

Inserting $\theta=0$ and substituting (16) in (14), we get

$$Q^*_0(z,0)=\frac{(1-A^*(\lambda+\eta))(Q_1(z,0)-Q_{1,0}(0))}{\lambda+\eta}\tag{17}$$

Inserting $\theta=(\lambda-\lambda z+\eta)$ and substituting (16) and (17) in (15), we get

$$Q_1(z,0)=\frac{S^*_v(\lambda-\lambda z+\eta)[\lambda z(\lambda+\eta)Q_{0,0}-(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z)Q_{1,0}(0)]}{z(\lambda+\eta)-S^*_v(\lambda-\lambda z+\eta)(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z)}\tag{18}$$

Substituting (18) in (16), we get

$$Q_0(z,0)=\frac{zA^*(\lambda+\eta)(\lambda+\eta)[\lambda S^*_v(\lambda-\lambda z+\eta)Q_{0,0}-Q_{1,0}(0)]}{z(\lambda+\eta)-S^*_v(\lambda-\lambda z+\eta)(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z)}\tag{19}$$

Let $f(z)=z(\lambda+\eta)-S^*_v(\lambda-\lambda z+\eta)(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z)$, we find $f(0)<0$ and $f(1)>0$. This implies that there exist a real root $z_1\in(0,1)$ for the equation $f(z)=0$. Hence at $z=z_1$ the equation (19) becomes

$$Q_{1,0}(0)=\lambda S^*_v(\lambda-\lambda z_1+\eta)Q_{0,0}\tag{20}$$

Substituting (20) in (18), we get

$$Q_1(z,0)=\frac{Q_{0,0}\lambda S^*_v(\lambda-\lambda z+\eta)\times NR_1}{z(\lambda+\eta)-S^*_v(\lambda-\lambda z+\eta)(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z)}\tag{21}$$

where

$$NR_1 \;=\; \big[z(\lambda+\eta)-S^*_v(\lambda-\lambda z_1+\eta)\big(\lambda z+A^*(\lambda+\eta)(\lambda-\lambda z+\eta)\big)\big]$$

Substituting (20) in (19), we get

$$Q_0(z,0)=\frac{\big\{\lambda zA^*(\lambda+\eta)(\lambda+\eta)[S^*_v(\lambda-\lambda z+\eta)-S^*_v(\lambda-\lambda z_1+\eta)]\big\}Q_{0,0}}{z(\lambda+\eta)-S^*_v(\lambda-\lambda z+\eta)(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z)}\tag{22}$$

Substituting (20) and (21) in (17), we get

$$Q_0^*(z,0)=\frac{\big\{\lambda z(1-A^*(\lambda+\eta))[S_v^*(\lambda-\lambda z+\eta)-S_v^*(\lambda-\lambda z_1+\eta)]\big\}Q_{0,0}}{z(\lambda+\eta)-S_v^*(\lambda-\lambda z+\eta)(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z)} \tag{23}$$

Inserting $\theta=0$ and substituting (21), (22) and (23) in (15), we get

$$Q_1^*(z,0)=\frac{Q_{0,0}\lambda(1-S_v^*(\lambda-\lambda z+\eta))\times NR_2(z)}{(\lambda-\lambda z+\eta)[z(\lambda+\eta)-S_v^*(\lambda-\lambda z+\eta)(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z)]} \tag{24}$$

where

$$NR_2(z) \;=\; \big[z(\lambda+\eta)-S_v^*(\lambda-\lambda z_1+\eta)\big(\lambda z+A^*(\lambda+\eta)(\lambda-\lambda z+\eta)\big)\big]$$

Multiplying (11) with z^n and summed over n from 1 to ∞, we get

$$\theta\sum_{n=1}^{\infty}P_{0,n}^*(\theta)z^n-\sum_{n=1}^{\infty}P_{0,n}(0)z^n=\lambda\sum_{n=1}^{\infty}P_{0,n}^*(\theta)z^n-B^*(\theta)\sum_{n=1}^{\infty}P_{1,n}(0)z^n-\eta B^*(\theta)\sum_{n=1}^{\infty}Q_{0,n}^*(0)z^n$$

$$(\theta-\lambda)P_0^*(z,\theta)=P_0(z,0)-B^*(\theta)[P_1(z,0)-P_{1,0}(0)]-\eta B^*(\theta)Q_0^*(z,0) \tag{25}$$

Substituting $Q_{1,0}(0)=\lambda S_v^*(\lambda-\lambda z_1+\eta)Q_{0,0}$ in (1), we get $P_{1,0}(0)=\lambda(1-S_v^*(\lambda-\lambda z_1+\eta))Q_{0,0}$. Inserting $\theta=\lambda$ and substituting $P_{1,0}(0)=\lambda(1-S_v^*(\lambda-\lambda z_1+\eta))Q_{0,0}$ in (25), we get

$$P_0(z,0)=B^*(\lambda)[P_1(z,0)-\lambda(1-S_v^*(\lambda-\lambda z_1+\eta))Q_{0,0}+\eta Q_0^*(z,0)] \tag{26}$$

z^n times (13) is summed over n from 1 to ∞ and added up with (12) gives

$$\begin{aligned}\theta\sum_{n=0}^{\infty}P_{1,n}^*(\theta)z^n-\sum_{n=0}^{\infty}P_{1,n}(0)z^n=&\lambda\sum_{n=0}^{\infty}P_{1,n}^*(\theta)z^n-\lambda z\sum_{n=1}^{\infty}P_{1,n-1}^*(\theta)z^{n-1}\\&-S_b^*(\theta)\sum_{n=0}^{\infty}P_{0,n+1}(0)z^n-\eta S_b^*(\theta)\sum_{n=0}^{\infty}Q_{1,n}^*(0)z^n-\lambda S_b^*(\theta)\sum_{n=1}^{\infty}P_{0,n}^*(0)z^n\end{aligned}$$

$$[\theta-(\lambda-\lambda z)]P_1^*(z,\theta)=P_1(z,0)-S_b^*(\theta)\left[\frac{P_0(z,0)}{z}+\eta Q_1^*(z,0)+\lambda P_0^*(z,0)\right] \tag{27}$$

