International Journal of Mathematics And its Applications
Volume 4, Issue 2–D (2016), 57–65.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Equitable and Non-Equitable Zagreb Indices of Graphs

Research Article

Akram Alqesmah[1]*, Anwar Alwardi[1] and R. Rangarajan[1]

1 Department of Studies in Mathematics, University of Mysore, Mysore 570 006, India.

Abstract: The Zagreb indices have been introduced more than forty four years ago by Gutman and Trinajestic as the sum of the squares of the degrees of the vertices, and the sum of the products of the degrees of pairs of adjacent vertices, respectively, [7]. In this paper, we introduce the first and second equitable and non-equitable Zagreb indices as $M_1^e(G) = \sum_{u\in V(G)} \left[deg_e(u)\right]^2$, $M_2^e(G) = \sum_{uv\in E(G)} deg_e(u)deg_e(v)$, $M_1^{ne}(G) = \sum_{u\in V(G)} \left[deg_{ne}(u)\right]^2$ and $M_2^{ne}(G) = \sum_{uv\in E(G)} deg_{ne}(u)deg_{ne}(v)$, respectively, where $deg_e(u)$ and $deg_{ne}(u)$ denotes the equitable and non-equitable degrees of vertex u. Exact values for wheel, firecracker and firefly graph families are obtained, some properties of the equitable and non-equitable Zagreb indices are established.

MSC: 05C69.

Keywords: First equitable Zagreb index, Second equitable Zagreb index, First non-equitable Zagreb index, Second non-equitable Zagreb index.

1. Introduction

In this research work, we concerned about simple graphs which are finite, undirected with no loops and multiple edges. Throughout this paper, we denote $p = |V(G)|$ and $q = |E(G)|$. The complement of G, denoted by $\overline{G}$, is a simple graph on the same set of vertices $V(G)$ in which two vertices u and v are adjacent if and only if they are not adjacent in G. Obviously, $E(G) \cup E(\overline{G}) = E(K_p)$ and $\overline{q} = \binom{p}{2} - q$. The open neighborhood and the closed neighborhood of v are denoted by $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. The degree of a vertex v in G, is denoted by $deg(v)$, and is defined to be the number of edges incident with v, shortly $deg(v) = |N(v)|$, the degree of the same vertex in $\overline{G}$ is then given by $deg_{\overline{G}}(v) = p - 1 - deg(v)$. The minimum degree of G is denoted by δ, and the maximum degree is denoted by Δ. If $\delta = \Delta = k$ for any graph G, we say G is a regular graph of degree k. A graph G is called bi-regular graph with degrees (l, k) if the degrees of all the vertices of G either l or k. Any two vertices u and v in G are side to be equitable adjacent if they are adjacent and $|deg(u) - deg(v)| \leq 1$. The open equitable neighborhood of a vertex $v \in V(G)$ denoted by $N_e(v) = \{u \in V(G) : u \text{ is equitable to } v\}$, and the equitable degree of v is denoted by $deg_e(v) = |N_e(v)|$. The maximum and minimum equitable degree of G are defined by $\Delta_e(G) = max\{deg_e(v) : v \in V(G)\}$ and $\delta_e(G) = min\{deg_e(v) : v \in V(G)\}$, respectively. The equitable complete graph is a connected graph which all its edges are equitable edges. For a connected graph G, a vertex v is called equitable isolated if $|deg(v) - deg(u)| \geq 2$, $\forall u \in N(v)$. A graph G is called an equitable edge-free graph if for any two adjacent vertices u and v in G, $|deg(u) - deg(v)| \geq 2$. The sum of the equitable degrees of a graph G is twice the number of equitable edges in it, that is $\sum_{v\in V(G)} deg_e(v) = 2q_e$, [1].

* E-mail: aalqesmah@gmail.com

We define the set of non-equitable neighborhood of a vertex v as $N_{ne}(v) = \{u \in N(v) : |deg(v) - deg(u)| \geq 2\}$ and denote $deg_{ne}(v) = |N_{ne}(v)|$. It is clear that, $deg_{ne}(v) = deg(v) - deg_e(v)$. All the definitions and terminologies about graph in this paper available in [8].

