International Journal of Mathematics And its Applications
Volume 4, Issue 2–D (2016), 75–80.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Formula for finding n^{th} Term of Fibonacci-Like Sequence of Higher Order

Research Article

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Abstract: The Fibonacci and Lucas sequences are well-known examples of second order recurrence sequences. The Fibonacci and Lucas sequences have been discussed in so many articles and books. The well-known Fibonacci sequence $\{F_n\}$ is defined as $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ and $F_0 = 0$, $F_1 = 1$, where F_n is a n^{th} number of sequence. Many authors have defined Fibonacci pattern based sequences which are popularized and known as Fibonacci-Like sequences. These are similar to Fibonacci sequences in pattern, but initial conditions are different. In this paper, we present formula for finding n^{th} term of Fibonacci-Like Sequence of Higher Order.

MSC: 11B37, 11B39.

Keywords: Fibonacci sequence, Fibonacci-Like sequence, Fibonacci-like sequence of higher order.

1. Introduction

The Fibonacci and Lucas sequences are well-known examples of second order recurrence sequences. The Fibonacci sequence, Lucas numbers and their generalization have many interesting properties and applications to almost every field. The Fibonacci sequence has been discussed in so many articles and books [10–12]. The well-known Fibonacci sequence is defined by

$$F_{\mathrm{n}} = F_{\mathrm{n}-1} + F_{\mathrm{n}-2},\ \mathrm{n} \geq 2 \text{ with } \mathrm{F}_0 = 0 \text{ and } \mathrm{F}_1 = 1. \tag{1}$$

The similar interpretation also exists for Lucas sequence. Lucas sequence is defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2},\ n \geq 2 \text{ with } L_0 = 2,\ L_1 = 1. \tag{2}$$

Many authors have been defined Fibonacci pattern based sequences which are known as Fibonacci-Like sequences. The Fibonacci-Like sequence [4] is defined by recurrence relation

$$S_n = S_{n-1} + S_{n-2},\ n \geq 2 \text{ with } S_0 = 2,\ S_1 = 2. \tag{3}$$

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The associated initial conditions S_0 and S_1 are the sum of initial conditions of Fibonacci and Lucas sequence respectively. That is $S_0 = F_0 + L_0$ and $S_1 = F_1 + L_1$. Fibonacci-Like sequence [5] is defined by the recurrence relation

$$H_n = 2H_{n-1} + H_{n-2} \text{ for } n \geq 2 \text{ with } H_0 = 2\ ,\ H_1 = 1. \tag{4}$$

Various properties of Fibonacci-Like sequence of order two have been presented in the paper [4]. In [7] Natividad's derived a formula in solving a Fibonacci-Like sequence using the Binet's formula and Bueno [1] gives the formula for the k^{th} term of Natividad's Fibonacci-Like sequence. Tribonacci (order 3), Tetranacci (order 4), Pentanacci (order 5) sequences and among others are also called Fibonacci sequences of higher order. When initial terms will be considered arbitrary with recurrence relations, then these will be known as Fibonacci-like sequences of higher order. Natividad [8, 9] established a formula in solving the n^{th} term of the Tribonacci-Like sequence and Fibonacci-Like sequence of fourth, fifth and sixth order. Natividad and Policarpio [8] provided and proved a formula for $n \geq 4$ Concerning Fibonacci-Like sequence of order 3. They found out that given the arbitrary terms s_1, s_2 and s_3 the formula to find the n^{th} term or s_n is

$$S_n = T_{n-2}S_1 + \ (T_{n-2} + T_{n-3})\, S_2 + \ T_{n-1}S_3, \qquad n \geq 4. \tag{5}$$

Natividad [9] derived formula for Fibonacci-Like sequence of higher orders. The formula to find n^{th} term of Tetranacci-Like sequence (Z_1, Z_2, Z_3 and Z_4 are real numbers) is

$$Z_n = Y_{n-2}Z_1 + (Y_{n-2} + Y_{n-3})\, Z_2 + (Y_{n-2} + Y_{n-3} + Y_{n-4})\, Z_3 + \ Y_{n-1}Z_4, \qquad n \geq 5. \tag{6}$$