Inserting $\theta=0$ and substituting (26) and $P_{1,0}(0)=\lambda(1-S_v^*(\lambda-\lambda z_1+\eta))Q_{0,0}$ in (25), we get

$$P_0^*(z,0)=\frac{[1-B^*(\lambda)]}{\lambda}[P_1(z,0)-\lambda(1-S_v^*(\lambda-\lambda z_1+\eta))Q_{0,0}+\eta Q_0^*(z,0)] \tag{28}$$

Inserting $\theta=(\lambda-\lambda z)$ and substituting (26) and (28) in (27), we get

$$P_1(z,0)=\frac{\left\{\begin{aligned}&S_b^*(\lambda-\lambda z)\big[\eta z Q_1^*(z,0)+\eta[(1-z)B^*(\lambda)+z]Q_0^*(z,0)\\&\qquad-[(1-z)B^*(\lambda)+z](1-S_v^*(\lambda-\lambda z_1+\eta))\lambda Q_{0,0}\big]\end{aligned}\right\}}{z-S_b^*(\lambda-\lambda z)(z+(1-z)B^*(\lambda))} \tag{29}$$

Substituting (29) in (26), we get

$$P_0(z,0)=\frac{\left\{\begin{aligned}&zB^*(\lambda)\big[\eta S_b^*(\lambda-\lambda z)Q_1^*(z,0)+\eta Q_0^*(z,0)\\&\qquad\qquad-\lambda(1-S_v^*(\lambda-\lambda z_1+\eta))Q_{0,0}\big]\end{aligned}\right\}}{z-S_b^*(\lambda-\lambda z)\big[(1-z)B^*(\lambda)+z\big]} \tag{30}$$

Substituting (29) in (28), we get

$$P_0^*(z,0) = \frac{\begin{Bmatrix} z(1-B^*(\lambda))[\eta S_b^*(\lambda-\lambda z)Q_1^*(z,0)+\eta Q_0^*(z,0) \\ -\lambda(1-S_v^*(\lambda-\lambda z_1+\eta))Q_{0,0}] \end{Bmatrix}}{\lambda\big[z-S_b^*(\lambda-\lambda z)\big((1-z)B^*(\lambda)+z\big)\big]} \tag{31}$$

Inserting $\theta = 0$ and substituting (29), (30) and (31) in (27), we get

$$P_1^*(z,0) = \frac{\begin{Bmatrix} (1-S_b^*(\lambda-\lambda z))\big[\eta z Q_1^*(z,0)+\eta[(1-z)B^*(\lambda)+z]Q_0^*(z,0) \\ -[(1-z)B^*(\lambda)+z](1-S_v^*(\lambda-\lambda z_1+\eta))\lambda Q_{0,0}\big] \end{Bmatrix}}{(\lambda-\lambda z)\Big[z-S_b^*(\lambda-\lambda z)(z+(1-z)B^*(\lambda))\Big]} \tag{32}$$

Substituting (23) and (24) in (31), we get

$$\begin{aligned} P_0^*(z,0) &= \frac{z(1-B^*(\lambda))Q_{0,0}}{(\lambda-\lambda z+\eta)D_1(z)D_2(z)}\big\{\eta S_b^*(\lambda-\lambda z)(1-S_v^*(\lambda-\lambda z+\eta)) \\ &\times\big[(\lambda+\eta)z-S_v^*(\lambda-\lambda z_1+\eta)[\lambda z+(\lambda-\lambda z+\eta)A^*(\lambda+\eta)]\big] \\ &+\eta z(\lambda-\lambda z+\eta)(1-A^*(\lambda+\eta))(S_v^*(\lambda-\lambda z+\eta)-S_v^*(\lambda-\lambda z_1+\eta)) \\ &-(\lambda-\lambda z+\eta)(1-S_v^*(\lambda-\lambda z_1+\eta))\times\big[(\lambda+\eta)z-S_v^*(\lambda-\lambda z+\eta)[\lambda z+(\lambda-\lambda z+\eta)A^*(\lambda+\eta)]\big]\big\} \end{aligned} \tag{33}$$

where

$$D_1(z) = z(\lambda+\eta)-S_v^*(\lambda-\lambda z+\eta)(\lambda z+A^*(\lambda+\eta)(\lambda-\lambda z+\eta)) \tag{34}$$

$$D_2(z) = z-S_b^*(\lambda-\lambda z)[(1-z)B^*(\lambda)+z] \tag{35}$$

Substituting (23),(24) in (32), we get

$$\begin{aligned} P_1^*(z,0) &= \frac{(1-S_b^*(\lambda-\lambda z))Q_{0,0}}{(\lambda-\lambda z+\eta)D_1(z)D_2(z)}\big\{\eta z[\lambda z+(\lambda-\lambda z+\eta)B^*(\lambda)][1-A^*(\lambda+\eta)] \\ &\times[S_v^*(\lambda-\lambda z+\eta)-S_v^*(\lambda-\lambda z_1+\eta)]-(1-S_v^*(\lambda-\lambda z_1+\eta)) \\ &\times[\lambda z+(\lambda-\lambda z+\eta)B^*(\lambda)][(\lambda+\eta)z-S_v^*(\lambda-\lambda z+\eta) \\ &\times[\lambda z+(\lambda-\lambda z+\eta)A^*(\lambda+\eta)]]+\eta z(\lambda+\eta)(S_v^*(\lambda-\lambda z+\eta)-S_v^*(\lambda-\lambda z_1+\eta))A^*(\lambda+\eta)\big\} \end{aligned} \tag{36}$$