The Zagreb indices have been introduced by Gutman and Trinajestic [7]. They are defined as:

$$M_1(G) = \sum_{u\in V(G)} \big[deg(u)\big]^2 = \sum_{u\in V(G)} \sum_{v\in N(u)} deg(v) = \sum_{uv\in E(G)} \big[deg(u) + deg(v)\big].$$
$$M_2(G) = \sum_{uv\in E(G)} deg(u)deg(v) = \frac{1}{2} \sum_{u\in V(G)} deg(u) \sum_{v\in N(u)} deg(v).$$

Here $M_1(G)$ and $M_2(G)$ denote the first and the second Zagreb indices, respectively. For more information on Zagreb indices, we refer to [3, 6, 10–14].

Degree equitable adjacency has interesting applications in the context of social networks. In a network, nodes with nearly equal capacity may interact with each other in a better way. In society, persons with nearly equal status, tend to be friendly. In industry, employees with nearly equal powers form associations and move closely. Equitability among citizens in terms of wealth, health, status, etc is the goal of a democratic nation, [1]. This ideas and the Zagreb indices of graphs motivated us in this paper to introduce the equitable Zagreb indices of graphs. Exact values for wheel, firecracker and firefly graph families are obtained, some properties of the equitable and non-equitable Zagreb indices are established.

2. Some properties of the equitable and non-equitable Zagreb indices of graphs

In this section, we define the first and second equitable, relative equitable, non-equitable and relative non-equitable Zagreb indices of graphs and study some of their properties and exact values of some standard graphs.

Definition 2.1. *Let $G = (V, E)$ be a graph. Then the first and second equitable Zagreb indices of G are defined by*

(1). $M_1^e(G) = \sum_{u\in V(G)} \big[deg_e(u)\big]^2 = \sum_{u\in V(G)} \sum_{v\in N_e(u)} deg_e(v).$

(2). $M_2^e(G) = \sum_{uv\in E(G)} deg_e(u)deg_e(v) = \frac{1}{2} \sum_{u\in V(G)} deg_e(u) \sum_{v\in N(u)} deg_e(v).$

Definition 2.2. *For a graph G, the first and second relative equitable Zagreb indices of G are defined by*

(1). $RM_1^e(G) = \sum_{uv\in E(G)} [deg_e(u) + deg_e(v)] = \sum_{u\in V(G)} deg_e(u)deg(u)$

$$= \sum_{u\in V(G)} \sum_{v\in N(u)} deg_e(v).$$

(2). $RM_2^e(G) = \sum_{uv\in E(G)} \big[deg_e(u)deg(v) + deg(u)deg_e(v)\big]$

$$= \sum_{u\in V(G)} deg_e(u) \sum_{v\in N(u)} deg(v) = \sum_{u\in V(G)} deg(u) \sum_{v\in N(u)} deg_e(v).$$

Definition 2.3. *Let $G = (V, E)$ be a graph. Then the first and second non-equitable Zagreb indices of G are defined by*

(1). $M_1^{ne}(G) = \sum_{u\in V(G)} \big[deg_{ne}(u)\big]^2 = \sum_{u\in V(G)} \sum_{v\in N_{ne}(u)} deg_{ne}(v).$

(2). $M_2^{ne}(G) = \sum_{uv\in E(G)} deg_{ne}(u)deg_{ne}(v) = \frac{1}{2} \sum_{u\in V(G)} deg_{ne}(u) \sum_{v\in N(u)} deg_{ne}(v).$

Definition 2.4. *For a graph G, the first and second relative non-equitable Zagreb indices of G are defined by*