The formula to find n^{th} term of Pentanacci-Like sequence (Q_1, Q_2, Q_3, Q_4 and Q_5 are real numbers) is

$$Q_n = P_{n-2}Q_1 + (P_{n-2} + P_{n-3})\, Q_2 + (P_{n-2} + P_{n-3} + P_{n-4})\, Q_3 + (P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5})\, Q_4 + P_{n-1}Q_5, \quad n \geq 6 \tag{7}$$

The formula to find n^{th} term of Hexanacci-Like sequence (M_1, M_2, M_3, M_4, M_5 and M_6 are real numbers) is

$$\begin{aligned} M_n = {} & R_{n-2}M_1 + (R_{n-2} + R_{n-3})\, M_2 + (R_{n-2} + R_{n-3} + R_{n-4})\, M_3 + (R_{n-2} + R_{n-3} + R_{n-4} + R_{n-5})\, M_4 \\ & + (R_{n-2} + R_{n-3} + R_{n-4} + R_{n-5} + R_{n-6})\, M_5 + R_{n-1}M_6, \quad n \geq 7. \end{aligned} \tag{8}$$

Definition 1.1. *The sequence $H_1, H_2, H_3, \ldots, H_n$ in which $H_n = H_{n-1} + H_{n-2} + H_{n-3} + H_{n-4} + H_{n-5} + H_{n-6} + H_{n-7}$ is a generalization for the Fibonacci-Like sequence of seventh order(Heptanacci-Like sequence). This sequence follows the pattern of Fibonacci sequence of seventh order (Heptanacci sequence). The Heptanacci sequence $\{I_n\}$ defined by the recurrence relation*

$$I_n = I_{n-1} + I_{n-2} + I_{n-3} + I_{n-4} + I_{n-5} + I_{n-6} + I_{n-7} \quad for\ n \geq 8, \tag{9}$$

where $I_0 = I_1 = I_2 = I_3 = I_4 = I_5 = I_6 = 0$, $I_7 = 1$. First few terms are Heptanacci sequence are 0, 0, 0, 0, 0, 0, 1, 1, 2, 4, 8, 16, 32, 64, 127,...

Definition 1.2. *The sequence $u_1, u_2, u_3, u_4, \ldots, u_n$ in which $u_n = u_{n-1} + u_{n-2} + u_{n-3} + u_{n-4} + \cdots + u_{n-m}$ is generalization for the Fibonacci-like sequence of m^{th} order (m-nacci-like sequence). This sequence follows the pattern of Fibonacci sequence of m^{th} order (m-nacci sequence). The m-nacci sequence $\{x_n\}$ defined by recurrence relation*

$$x_n = x_{n-1} + x_{n-2} + x_{n-3} + x_{n-4} + \ldots + x_{n-m} \quad for\ n \geq m, \tag{10}$$

where $x_0 = x_1 = x_2 = x_3 = \cdots = x_{m-2} = 0$, $x_{m-1} = 1$.

2. Main Result

In this section we present formula for finding n^{th} term of Fibonacci-Like sequence of higher order.

Theorem 2.1. *For any real numbers H_1, H_2, H_3, H_4, H_5, H_6, H_7, the formula for finding n^{th} term of Fibonacci-Like sequence of seventh order (Heptanacci-Like sequence) is*

$$H_n = I_{n-2}H_1 + \left(\sum_{i=2}^{3} I_{n-i}\right) H_2 + \left(\sum_{i=2}^{4} I_{n-i}\right) H_3 + \left(\sum_{i=2}^{5} I_{n-i}\right) H_4 + \left(\sum_{i=2}^{6} I_{n-i}\right) H_5 + \left(\sum_{i=2}^{7} I_{n-i}\right) H_6 + I_{n-1}H_7, \quad (11)$$

where H_n is n^{th} term of Heptanacci-Like sequence and I_{n-1}, I_{n-2}, I_{n-3}, I_{n-4}, I_{n-5}, I_{n-6}, I_{n-7} are corresponding Fibonacci numbers of seventh order (Heptanacci sequence).