where $D_1(z)$ and $D_2(z)$ are given in (34) and (35) respectively. We define $P_v(z) = Q_0^*(z,0)+Q_1^*(z,0)+Q_{0,0}$

$$\begin{aligned} P_v(z) &= \frac{Q_{0,0}}{(\lambda-\lambda z+\eta)D_1(z)}\big\{\lambda z(\lambda-\lambda z+\eta)(1-A^*(\lambda+\eta))(S_v^*(\lambda-\lambda z+\eta) \\ &-S_v^*(\lambda-\lambda z_1+\eta))+\lambda(1-S_v^*(\lambda-\lambda z+\eta))[z(\lambda+\eta)-S_v^*(\lambda-\lambda z_1+\eta)\times(\lambda z+A^*(\lambda+\eta)(\lambda-\lambda z+\eta))] \\ &+(\lambda-\lambda z+\eta)[z(\lambda+\eta)-S_v^*(\lambda-\lambda z+\eta)(\lambda z+A^*(\lambda+\eta)(\lambda-\lambda z+\eta))]\big\} \end{aligned} \tag{37}$$

as the probability generating function for the number of customers in the orbit when the server is on working vacation period,where $D_1(z)$ is given in (34) and $P_B(z) = P_0^*(z,0)+P_1^*(z,0)$

$$\begin{aligned} P_B(z) &= \frac{Q_{0,0}}{(\lambda-\lambda z+\eta)(D_1(z)D_2(z))}\big\{z(1-B^*(\lambda))\big\{\eta S_b^*(\lambda-\lambda z)\times(1-S_v^*(\lambda-\lambda z+\eta))\big[(\lambda+\eta)z-S_v^*(\lambda-\lambda z_1+\eta) \\ &\times[\lambda z+(\lambda-\lambda z+\eta)A^*(\lambda+\eta)]\big]+\eta z(\lambda-\lambda z+\eta)(1-A^*(\lambda+\eta))(S_v^*(\lambda-\lambda z+\eta)-S_v^*(\lambda-\lambda z_1+\eta)) \\ &-(\lambda-\lambda z+\eta)(1-S_v^*(\lambda-\lambda z_1+\eta)) \end{aligned}$$

$$\times\big[(\lambda+\eta)z - S_v^*(\lambda-\lambda z+\eta)[\lambda z+(\lambda-\lambda z+\eta)A^*(\lambda+\eta)]\big]\big\} + (1-S_b^*(\lambda-\lambda z))\big\{\eta z[\lambda z+(\lambda-\lambda z+\eta)B^*(\lambda)][1-A^*(\lambda+\eta)]$$
$$\times[S_v^*(\lambda-\lambda z+\eta) - S_v^*(\lambda-\lambda z_1+\eta)] - (1-S_v^*(\lambda-\lambda z_1+\eta)) \times [\lambda z+(\lambda-\lambda z+\eta)B^*(\lambda)][(\lambda+\eta)z - S_v^*(\lambda-\lambda z+\eta)$$
$$\times[\lambda z+(\lambda-\lambda z+\eta)A^*(\lambda+\eta)]] + \eta z(\lambda+\eta)(S_v^*(\lambda-\lambda z+\eta) - S_v^*(\lambda-\lambda z_1+\eta))A^*(\lambda+\eta)\big\}\big\} \tag{38}$$

as the probability generating function for the number of customers in the orbit when the server is on not working vacation period where $D_1(z)$ and $D_2(z)$ are given in (34) and (35) respectively. Again we define$P(z) = P_B(z) + P_v(z)$

$$\begin{aligned} P(z) \;=\; & \frac{Q_{0,0}}{(\lambda-\lambda z+\eta)D_1(z)D_2(z)}\big\{z(1-B^*(\lambda))\big\{\eta S_b^*(\lambda-\lambda z)\times(1-S_v^*(\lambda-\lambda z+\eta)) \\ & \times\big[(\lambda+\eta)z - S_v^*(\lambda-\lambda z_1+\eta)[\lambda z+(\lambda-\lambda z+\eta)A^*(\lambda+\eta)]\big] \\ & +\eta z(\lambda-\lambda z+\eta)(1-A^*(\lambda+\eta))(S_v^*(\lambda-\lambda z+\eta) - S_v^*(\lambda-\lambda z_1+\eta)) - (\lambda-\lambda z+\eta)(1-S_v^*(\lambda-\lambda z_1+\eta)) \\ & \times\big[(\lambda+\eta)z - S_v^*(\lambda-\lambda z+\eta)[\lambda z+(\lambda-\lambda z+\eta)A^*(\lambda+\eta)]\big]\big\} \\ & +(1-S_b^*(\lambda-\lambda z))\big\{\eta z[\lambda z+(\lambda-\lambda z+\eta)B^*(\lambda)][1-A^*(\lambda+\eta)] \\ & \times[S_v^*(\lambda-\lambda z+\eta) - S_v^*(\lambda-\lambda z_1+\eta)] - (1-S_v^*(\lambda-\lambda z_1+\eta)) \\ & \times[\lambda z+(\lambda-\lambda z+\eta)B^*(\lambda)][(\lambda+\eta)z - S_v^*(\lambda-\lambda z+\eta)\times[\lambda z+(\lambda-\lambda z+\eta)A^*(\lambda+\eta)]] \\ & +\eta z(\lambda+\eta)(S_v^*(\lambda-\lambda z+\eta) - S_v^*(\lambda-\lambda z_1+\eta))A^*(\lambda+\eta)\big\}\big\} \\ & +\Big\{\lambda z(\lambda-\lambda z+\eta)(1-A^*(\lambda+\eta))(S_v^*(\lambda-\lambda z+\eta) - S_v^*(\lambda-\lambda z_1+\eta)) + \lambda(1-S_v^*(\lambda-\lambda z+\eta)) \\ & \times[z(\lambda+\eta) - S_v^*(\lambda-\lambda z_1+\eta)(\lambda z+A^*(\lambda+\eta)(\lambda-\lambda z+\eta))] \\ & +(\lambda-\lambda z+\eta)[z(\lambda+\eta) - S_v^*(\lambda-\lambda z+\eta)(\lambda z+A^*(\lambda+\eta) \\ & \times(\lambda-\lambda z+\eta))]\big\}\big\{z - S_b^*(\lambda-\lambda z)[z+(1-z)B^*(\lambda)]\big\}\big\} \end{aligned} \tag{39}$$