(1). $RM_1^{ne}(G) = \sum_{uv\in E(G)} [deg_{ne}(u) + deg_{ne}(v)] = \sum_{u\in V(G)} deg_{ne}(u)deg(u)$

$$= \sum_{u\in V(G)} \sum_{v\in N(u)} deg_{ne}(v).$$

(2). $RM_2^{ne}(G) = \sum_{uv\in E(G)} \big[deg_{ne}(u)deg(v) + deg(u)deg_{ne}(v)\big]$

$$= \sum_{u\in V(G)} deg_{ne}(u) \sum_{v\in N(u)} deg(v) = \sum_{u\in V(G)} deg(u) \sum_{v\in N(u)} deg_{ne}(v).$$

Example 2.5. *Let G be a graph as in Figure 1,*

(1). $M_1^e(G) = \sum_{u\in V(G)} \big(deg_e(u)\big)^2 = \sum_{i=1}^{5} \big(deg_e(v_i)\big)^2 = 24.$

(2). $M_2^e(G) = \sum_{uv\in E(G)} deg_e(u)deg_e(v) = 31.$

(3). $RM_1^e(G) = \sum_{uv\in E(G)} [deg_e(u) + deg_e(v)] = 30.$

(4). $RM_2^e(G) = \sum_{u\in V(G)} deg_e(u) \sum_{v\in N(u)} deg(v) = 88.$

(5). $M_1^{ne}(G) = \sum_{u\in V(G)} \big(deg_{ne}(u)\big)^2 = \sum_{i=1}^{5} \big(deg_{ne}(v_i)\big)^2 = 6.$

(6). $M_2^{ne}(G) = \sum_{uv\in E(G)} deg_{ne}(u)deg_{ne}(v) = 4.$

(7). $RM_1^{ne}(G) = \sum_{uv\in E(G)} [deg_{ne}(u) + deg_{ne}(v)] = 12.$

(8). $RM_2^{ne}(G) = \sum_{u\in V(G)} deg_{ne}(u) \sum_{v\in N(u)} deg(v) = 34.$

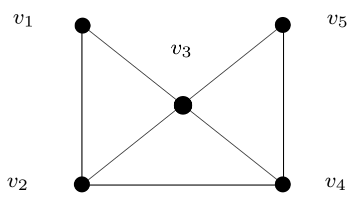

Figure 1. **Graph G**

Note that, $M_1^e(G) = M_1(G_e)$ and $M_2^e(G) \geq M_2(G_e)$, where G_e is the equitable graph of the graph G which has the same vertices of G and any two vertices are adjacent in G_e if they are equitable adjacent, [1]. According to Definitions (2.1, 2.2, 2.3, 2.4) and the Definition of Zagreb indices, the first and second equitable and non-equitable Zagreb indices can be expressed as follows.

Theorem 2.6. *For any graph G,*

(1). $M_1^e(G) = M_1(G) + M_1^{ne}(G) - 2RM_1^{ne}(G)$.

(2). $M_2^e(G) = M_2(G) + M_2^{ne}(G) - RM_2^{ne}(G)$.

(3). $M_1^{ne}(G) = M_1(G) + M_1^e(G) - 2RM_1^e(G)$.

(4). $M_2^{ne}(G) = M_2(G) + M_2^e(G) - RM_2^e(G)$.

Proof.

(1). $$M_1^e(G) = \sum_{u\in V(G)} \big(deg_e(u)\big)^2 = \sum_{u\in V(G)} \big(deg(u) - deg_{ne}(u)\big)^2$$
$$= M_1(G) + M_1^{ne}(G) - 2RM_1^{ne}(G).$$

(2). $$M_2^e(G) = \sum_{uv\in E(G)} deg_e(u)deg_e(v) = \sum_{uv\in E(G)} \big(deg(u) - deg_{ne}(u)\big)\big(deg(v) - deg_{ne}(v)\big)$$
$$= M_2(G) + M_2^{ne}(G) - RM_2^{ne}(G).$$

(3). $$M_1^{ne}(G) = \sum_{u\in V(G)} \big(deg_{ne}(u)\big)^2 = \sum_{u\in V(G)} \big(deg(u) - deg_e(u)\big)^2$$
$$= M_1(G) + M_1^e(G) - 2RM_1^e(G).$$

(4). $$M_2^{ne}(G) = \sum_{uv\in E(G)} deg_{ne}(u)deg_{ne}(v) = \sum_{uv\in E(G)} \big(deg(u) - deg_e(u)\big)\big(deg(v) - deg_e(v)\big)$$
$$= M_2(G) + M_2^e(G) - RM_2^e(G).$$

□

Corollary 2.7. *For any graph G,*

(1). $M_1(G) = RM_1^e(G) + RM_1^{ne}(G)$.