Proof. Let the first seven terms of Heptanacci-Like sequence be H_1, H_2, H_3, H_4, H_5, H_6, H_7. Now, we derive an explicit formula for H_n. The numerical coefficients of H_1, H_2, H_3, H_4, H_5, H_6, H_7 for $n \geq 8$ are tabulated below:

Number of terms	n^{th} Heptanacci-Like sequence	Coefficient						
		H_1	H_2	H_3	H_4	H_5	H_16	H_7
1	H_8	1	1	1	1	1	1	1
2	H_9	1	2	2	2	2	2	2
3	H_{10}	2	3	4	4	4	4	4
4	H_{11}	4	6	7	8	8	8	8
5	H_{12}	8	12	14	15	16	16	16
6	H_{13}	16	24	28	30	31	32	32
7	H_{14}	32	48	56	60	62	63	64
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

From above table, we note that coefficient of H_1 corresponds to I_{n-2}, H_2 corresponds to $I_{n-2} + I_{n-3}$, hence the n^{th} term H_n is given by

$$H_n = I_{n-2}H_1 + \left(\sum_{i=2}^{3} I_{n-i}\right) H_2 + \left(\sum_{i=2}^{4} I_{n-i}\right) H_3 + \left(\sum_{i=2}^{5} I_{n-i}\right) H_4 + \left(\sum_{i=2}^{6} I_{n-i}\right) H_5 + \left(\sum_{i=2}^{7} I_{n-i}\right) H_6 + I_{n-1}H_7.$$

By using mathematical induction, the formula can be validating for any values of n. The formula can be easily verify using $n = 8, 9, 10$ and so on.

Let $P(n)$ be taken as

$$H_n = I_{n-2}H_1 + \left(\sum_{i=2}^{3} I_{n-i}\right) H_2 + \left(\sum_{i=2}^{4} I_{n-i}\right) H_3 + \left(\sum_{i=2}^{5} I_{n-i}\right) H_4 + \left(\sum_{i=2}^{6} I_{n-i}\right) H_5 + \left(\sum_{i=2}^{7} I_{n-i}\right) H_6 + I_{n-1}H_7.$$

Now, we assume that theorem is true for some integer $k \geq 8$ then $P(k)$ is

$$P(k) : H_k = I_{k-2}H_1 + \left(\sum_{i=2}^{3} I_{k-i}\right) H_2 + \left(\sum_{i=2}^{4} I_{k-i}\right) H_3 + \left(\sum_{i=2}^{5} I_{k-i}\right) H_4 + \left(\sum_{i=2}^{6} I_{k-i}\right) H_5 + \left(\sum_{i=2}^{7} I_{k-i}\right) H_6 + I_{k-1}H_7. \quad (12)$$

We shall now prove that $P(k+1)$ is true whenever $P(k)$ is true that is

$$P(k+1) : H_{k+1} = I_{k-1}H_1 + \left(\sum_{i=1}^{3} I_{k-i}\right) H_2 + \left(\sum_{i=1}^{4} I_{k-i}\right) H_3 + \left(\sum_{i=1}^{5} I_{k-i}\right) H_4 + \left(\sum_{i=1}^{6} I_{k-i}\right) H_5 + \left(\sum_{i=1}^{7} I_{k-i}\right) H_6 + I_k H_7. \quad (13)$$

Now to verify, we will provide the assumption of $P(k)$ implies the truth of $P(k+1)$. We shall add H_{k-1}, H_{k-2}, H_{k-3} , H_{k-4}, H_{k-5} and H_{k-6} to both side of $P(k)$ the equation (12) will become

$$\sum_{i=0}^{6} H_{k-i} = I_{k-2}H_1 + \left(\sum_{i=2}^{3} I_{k-i}\right) H_2 + \left(\sum_{i=2}^{4} I_{k-i}\right) H_3 + \left(\sum_{i=2}^{5} I_{k-i}\right) H_4 + \left(\sum_{i=2}^{6} I_{k-i}\right) H_5 + \left(\sum_{i=2}^{7} I_{k-i}\right) H_6$$
$$+ I_{k-1}H_7 + H_{k-1} + H_{k-2} + H_{k-3} + H_{k-4} + H_{k-5} + H_{k-6} \quad (14)$$

