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Best Proximity Point Theorems for Generalized Weakly Contractive Mappings

Research Article

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Abstract: Recently, J.Calallero(Fixed Point Theory and Applications 2012, 2012:231) observed best proximity results for Geraghty-contractions by using the P-property. In this paper we introduce the notion of Boyd and wong result and Generalized weakly contractive mapping and show the existence and uniqueness of the best proximity point of such contractions in the setting of a metric space.

Keywords: Best proximity point, P-property, Boyd and Wong contraction, Generalized weakly contractive

1. Introduction

In nonlinear functional analysis, fixed point theory and best proximity point theory play an important role in the establishment of the existence of a certain differential and integral equations. As a consequence fixed point theory is very much useful for various quantitative sciences that involve such equations. The most remarkable paper in this field was reported by Banach in 1922 [2]. In his paper Banach proved that every contraction in a complete metric space has a unique fixed point. Following this paper many have extended and generalized this remarkable fixed point theorem of Banach by changing either the conditions of the mappings or the construction of the space. In particular, one of the notable generalizations of Banach fixed point theorem was introduced by Geraghty [8].

Theorem 1.1 ([8])**.** *Let (X, d) be a complete metric space and $f : X \to X$ be an operator. Suppose that there exists $\beta : [0, \infty) \to (0, 1)$ satisfying if f satisfies the following inequality:*

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y) \text{ for any } x, y \in X,$$

then f has unique fixed point.

It is natural that some mapping, especially non-self mappings defined on a complete metric space (X, d), do not necessarily possess a fixed point, that is $d(x, f(x)) > 0$ for all $x \in X$. In such situations it is reasonable to search for the existence and uniqueness of a point $x^* \in X$ such that $d(x^*, f(x^*))$ is an approximation of an $x \in X$ such that $d(x, f(x)) = 0$.

In other words one speculates to determine an approximate solution x^* that is optimal in the sense that the distance between x^* and $f(x^*)$ is minimum. Here the point x^* is called the best proximity point. In this paper we generalize and improve certain results of J.Caballero et al [7].

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2. Preliminaries

Let (X, d) be a metric space and A and B be nonempty subsets of a metric space X. A mapping $f : A \to B$ is called a k-contraction if there exists $k \in (0,1)$ such that $d(f(x),\ f(y)) \leq kd(x,y)$ for any $x, y \in A$. It is clear that a k-contraction coincides with the celebrated Banach fixed point theorem if one takes $A = B$ where A is a complete subset of X. Let A and B be nonempty subsets of a metric space (X, d). we denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A : d(x,y) = d(A,B), \quad \text{for some} \quad y \in B\}$$
$$B_0 = \{y \in B : d(x,y) = d(A,B), \quad \text{for some} \quad x \in A\} \text{ where } d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}.$$

Definition 2.1 ([9]). *Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$; $d(x_1, y_1) = d(A, B)$ and $d(x_2, y_2) = d(A, B)$ implies that $d(x_1, x_2) = d(y_1, y_2)$.*

It can be easily seen that for any nonempty subset A of (X, d), the pair (A, A) has the P-property. In [11] V.Sankarraj has proved that any pair (A, B) of nonempty closed convex subsets of a real Hilbert space H satisfies P-property. Now we introduce the class of those functions $\beta : [0,\infty) \to [0,1)$ satisfying the following condition: $\beta(t_n) \to 1 \Rightarrow t_n \to 0$.

Definition 2.2 ([7]). *Let A and B be nonempty subsets of a metric space (X, d). A mapping $f : A \to B$ is said to be a Geraghty contraction if there exists $\beta \in F$ such that $d(f(x), f(y)) \leq \beta(d(x,y))d(x,y)$ for any $x, y \in A$.*

Remark 2.1. *Notice that since $\beta : [0,\infty) \to (0,1)$, we have $d(f(x), f(y)) \leq \beta(d(x,y))d(x,y) < d(x,y)$ for any $x, y \in A$ with $x \neq y$.*

Theorem 2.3 ([7]). *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $f : A \to B$ be a continuous Geraghty contraction satisfying $f(A_0) \subseteq B_0$. Suppose that the pair (A, B) has the P-property. Then there exists a unique $x^* \in A$ such that $d(x^*, f(x^*)) = d(A, B)$.*

We would like to extend the result of J.Caballero and explore the best proximity point based on the well known result of Boyd and Wong [5].