(2). $2M_2(G) = RM_2^e(G) + RM_2^{ne}(G)$.

From Definitions (2.1, 2.2, 2.3, 2.4), it is easy to check the following proposition:

Proposition 2.8.

(1). For any graph G, $M_1^e(G) \le RM_1^e(G)$ and $M_1^{ne}(G) \le RM_1^{ne}(G)$, where the equalities hold if and only if G is either an equitable edge-free graph or an equitable complete graph.

(2). For any graph G, $M_1^e(G) \le M_1(G)$ and $M_2^e(G) \le M_2(G)$ with the equality if and only if G is an equitable complete graph.

(3). For any graph G, $M_1^{ne}(G) \le M_1(G)$ and $M_2^{ne}(G) \le M_2(G)$ with the equality if and only if G is equitable edge-free graph.

Proposition 2.9. *For any regular or bi-regular with degrees $(k, k+1)$ graph G, $M_1(G) = M_1^e(G)$ and $M_2(G) = M_2^e(G)$.*

Proposition 2.10. *For any complete bipartite graph $K_{r,m}$,*

(1). $$M_1(K_{r,m}) = \begin{cases} M_1^e(K_{r,m}), & \text{if } |r-m| \le 1; \\ M_1^{ne}(K_{r,m}), & \text{if } |r-m| \ge 2. \end{cases}$$

(2). $M_2(K_{r,m}) = \begin{cases} M_2^e(K_{r,m}), & \text{if } |r-m| \leq 1; \\ M_2^{ne}(K_{r,m}), & \text{if } |r-m| \geq 2. \end{cases}$

The equitable complement graph of a graph G denoted by $\overline{G}^e$ is the graph with the same vertices as G and any two vertices u, v are adjacent if u and v are not equitable adjacent in G, [1]. The relation between the complement of a graph and its equitable complement graph can be found in the following Lemma.

Lemma 2.11 ([1]). *For any graph G, $\overline{G} \subseteq \overline{G}^e$. Furthermore, $\overline{G} \cong \overline{G}^e$ if and only if G is isomorphic to an equitable complete graph.*

Theorem 2.12. *For any graph G,*

(1). $M_1^e(\overline{G}) \leq p\big(p-1\big)^2 - 4q_e(p-1) + M_1^e(G)$,

(2). $M_2^e(\overline{G}) \leq \frac{1}{2}(2p-3)M_1^e(G) - M_2(G_e) + \frac{1}{2}\big(p-1\big)^2\big(p(p-1) - 6q_e\big) + 2q_e^2$,

where the equalities are attained if and only if G is isomorphic to an equitable complete graph.

Proof. Let G be a graph and $\overline{G}$ its complement. According to Lemma 2.11, we have

$$(1).\ M_1^e(\overline{G}) \leq M_1^e(\overline{G}^e) = \sum_{u\in V(G)} \big(deg_e^{\overline{G}^e}(u)\big)^2 = \sum_{u\in V(G)} \big(p-1-deg_e(u)\big)^2 = p\big(p-1\big)^2 - 4q_e(p-1) + M_1^e(G).$$