as the probability generating function for the number of customers in the orbit where $D_1(z)$ and $D_2(z)$ are given in (34) and (35) respectively . We shall now use the normalizing condition $P(1) = 1$ to determine the unknown $Q_{0,0}$ which appears in (39). Substituting $z = 1$ in (39) and using L'Hospitals rule, we obtain

$$Q_{0,0} = \frac{(1-\rho_b)}{\left\{\begin{array}{r} \left[\dfrac{(\lambda-\lambda S_v^*(\lambda-\lambda z_1+\eta)+\eta)[\lambda+\eta B^*(\lambda) - S_v^*(\eta)(\lambda+\eta A^*(\lambda+\eta))]}{\eta B^*(\lambda)[\lambda+\eta-S_v^*(\eta)(\lambda+\eta A^*(\lambda+\eta))]}\right] \\ -\left[\dfrac{\lambda E(S_b)S_v^*(\eta)[\lambda+\eta-S_v^*(\lambda-\lambda z_1+\eta)(\lambda+\eta A^*(\lambda+\eta))]}{B^*(\lambda)[\lambda+\eta-S_v^*(\eta)(\lambda+\eta A^*(\lambda+\eta))]}\right] \\ +\left[\dfrac{\eta S_v^*(\lambda-\lambda z_1+\eta)A^*(\lambda+\eta)(1-B^*(\lambda))}{B^*(\lambda)[\lambda+\eta-S_v^*(\eta)(\lambda+\eta A^*(\lambda+\eta))]}\right] \end{array}\right\}} \tag{40}$$

where $\rho_b = \dfrac{\lambda E(S_b)}{B^*(\lambda)}$, $E(S_b)$ is the mean service time. From (40) we obtain the system stability condition

$$\rho_b < 1 \tag{41}$$

3.1. Particular Cases

Case i: Suppose that there is no retrial time in the system that is the retrial time is 0 (by setting $B^*(\lambda) = 1, A^*(\lambda+\eta) = 1$ in (39)) then our system is reduced to the $M/G/1$ queue with multiple working vacation (Takagi (2006))irrespective of the notations.

$$P(z) = P_V(z) + P_B(z) \tag{42}$$

where

$$P_V(z)=\frac{Q_{0,0}\{\lambda z(1-S_v^*(\lambda-\lambda z+\eta))(z-S_v^*(\lambda-\lambda z_1+\eta))+(\lambda-\lambda z+\eta)(z-S_v^*(\lambda-\lambda z+\eta))\}}{(\lambda-\lambda z+\eta)(z-S_v^*(\lambda-\lambda z+\eta))}$$

$$P_B(z)=\frac{\left\{\begin{array}{c}Q_{0,0}z(1-S_b^*(\lambda-\lambda z))\big[\eta S_v^*(\lambda-\lambda z+\eta)(1-S_v^*(\lambda-\lambda z_1+\eta))\\ -\eta z(1-S_v^*(\lambda-\lambda z+\eta))-\lambda(1-S_v^*(\lambda-\lambda z_1+\eta))\times(z-S_v^*(\lambda-\lambda z+\eta))\big]\end{array}\right\}}{(\lambda-\lambda z+\eta)(z-S_v^*(\lambda-\lambda z+\eta))(z-S_b^*\lambda-\lambda z)}$$

$$Q_{0,0}=\frac{(1-\lambda E(S_b))}{\left[\frac{(\lambda-\lambda S_v^*(\lambda-\lambda z_1+\eta)+\eta)}{\eta}-\frac{\lambda E(S_b)S_v^*(\eta)(1-S_v^*(\lambda-\lambda z_1+\eta))}{(1-S_v^*(\eta))}\right]}$$

Case ii: If the server never takes the vacation then taking the limit $\eta\to\infty$ in (39), we get

$$P(z)=\frac{(B^*(\lambda)-\lambda E(S_b))(1-z)S_b^*(\lambda-\lambda z)}{B^*(\lambda)(1-z)S_b^*(\lambda-\lambda z)-z(1-S_b^*(\lambda-\lambda z))} \tag{43}$$

Equation (43) is well known probability generating function of the steady state system length distribution of an $M/G/1$ retrial queue (Equation (16) of Gomez-Corral (1999)) irrespective of the notations.

Case iii: If the server never takes the vacation and there is no retrial time in the system then taking the limit $\eta\to\infty$ and putting $B^*(\lambda)=1$ and $A^*(\lambda+\eta)=1$ in (39)

We get

$$P(z)=\frac{S_b^*(\lambda-\lambda z)(1-z)(1-\lambda E(S_b))}{S_b^*(\lambda-\lambda z)-z} \tag{44}$$

Equation (44) is well known probability generating function of the steady state system length distribution of an $M/G/1$ queue (Medhi (1982)) irrespective of the notations.