But since

$$H_k = I_{k-1}H_1 + \left(\sum_{i=1}^{3} I_{k-i}\right) H_2 + \left(\sum_{i=1}^{4} I_{k-i}\right) H_3 + \left(\sum_{i=1}^{5} I_{k-i}\right) H_4 + \left(\sum_{i=1}^{6} I_{k-i}\right) H_5 + \left(\sum_{i=1}^{7} I_{k-i}\right) H_6 + I_k H_7.$$

Substituting and rearranging the terms in equation (14) we have

$$\begin{aligned} H_{k+1} = &\left(\sum_{i=2}^{8} I_{k-i}\right) H_1 + \left[\left(\sum_{i=2}^{8} I_{k-i}\right) + \left(\sum_{i=3}^{9} I_{k-i}\right)\right] H_2 + \left[\left(\sum_{i=2}^{8} I_{k-i}\right) + \left(\sum_{i=3}^{9} I_{k-i}\right) + \left(\sum_{i=4}^{10} I_{k-i}\right)\right] H_3 \\ &+ \left[\left(\sum_{i=2}^{8} I_{k-i}\right) + \left(\sum_{i=3}^{9} I_{k-i}\right) + \left(\sum_{i=4}^{10} I_{k-i}\right) + \left(\sum_{i=5}^{11} I_{k-i}\right)\right] H_4 \\ &+ \left[\left(\sum_{i=2}^{8} I_{k-i}\right) + \left(\sum_{i=3}^{9} I_{k-i}\right) + \left(\sum_{i=4}^{10} I_{k-i}\right) + \left(\sum_{i=5}^{11} I_{k-i}\right) + \left(\sum_{i=6}^{12} I_{k-i}\right)\right] H_5 \\ &+ \left[\left(\sum_{i=2}^{8} I_{k-i}\right) + \left(\sum_{i=3}^{9} I_{k-i}\right) + \left(\sum_{i=4}^{10} I_{k-i}\right) + \left(\sum_{i=5}^{11} I_{k-i}\right) + \left(\sum_{i=6}^{12} I_{k-i}\right) + \left(\sum_{i=7}^{13} I_{k-i}\right)\right] H_6 + \sum_{i=1}^{7} I_{k-i} H_7. \end{aligned}$$

Now by definition of Heptanacci sequence equation (8) we have

$$\begin{aligned} H_{k+1} = &I_{k-1}H_1 + [I_{k-1} + I_{k-2}] H_2 + [I_{k-1} + I_{k-2} + I_{k-3}] H_3 + [I_{k-1} + I_{k-2} + I_{k-3} + I_{k-4}] H_4 \\ &+ [I_{k-1} + I_{k-2} + I_{k-3} + I_{k-4} + I_{k-5}] H_5 + [I_{k-1} + I_{k-2} + I_{k-3} + I_{k-4} + I_{k-5} + I_{k-6}] H_6 + I_k H_7. \end{aligned} \tag{15}$$

Thus by mathematical induction $P(k+1)$ is true, whenever $P(k)$ is true. Hence

$$H_n = I_{n-2}H_1 + \left(\sum_{i=2}^{3} I_{n-i}\right) H_2 + \left(\sum_{i=2}^{4} I_{n-i}\right) H_3 + \left(\sum_{i=2}^{5} I_{n-i}\right) H_4 + \left(\sum_{i=2}^{6} I_{n-i}\right) H_5 + \left(\sum_{i=2}^{7} I_{n-i}\right) H_6 + I_{n-1} H_7.$$