$$\begin{aligned}(2).\ M_2^e(\overline{G}) \leq M_2^e(\overline{G}^e) =& \frac{1}{2} \sum_{u\in V(G)} deg_e^{\overline{G}^e}(u) \sum_{v\in N^{\overline{G}^e}(u)} deg_e^{\overline{G}^e}(v) \\ =& \frac{1}{2} \sum_{u\in V(G)} \big(p-1-deg_e(u)\big) \sum_{v\in N^{\overline{G}^e}(u)} \big(p-1-deg_e(v)\big) \\ =& \frac{1}{2} \sum_{u\in V(G)} \big(p-1-deg_e(u)\big)\Big[\big(p-1\big)^2 - (p-2)deg_e(u) - 2q_e + \sum_{v\in N_e(u)} deg_e(v)\Big] \\ =& \frac{1}{2}(2p-3)M_1^e(G) - M_2(G_e) + \frac{1}{2}\big(p-1\big)^2\big(p(p-1) - 6q_e\big) + 2q_e^2.\end{aligned}$$

Note that, the equality $\sum_{v\in N^{\overline{G}^e}(u)} deg_e(v) = 2q_e - deg_e(u) - \sum_{v\in N_e(u)} deg_e(v)$ is used. The equalities of the bounds are straightforward by Lemma 2.11. □

3. First and Second equitable Zagreb indices for some graphs families

In this section, we will compute the first and second equitable Zagreb indices for the wheel, firecracker and firefly graphs as follows.

Proposition 3.1. *For the wheel graph W_p, where $p \geq 3$,*

(1). $M_1^e(W_p) = \begin{cases} M_1(W_p), & \text{if } p \leq 5; \\ M_1(C_{p-1}), & \text{otherwise.} \end{cases}$

(2). $M_2^e(W_p) = \begin{cases} M_2(W_p), & \text{if } p \leq 5; \\ M_2(C_{p-1}), & \text{otherwise.} \end{cases}$

Proof. If $p \leq 5$, then the wheel graph becomes regular (with degree 2 or 3) or bi-regular graph with degrees $(3,4)$. Thus by Proposition 2.9, $M_1^e(W_p) = M_1(W_p)$ and $M_2^e(W_p) = M_2(W_p)$. On the other hand, if $p \geq 6$, the center vertex in W_p has equitable degree zero, so $(W_p)_e \cong C_{p-1}$. □

An (n,k)-firecracker graph denoted by $F_{n,k}$ is a graph obtained by the concatenation of n k-stars by linking one leaf from each, [4].

Proposition 3.2. *For the firecracker graph $F_{n,k}$, where $n,k \geq 2$, the first and second equitable Zagreb indices are given by,*

$$\text{(1). } M_1^e(F_{n,k}) = \begin{cases} 4n+2, & \text{if } k=2; \\ 14n-19, & \text{if } k=3; \\ 10(n-1), & \text{if } k=4; \\ 2(5n-9), & \text{if } k=5; \\ 4n-6, & \text{if } k \geq 6. \end{cases}$$

$$\text{(2). } M_2^e(F_{n,k}) = \begin{cases} 4n, & \text{if } k=2; \\ 17n-19, & \text{if } k=3; \\ 12n-17, & \text{if } k=4 \text{ and } n \geq 3; \\ 3(4n-9), & \text{if } k=5 \text{ and } n \geq 3; \\ 4n-8, & \text{if } k \geq 6. \end{cases}$$

Proof. From the definition of the firecracker graph, one can see that $F_{n,k}$ consists of n k-stars and one path P_n linking them. So, we have the following cases:

Case 1. If $k=2$, then all the leafs whose adjacent to the middle vertices of P_n have equitable degrees zero. Thus, $(F_{n,k})_e \cong P_{n+2}$. Hence, $M_1^e(F_{n,2}) = M_1(P_{n+2}) = 4(n+2)-6 = 4n+2$ and $M_2^e(F_{n,2}) = M_2(P_{n+2}) = 4(n+2)-8 = 4n$.

Case 2. If $k=3$, then $F_{n,3}$ is an equitable complete graph, so by Proposition 2.8, $M_1^e(F_{n,3}) = M_1(F_{n,3}) = 14n-19$ and $M_2^e(F_{n,3}) = M_2(F_{n,3}) = 17n-19$.