3.2. Performance Measures

Mean Orbit Size

Let L_v an d L_b denotes the mean orbit size during the working vacation and regular service period respectively and let W_v and W_b be the mean waiting time of the customer in the orbit during WV period and regular service period respectively.

$$\begin{aligned}
L_v &= \frac{d}{dz}P_v(z)\big|_{z=1}\\
&= \frac{d}{dz}\left[Q_1^*(z,0)+Q_0^*(z,0)\right]\big|_{z=1}\\
&= \frac{d}{dz}\left[\frac{A(z)}{(\lambda-\lambda z+\eta)D_1(z)}+\frac{B(z)}{D_1(z)}\right]Q_{0,0}\big|_{z=1}\\
&= \left[\frac{(\lambda-\lambda z+\eta)D_1(z)A'(z)-A(z)[(\lambda-\lambda z+\eta)D_1'(z)-\lambda D_1(z)]}{((\lambda-\lambda z+\eta)D_1(z))^2}\right]Q_{0,0}\big|_{z=1}+\left[\frac{D_1(z)B'(z)-B(z)D_1'(z)}{(D_1(z))^2}\right]Q_{0,0}\big|_{z=1}
\end{aligned}$$

where

$$A(z)=\lambda(1-S_v^*(\lambda-\lambda z+\eta))[z(\lambda+\eta)-S_v^*(\lambda-\lambda z_1+\eta)(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z)]$$

$$B(z)=\lambda z(1-A^*(\lambda+\eta))(S_v^*(\lambda-\lambda z+\eta)-S_v^*(\lambda-\lambda z_1+\eta))$$

$$D_1(z)=z(\lambda+\eta)-S_v^*(\lambda-\lambda z+\eta)(A^*(\lambda+\eta)(\lambda-\lambda z+\eta)+\lambda z)$$

At $z=1$ the formula L_v becomes

$$L_v=\left[\frac{\eta D_1(1)A'(1)-A(1)[\eta D_1'(1)-\lambda D_1(1)]}{(\eta D_1(1))^2}+\frac{D_1(1)B'(1)-B(1)D_1'(1)}{(D_1(1))^2}\right]Q_{0,0}$$

Using Little's formula, we get $W_v = \dfrac{L_v}{\lambda}$, where

$$
\begin{aligned}
A(1) &= \lambda(1 - S_v^*(\eta))[\lambda + \eta - S_v^*(\lambda - \lambda z_1 + \eta)(\lambda + \eta A^*(\lambda + \eta))] \\
A'(1) &= \lambda^2 S_v^{*'}(\eta)[\lambda + \eta - S_v^*(\lambda - \lambda z_1 + \eta)(\lambda + \eta A^*(\lambda + \eta))] + \lambda(1 - S_v^*(\eta))[\lambda + \eta - S_v^*(\lambda - \lambda z_1 + \eta)(\lambda - \lambda A^*(\lambda + \eta))] \\
B(1) &= \lambda(1 - A^*(\lambda + \eta))(S_v^*(\eta) - S_v^*(\lambda - \lambda z_1 + \eta)) \\
B'(1) &= \lambda(1 - A^*(\lambda + \eta))[S_v^*(\eta) - S_v^*(\lambda - \lambda z_1 + \eta) - \lambda S_v^{*'}(\eta)] \\
D_1(1) &= \lambda + \eta - S_v^*(\eta)(\lambda + \eta A^*(\lambda + \eta)) \\
D_1'(1) &= \lambda + \eta + \lambda S_v^{*'}(\eta)(\lambda + \eta A^*(\lambda + \eta)) - S_v^*(\eta)(\lambda - \lambda A^*(\lambda + \eta)) \\
L_b &= \frac{d}{dz} P_B(z)\big|_{z=1} \\
&= \frac{d}{dz}[P_0^*(z,0) + P_1^*(z,0)]\big|_{z=1} \\
&= \frac{d}{dz}\Big[\frac{(1 - B^*(\lambda))N_1(z) + N_2(z)N_3(z)}{(\lambda - \lambda z + \eta)D_1(z)D_2(z)}\Big] Q_{0,0}\big|_{z=1}
\end{aligned}
$$

and therefore

$$
L_b = \frac{NR_3(z)}{2((\lambda - \lambda z + \eta)D_1(z)D_2'(z))^2} Q_{0,0}\big|_{z=1}
$$