□

Theorem 2.2. *For any real numbers $u_1,\ u_2,\ u_3,\ u_4, \ldots,\ u_m$, formula for finding n^{th} term of Fibonacci-Like sequence of m^{th} order (m-nacci-like sequence) is*

$$U_n = \sum_{j=1}^{m-1} \left\{ \sum_{i=2}^{j+1} X_{n-i} \right\} U_j + X_{n-1} U_m, \quad n \geq m+1, \tag{16}$$

where U_n is the n^{th} term of m-nacci-like sequence $u_1,\ u_2,\ u_3,\ u_4, \ldots,\ u_m$ is the first, second, third, fourth, $\ldots$, m^{th} term and $X_{n-1},\ X_{n-2},\ \ldots,\ X_{n-m}$ are the corresponding Fibonacci numbers of m^{th} order (m-nacci numbers).

Proof. Let the first m^{th} term of m-nacci-like sequence are $u_1,\ u_2,\ u_3,\ u_4, \ldots,\ u_m$. Then we will derived an explicit formula for U_n given the m^{th} terms. We begin by computing the numerical coefficient for the first m term of the m-nacci like sequence $u_1,\ u_2,\ u_3,\ u_4, \ldots, u_n$. Equation were derived and coefficient are given for $n \geq m+1$. Each coefficient corresponds to the m-nacci number. We observe that the coefficient of U_1 correspond to X_{n-2}, U_2 correspond to $X_{n-2} + X_{n-3}$, U_3 correspond to $X_{n-2} + X_{n-3} + X_{n-4}$. So we conclude that the n^{th} term U_n is given by

$$U_n = \sum_{j=1}^{m-1} \left\{ \sum_{i=2}^{j+1} X_{n-i} \right\} U_j + X_{n-1} U_m.$$

By using mathematical induction, the formula can be validated in any values of n. The formula can be easily verified using $n = m+1,\ m+2,\ m+2$ and so on.

Let $P(n)$ be taken as

$$U_n = \sum_{j=1}^{m-1} \left\{ \sum_{i=2}^{j+1} X_{n-i} \right\} U_j + X_{n-1} U_m.$$

Then $P(k)$ is

$$U_k = \sum_{j=1}^{m-1} \left\{ \sum_{i=2}^{j+1} X_{k-i} \right\} U_j + X_{k-1} U_m. \tag{17}$$

It also follows $P(k+1)$ is

$$U_{k+1} = \sum_{j=1}^{m-1} \left\{ \sum_{i=2}^{j+1} X_{k+1-i} \right\} U_j + X_k U_m. \tag{18}$$

Now, we shall add u_{k-1}, u_{k-2}, $u_{k-3}, \ldots$, $u_{k-(m-1)}$ to both side of $P(k)$

$$\sum_{i=0}^{m-1} U_{k-i} = \sum_{j=1}^{m-1} \left\{ \sum_{i=2}^{j+1} X_{k-i} \right\} U_j + X_{k-1} U_m + U_{k-1} + U_{k-2} + U_{k-3} + \cdots + U_{k-(m-1)}. \tag{19}$$

Substitute values of u_{k-1}, u_{k-2}, $u_{k-3}, \ldots, u_{k-(m-1)}$ and rearranging the terms by definition of m-nacci sequence equation (10). Then the equation (19) becomes

$$U_{k+1} = \sum_{j=1}^{m-1} \left\{ \sum_{i=2}^{j+1} X_{k+1-i} \right\} U_j + X_k U_m.$$

The resulting equation is exactly our $P(k+1)$, hence the formula is valid for any value of n.

$$U_n = \sum_{j=1}^{m-1} \left\{ \sum_{i=2}^{j+1} X_{n-i} \right\} U_j + X_{n-1} U_m, \quad n \geq m+1.$$

$\square$

3. Conclusion

In this paper, we have presented formula for finding nth term of Fibonacci-Like Sequence of Higher Order. Further, the formula illustrated in general form to understand the terms in simple manner.

Acknowledgement

We would like to thank the anonymous referee for numerous helpful suggestions.

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