Case 3. If $k=4$, then all the leafs in $F_{n,4}$ have equitable degrees zero and all the center vertices of the n 4-stars have equitable degrees one. Hence, $M_1^e(F_{n,4}) = n+8+9(n-2) = 10(n-1)$ and if $n \geq 3$, $M_2^e(F_{n,4}) = 3(n-2)+9(n-3)+4+12 = 12n-17$.

Case 4. If $k=5$, then all the leafs in $F_{n,5}$ and the center vertices of the first and last 5-stars have equitable degrees zero. Hence, $M_1^e(F_{n,5}) = n+9(n-2) = 2(5n-9)$ and if $n \geq 3$, $M_2^e(F_{n,5}) = 3(n-2)+9(n-3)+6 = 3(4n-9)$.

Case 5. If $k \geq 6$, then all the leafs in $F_{n,k}$ and the center vertices of the n k-stars have equitable degrees zero. Thus $(F_{n,k})_e \cong P_n$. Hence, $M_1^e(F_{n,k}) = M_1(P_n) = 4n-6$ and $M_2^e(F_{n,2}) = M_2(P_n) = 4n-8$. □

A firefly graph $F_{s,t,p-2s-2t-1}$ ($s \geq 0$, $t \geq 0$ and $p-2s-2t-1 \geq 0$) is a graph of order p that consists of s triangles, t pendent paths of length 2 and $p-2s-2t-1$ pendent edges, sharing a common vertex.

Let $\mathfrak{F}_p$ be the set of all firefly graphs $F_{s,t,p-2s-2t-1}$. Note that $\mathfrak{F}_p$ contains the stars S_p ($\cong F_{0,0,p-1}$), stretched stars ($\cong F_{0,t,p-2t-1}$), friendship graphs ($\cong F_{\frac{p-1}{2},0,0}$) and butterfly graphs ($\cong F_{s,0,p-2s-1}$), [9]. In the following, we will compute the first and second equitable Zagreb indices of the firefly graph in cases $s,t > 0$, $s=0$ and $t>0$, $s>0$ and $t=0$ and $s=t=0$.

Proposition 3.3. *For the firefly graph $F_{s,t,p-2s-2t-1}$, where $s,t > 0$,*

$$\text{(1). } M_1^e(F_{s,t,p-2s-2t-1}) = \begin{cases} 22, & \text{if } p=5 \text{ and } s=t=1; \\ 2(s+t), & \text{otherwise.} \end{cases}$$

$$\text{(2). } M_2^e(F_{s,t,p-2s-2t-1}) = \begin{cases} 24, & \text{if } p=5 \text{ and } s=t=1; \\ s+t, & \text{otherwise.} \end{cases}$$

Proof. We have $F_{s,t,p-2s-2t-1}$, where $s,t > 0$. By the definition of $F_{s,t,p-2s-2t-1}$, if $s=t=1$, then $p \geq 5$. So, if $p=5$, then $F_{1,1,5}$ is an equitable complete graph. Hence, $M_1^e(F_{1,1,5}) = M_1(F_{1,1,5}) = 22$ and $M_2^e(F_{1,1,5}) = M_2(F_{1,1,5}) = 24$. Otherwise, $\left(F_{s,t,p-2s-2t-1}\right)_e \cong (s+t)P_2$. Hence, $M_1^e(F_{s,t,p-2s-2t-1}) = 2(s+t)$ and $M_2^e(F_{s,t,p-2s-2t-1}) = s+t$. □