where

$$
\begin{aligned}
NR_3(z) &= (1 - B^*(\lambda))\big[2N_1'(z)D_2'(z)(\lambda D_1(z) - (\lambda - \lambda z + \eta)D_1'(z)) + (\lambda - \lambda z + \eta) \times D_1(z)(D_2'(z)N_1''(z) - N_1'(z)D_2''(z))\big] \\
&\quad + 2(\lambda - \lambda z + \eta)N_2'(z)D_2'(z)(D_1(z)N_3'(z) - N_3(z)D_1'(z)) + N_3(z)D_1(z) \\
&\quad \times [2\lambda N_2'(z)D_2'(z) + (\lambda - \lambda z + \eta)D_2'(z)N_2''(z) - (\lambda - \lambda z + \eta)N_2'(z)D_2''(z)] \\
N_1(z) &= \eta z S_b^*(\lambda - \lambda z)(1 - S_v^*(\lambda - \lambda z + \eta))\big\{(\lambda + \eta)z - S_v^*(\lambda - \lambda z_1 + \eta) \\
&\quad \times [(\lambda - \lambda z + \eta)A^*(\lambda + \eta) + \lambda z]\big\} + (\lambda - \lambda z + \eta)\eta z^2(1 - A^*(\lambda + \eta)) \\
&\quad \times [S_v^*(\lambda - \lambda z + \eta) - S_v^*(\lambda - \lambda z_1 + \eta)] - (1 - S_v^*(\lambda - \lambda z_1 + \eta))z(\lambda - \lambda z + \eta) \\
&\quad \times \big\{(\lambda + \eta)z - S_v^*(\lambda - \lambda z + \eta)[(\lambda - \lambda z + \eta)A^*(\lambda + \eta) + \lambda z]\big\} \\
N_2(z) &= (1 - S_b^*(\lambda - \lambda z)) \\
N_3(z) &= \eta z[(\lambda - \lambda z + \eta)B^*(\lambda) + \lambda z](1 - A^*(\lambda + \eta))(S_v^*(\lambda - \lambda z + \eta) \\
&\quad - S_v^*(\lambda - \lambda z_1 + \eta)) - (1 - S_v^*(\lambda - \lambda z_1 + \eta))[(\lambda - \lambda z + \eta)B^*(\lambda) + \lambda z] \\
&\quad \times \{(\lambda + \eta)z - S_v^*(\lambda - \lambda z + \eta)((\lambda - \lambda z + \eta)A^*(\lambda + \eta) + \lambda z)\} \\
&\quad + \eta z(\lambda + \eta)(S_v^*(\lambda - \lambda z + \eta) - S_v^*(\lambda - \lambda z_1 + \eta))A^*(\lambda + \eta) \\
D_1(z) &= (\lambda + \eta)z - S_v^*(\lambda - \lambda z + \eta)[(\lambda - \lambda z + \eta)A^*(\lambda + \eta) + \lambda z] \\
D_2(z) &= z - S_b^*(\lambda - \lambda z)[(1 - z)B^*(\lambda) + z]
\end{aligned}
$$

At $z = 1$ the formula L_b becomes

$$
L_b = \frac{\left\{\begin{array}{r} (1 - B^*(\lambda))\big[2N_1'(1)D_2'(1)(\lambda D_1(1) - \eta D_1'(1)) + \eta D_1(1)(D_2'(1)N_1''(1) - N_1'(1)D_2''(1))\big] + 2\eta N_2'(1)D_2'(1)(D_1(1)N_3'(1) \\ - N_3(1)D_1'(1)) + N_3(1)D_1(1)[2\lambda N_2'(1)D_2'(1) + \eta D_2'(1)N_2''(1) - \eta N_2'(1)D_2''(1)] \end{array}\right\}}{2[\eta D_1(1)D_2'(1)]^2} Q_{0,0}
$$

Applying Little's formula we get $W_b = \dfrac{L_b}{\lambda}$, where

$N_1'(1) = -\eta\lambda S_v^*(\eta) + \eta\lambda E(S_b)(1 - S_v^*(\eta))[\lambda + \eta - \eta S_v^*(\lambda - \lambda z_1 + \eta)A^*(\lambda + \eta)$

$$- \lambda S_v^*(\lambda - \lambda z_1 + \eta)] - \eta^2 A^*(\lambda + \eta) S_v^*(\eta) + \eta^2 S_v^*(\lambda - \lambda z_1 + \eta) A^*(\lambda + \eta)$$
$$+ \lambda[\lambda + \eta - \eta S_v^*(\eta) A^*(\lambda + \eta) - \lambda S_v^*(\eta)] - \lambda^2 S_v^*(\lambda - \lambda z_1 + \eta) + \lambda^2 S_v^*(\lambda - \lambda z_1 + \eta) S_v^*(\eta) + \lambda \eta S_v^*(\lambda - \lambda z_1 + \eta) S_v^*(\eta) A^*(\lambda + \eta)$$

$$N_1''(1) = \{(1 - S_v^*(\eta))(2\eta\lambda E(S_b) + \eta\lambda^2 E(S_b^2)) + 2\eta\lambda^2 E(S_b) S_v^{*'}(\eta)\}$$
$$\times [\lambda + \eta - S_v^*(\lambda - \lambda z_1 + \eta)(\lambda + \eta A^*(\lambda + \eta))] + 2\eta\lambda E(S_b)(1 - S_v^*(\eta))$$
$$\times [\lambda + \eta + \lambda S_v^*(\lambda - \lambda z_1 + \eta)(A^*(\lambda + \eta) - 1)] + 2\eta\lambda[S_v^{*'}(\eta)(\lambda + \eta A^*(\lambda + \eta))$$
$$+ (1 - S_v^*(\eta))(2 - A^*(\lambda + \eta) S_v^*(\lambda - \lambda z_1 + \eta))] + 2\eta^2 A^*(\lambda + \eta)$$
$$\times (S_v^*(\lambda - \lambda z_1 + \eta) - S_v^*(\eta)) + 2\lambda^2(1 - S_v^*(\eta))(1 - S_v^*(\lambda - \lambda z_1 + \eta))$$
$$+ 2\lambda(1 - S_v^*(\lambda - \lambda z_1 + \eta))[\lambda + \lambda S_v^{*'}(\eta)(\lambda + \eta A^*(\lambda + \eta)) + \lambda S_v^*(\eta)(A^*(\lambda + \eta) - 1)]$$

$$N_2'(1) = -\lambda E(S_b)$$

$$N_2''(1) = -\lambda^2 E(S_b^2)$$

$$N_3(1) = (1 - S_v^*(\eta))\{\eta B^*(\lambda) S_v^*(\lambda - \lambda z_1 + \eta)(\lambda + \eta A^*(\lambda + \eta)) - \eta\lambda - \eta\lambda B^*(\lambda) - \eta^2 B^*(\lambda)\}$$
$$+ (1 - S_v^*(\lambda - \lambda z_1 + \eta))\{\eta\lambda S_v^*(\eta) A^*(\lambda + \eta) - \lambda^2(1 - S_v^*(\eta))\} + \eta^2 A^*(\lambda + \eta)(S_v^*(\eta) - S_v^*(\lambda - \lambda z_1 + \eta))$$