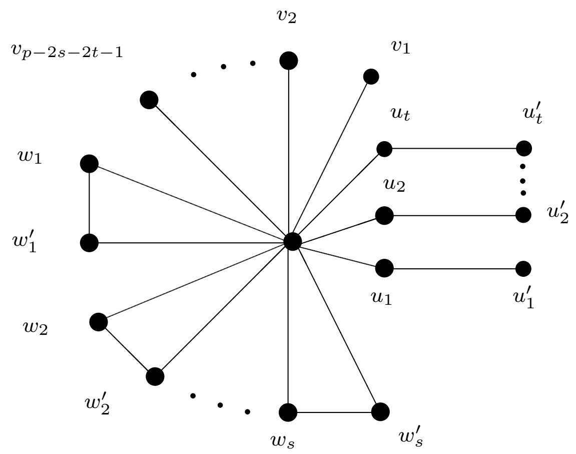

Figure 2. **Firefly graph** $F_{s,t,p-2s-2t-1}$

Proposition 3.4. *For the firefly graph* $F_{0,t,p-2t-1}$, *where* $t > 0$,

(1). If $t = 1$, *then*

$$(1).\ M_1^e(F_{0,1,p-3}) = \begin{cases} 6, & \text{if } p = 3 \text{ or } 5; \\ 10, & \text{if } p = 4; \\ 2, & \text{otherwise.} \end{cases}$$

$$(2).\ M_2^e(F_{0,1,p-3}) = \begin{cases} 4, & \text{if } p = 3 \text{ or } 5; \\ 8, & \text{if } p = 4; \\ 1, & \text{otherwise.} \end{cases}$$

(2). If $t = 2$, *then*

$$(1).\ M_1^e(F_{0,2,p-5}) = \begin{cases} 14, & \text{if } p = 5 \text{ or } 6; \\ 4, & \text{otherwise.} \end{cases}$$

$$(2).\ M_2^e(F_{0,2,p-5}) = \begin{cases} 12, & \text{if } p = 5 \text{ or } 6; \\ 2, & \text{otherwise.} \end{cases}$$

(3). If $t = 3$, *then*

$$(1).\ M_1^e(F_{0,3,p-7}) = \begin{cases} 24, & \text{if } p = 7; \\ 6, & \text{otherwise.} \end{cases}$$

$$(2).\ M_2^e(F_{0,3,p-7}) = \begin{cases} 24, & \text{if } p = 7; \\ 3, & \text{otherwise.} \end{cases}$$

(4). If $t \geq 4$, *then* $M_1^e(F_{0,t,p-2t-1}) = 2t$ *and* $M_2^e(F_{0,t,p-2t-1}) = t$.

Proof. We have four cases:

Case 1. Let $t = 1$. If $p = 3$ or 5, then $\left(F_{0,1,p-3}\right)_e \cong P_3$. Also, if $p = 4$, then $\left(F_{0,1,1}\right)_e \cong P_4$ and if $p \geq 6$, then $\left(F_{0,1,p-3}\right)_e \cong P_2$. Hence the result.

Case 2. Suppose $t = 2$. If $p = 5$ or 6, then $\left(F_{0,2,p-5}\right)_e \cong P_5$ and if $p \geq 7$, then $\left(F_{0,2,p-5}\right)_e \cong 2P_2$. Hence the result.

Case 3. Let $t = 3$. If $p = 7$, then $F_{0,3,0}$ is an equitable complete graph. Hence, $M_1^e(F_{0,3,0}) = M_1(F_{0,3,0}) = 24$ and

$M_2^e(F_{0,3,0}) = M_2(F_{0,3,0}) = 24$. Otherwise, $\left(F_{0,3,p-7}\right)_e \cong 3P_2$.

Case 4. If $t \geq 4$, then $\left(F_{0,t,p-2t-1}\right)_e \cong tP_2$. Hence the result. □

Proposition 3.5. *For the firefly graph $F_{s,0,p-2s-1}$, where $s > 0$,*

(1). If $s = 1$, then

(1). $M_1^e(F_{1,0,p-3}) = \begin{cases} 12, & \text{if } p = 3 \text{ or } 4; \\ 2, & \text{otherwise.} \end{cases}$

(2). $M_2^e(F_{1,0,p-3}) = \begin{cases} 12, & \text{if } p = 3 \text{ or } 4; \\ 1, & \text{otherwise.} \end{cases}$

(2). If $s \geq 2$, then $M_1^e(F_{s,0,p-2s-1}) = 2s$ and $M_2^e(F_{s,0,p-2s-1}) = s$.

Proof. We have two cases:

Case 1. Let $s = 1$. If $p = 3$ or 4, then $\left(F_{1,0,p-3}\right)_e \cong C_3$ and if $p \geq 5$, then $\left(F_{1,0,p-3}\right)_e \cong P_2$. Hence the result.

Case 2. If $s \geq 2$, then $\left(F_{s,0,p-2s-1}\right)_e \cong sP_2$. Hence the result. □

Proposition 3.6. *For the firefly graph $F_{0,0,p-1}$,*

(1). $M_1^e(F_{0,0,p-1}) = \begin{cases} 4p-6, & \text{if } p = 2 \text{ or } 3; \\ 0, & \text{otherwise.} \end{cases}$

(2). $M_2^e(F_{0,0,p-1}) = \begin{cases} 1, & \text{if } p = 2; \\ 4, & \text{if } p = 3; \\ 0, & \text{otherwise.} \end{cases}$

Proof. It is clear that, $F_{0,0,p-1} \cong S_p$. So, if $p = 2$ or 3, then $F_{0,0,p-1}$ is a path. Otherwise, $\left(F_{0,0,p-1}\right)_e \cong \overline{K_p}$. □

4. Conclusion

In this paper, we initiate the study of the equitable Zagreb indices of graphs, and there are a lot of problems in this concept for future study, we mention some of them as follows:

(1). Classification the graphs G such that $M_1^e(G) = M_2^e(G)$.

(2). Calculate the equitable Zagreb indices of $G_1 + G_2$, $G_1 \Box G_2$, $G_1 \circ G_2, \ldots$ for any two graphs G_1 and G_2.

References

[1] A.N.Alkenani, N.D.Soner and Anwar Alwardi, *Graphs and degree equitability*, Scientific Research, Applied Mathematics, 4(2013) 1199-1203.

[2] A.R.Ashrafi, T.Došlić and A.Hamzeha, *The Zagreb coindices of graph operations*, Discrete Applied Mathematics, 158 (2010), 1571-1578.

[3] J.Braun, A.Kerber, M.Meringer and C.Rucker, *Similarity of molecular descriptors: the equivalence of Zagreb indices and walk counts*, MATCH Commun. Math. Comput. Chem., 54(2005), 163-176.

[4] W.C.Chen, H.I.Lü and Y.N.Yeh, *Operations of Interlaced Trees and Graceful Trees*, Southeast Asian Bull. Math., 21(1997), 337-348.

[5] T.Došlić, *Vertex-Weighted Wiener Polynomials for Composite Graphs*, Ars Math. Contemp., 1(2008), 66-80.

[6] I.Gutman and K.C.Das, *The first Zagreb index* 30 *years after*, MATCH Commun. Math. Comput. Chem., 50(2004), 83-92.

[7] I.Gutman and N.Trinajstic, *Graph theory and molecular orbitals*, Total π-electron energy of alternant hydrocarbons, Chem. Phys. Lett., 17(1972), 535-538.

[8] F.Harary, *Graph theory*, Addison-Wesley, Reading Mass, (1969).

[9] W.X.Hong and L.H.You, *On the eigenvalues of firefly graphs*, Transactions on Combinatorics, 3(3)(2014), 1-9.

[10] M.H.Khalifeh, H.Yousefi-Azari and A.R.Ashrafi, *The first and second Zagreb indices of some graph operations*, Discrete Applied Mathematics, 157(2009), 804-811.

[11] S.Nikolić, G.Kovačević, A.Miličević and N.Trinajstić, *The Zagreb indices* 30 *years after*, Croat. Chem. Acta, 76(2003), 113-124.

[12] B.Zhou and I.Gutman, *Further properties of Zagreb indices*, MATCH Commun. Math. Comput. Chem., 54(2005), 233-239.

[13] B.Zhou and I.Gutman, *Relations between Wiener, hyper-Wiener and Zagreb indices*, Chem. Phys. Lett., 394(2004), 93-95.

[14] B.Zhou, *Zagreb indices*, MATCH Commun. Math. Comput. Chem., 52(2004), 113-118.