$$N_3'(1) = (S_v^*(\eta) - S_v^*(\lambda - \lambda z_1 + \eta))[(1 - A^*(\lambda + \eta))(\eta\lambda(1 - B^*(\lambda)) + \eta^2 B^*(\lambda)) + \eta^2 A^*(\lambda + \eta)]$$
$$+ B^*(\lambda)(\lambda + \eta + \lambda S_v^*(\eta))(\eta S_v^*(\lambda - \lambda z_1 + \eta) - \lambda) - \eta\lambda A^*(\lambda + \eta)$$
$$\times (S_v^*(\eta)(1 - S_v^*(\lambda - \lambda z_1 + \eta)) + \eta S_v^{*'}(\eta)) + \lambda B^*(\lambda) S_v^*(\lambda - \lambda z_1 + \eta)$$
$$\times [\lambda + \eta - \eta S_v^*(\eta) A^*(\lambda + \eta) + \lambda S_v^*(\eta)] - \eta B^*(\lambda)[\lambda + \eta - \lambda^2 S_v^{*'}(\eta) + \lambda S_v^*(\eta)]$$
$$+ [\lambda S_v^{*'}(\eta)(\eta A^*(\lambda + \eta) - \lambda) + \lambda S_v^*(\eta) A^*(\lambda + \eta)] \times [\eta B^*(\lambda) S_v^*(\lambda - \lambda z_1 + \eta) - \lambda(1 - S_v^*(\lambda - \lambda z_1 + \eta))]$$

$$D_1(1) = \lambda + \eta - S_v^*(\eta)(\lambda + \eta A^*(\lambda + \eta))$$

$$D_1'(1) = \lambda + \eta + \lambda S_v^{*'}(\eta)(\lambda + \eta A^*(\lambda + \eta)) - \lambda S_v^*(\eta)(1 - A^*(\lambda + \eta))$$

$$D_2'(1) = B^*(\lambda) - \lambda E(S_b)$$

$$D_2''(1) = -2\lambda E(S_b)(1 - B^*(\lambda)) - \lambda^2 E(S_b^2)$$

3.3. Numerical Result

Fixing the values of $\mu_v = 5, \mu_b = 7, \mu_{v_r} = 2, \mu_{b_r} = 3$ and ranging the values of λ from 1.1 to 1.5 insteps of 0.1 and varying the values of η from 5.1 to 6.3 insteps of 0.3, we calculated the corresponding values of L_b and W_b for multiple working vacation and tabulated in Table 1 and in Table 2 respectively.

$\lambda\eta$	5.1	5.4	5.7	6.0	6.3
1.1	0.073915	0.072699	0.071203	0.069543	0.067794
1.2	0.095013	0.092932	0.090631	0.088219	0.085768
1.3	0.119968	0.116829	0.113550	0.110230	0.106936
1.4	0.149341	0.144929	0.140478	0.136076	0.131780
1.5	0.183801	0.177875	0.172034	0.166353	0.160875

Table 1. **Arrival rate (λ) versus mean orbit size (L_b) in regular service period**

$\lambda\eta$	5.1	5.4	5.7	6.0	6.3
1.1	0.067196	0.066090	0.064730	0.063221	0.061631
1.2	0.079177	0.077443	0.075526	0.073515	0.071473
1.3	0.092283	0.089869	0.087346	0.084792	0.082259
1.4	0.106672	0.103521	0.100341	0.097197	0.094128
1.5	0.122534	0.118583	0.114689	0.110902	0.107250

Table 2. **Arrival rate (λ) versus waiting time (W_b) in regular service period**

The corresponding graphs have been drawn for λ versus L_b and λ versus W_b and are shown in Figure 1 and in Figure 2 respectively. From the graphs it is seen that as λ increases both L_b and W_b increases for various values of η.

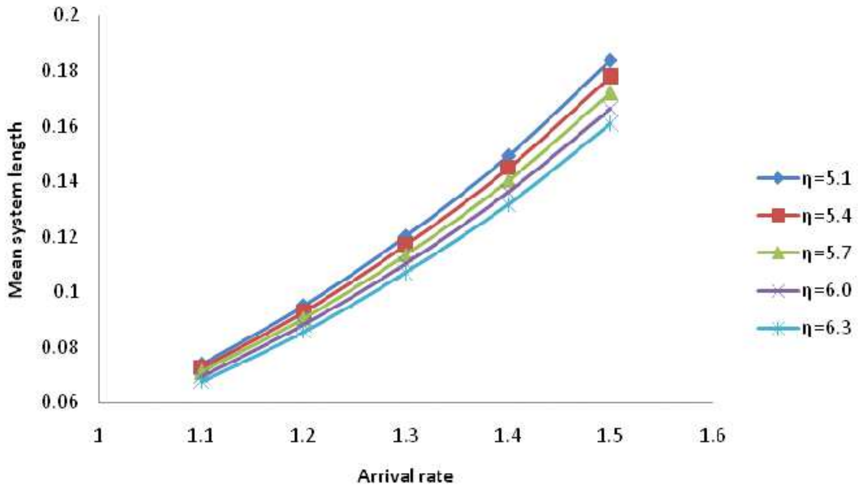

Figure 1. **Arrival rate (λ) versus mean orbit size (L_b) in regular service period**

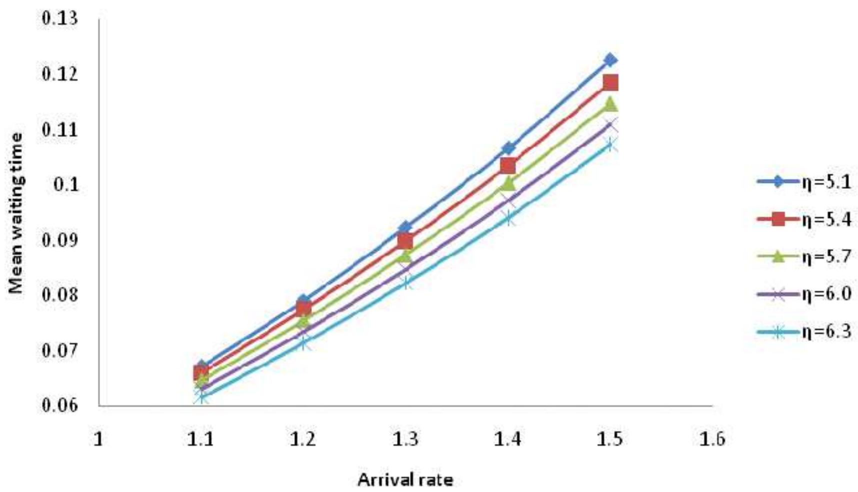

Figure 2. **Arrival rate (λ) versus mean waiting time (W_b) in regular service period**

Again fixing the values of $\mu_v = 4, \mu_b = 5, \mu_{v_r} = 1, \mu_{b_r} = 2$ and ranging the values of λ from 0.4 to 0.8 insteps of 0.1 and varying the values of η from 2.1 to 2.9 insteps of 0.2, we calculated the corresponding values of L_v and W_v for multiple working vacation and tabulated in Table 3 and in Table 4 respectively.

$\lambda\eta$	2.1	2.3	2.5	2.7	2.9
0.4	0.001163	0.001045	0.000921	0.000806	0.000705
0.5	0.002115	0.001893	0.001666	0.001458	0.001275
0.6	0.003383	0.003019	0.002654	0.002322	0.002032
0.7	0.004944	0.004404	0.003870	0.003387	0.002966
0.8	0.006753	0.006008	0.005279	0.004623	0.004052

Table 3. **Arrival rate (λ) versus mean orbit size (L_v) in WV period**

$\lambda\eta$	2.1	2.3	2.5	2.7	2.9
0.4	0.002907	0.002612	0.002302	0.002016	0.001763
0.5	0.004229	0.003785	0.003331	0.002915	0.002550
0.6	0.005638	0.005032	0.004424	0.003871	0.003387
0.7	0.007062	0.006291	0.005528	0.004838	0.004237
0.8	0.008441	0.007509	0.006599	0.005779	0.005065

Table 4. Arrival rate (λ) versus mean waiting time (W_v) in WV period

The corresponding graphs have been drawn for λ versus L_v and λ versus W_v and are shown in Figure 3 and in Figure 4 respectively. From the graphs it is seen that as λ increases both L_v and W_v increases for various values of η.

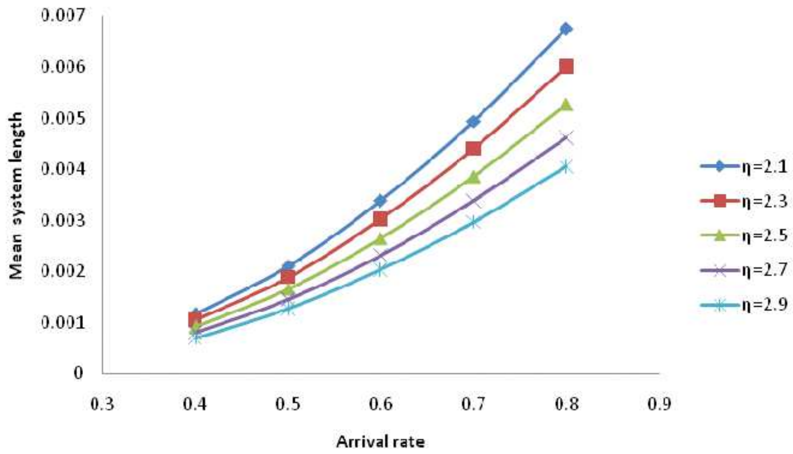

Figure 3. Arrival rate (λ) versus mean orbit size (L_v) in WV period

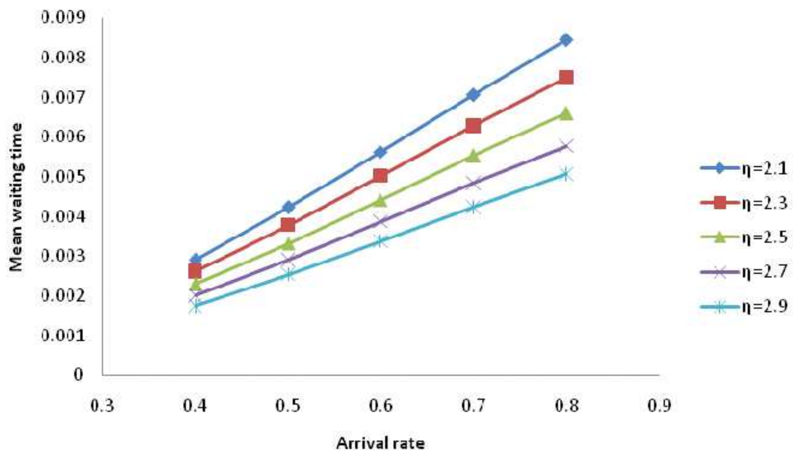

Figure 4. Arrival rate (λ) versus mean waiting time (W_v) in WV period

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