Theorem 2.4 ([1]). *Let X be a complete metric space and let $f : X \to X$ satisfy $d(f(x), f(y)) \leq \psi(d(x,y))$ where $\psi : R^+ \to R^+$ is upper semi-continuous from the right and satisfies $0 \leq \psi(t) < t$. Then f has a unique fixed point. Further if $x_0 \in X$ and $x_{n+1} = f(x_n)$, then $\{x_n\}$ converges to the fixed point. A mapping $f : X \to X$ is said to be contractive if*

$$d(f(x), f(y)) < d(x,y) \quad \textit{for each} \quad x, y \in X \quad \textit{with} \quad x \neq y. \tag{1}$$

3. Main Results

Theorem 3.1. *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$. Let $f : A \to B$ be such that $f(A_0) \subseteq B_0$. Suppose $d(f(x), f(y)) \leq \psi(d(x,y))$ for each $x, y \in A$, where $\psi : R^+ \to [0,\infty)$ is upper semi-continuous from the right satisfies $0 \leq \psi(t) < t$ for $t > 0$. Furthermore the pair (A, B) has the P-property. Then there exists a unique $x^* \in A$ such that $d(x^*, f(x^*)) = d(A, B)$.*

Proof. Regarding that A_0 is nonempty, we take $x_0 \in A_0$. Since $f(x_0) \in f(A_0) \subseteq B_0$, we can find $x_1 \in A_0$ such that $d(x_1, f(x_0)) = d(A, B)$. Analogously regarding the assumption $f(x_1) \in f(A_0) \subseteq B_0$, we determine $x_2 \in A_0$ such that $d(x_2, f(x_1)) = d(A, B)$. Recursively we obtain a sequence $\{x_n\}$ in A_0 satisfying

$$d(x_{n+1}, f(x_n)) = d(A, B) \quad \text{for any} \quad n \in N \tag{2}$$

Since (A, B) has the P-property we derive that

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \quad \text{for any} \quad n \in N \tag{3}$$

If there exists $n_0 \in N$ such that $d(x_{n_0},\ x_{n_0+1}) = 0$, then the proof is completed. Indeed

$$0 = d(x_{n_0}, x_{n_0+1}) = d(f(x_{n_0-1}),\ f(x_{n_0})) \tag{4}$$

and consequently $f(x_{n_0-1}) = f(x_{n_0})$. On the other hand due to (2) we have $d(x_{n_0}, f(x_{n_0-1})) = d(A, B)$. Therefore we conclude that

$$d(A, B) = d(x_{n_0}, f(x_{n_0-1})) = d(x_{n_0},\ f(x_{n_0})) \tag{5}$$

For the rest of the proof we suppose that $d(x_n, x_{n+1}) > 0$ for any $n \in N$. Since f is contractive, for any $n \in N$, we have that

$$d(x_{n+1}, x_{n+2}) = d(f(x_n),\ f(x_{n+1})) \le \psi(d(x_n, x_{n+1})) < d(x_n, x_{n+1}) \tag{6}$$

consequently $\{d(x_n, x_{n+1})\}$ is monotonically decreasing sequence and bounded below and so we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$ exists. Let $\lim_{n\to\infty} d(x_n, x_{n+1}) = r \ge 0$. Assume that $r > 0$. Then from (1) we have $d(x_{n+1}, x_{n+2}) \le \psi(d(x_n, x_{n+1}))$ which implies that $r \le \psi(r) \Rightarrow r = 0$. That is

$$\lim_{n\to\infty} d(x_n,\ x_{n+1}) = 0 \tag{7}$$

Notice that since $d(x_{n+1},\ f(x_n)) = d(A, B)$ for any $n \in N$, for fixed $p, q \in N$, we have $d(x_p, f(x_{p-1})) = d(x_q, f(x_{q-1})) = d(A, B)$ and since (A, B) satisfies the P-property, $d(x_p, x_q) = d(f(x_{p-1}), f(x_{q-1}))$. In what follows, we prove that $\{x_n\}$ is cauchy sequence. On the contrary, assume that we have

$$\epsilon = \limsup_{m,n\to\infty} d(x_n, x_m) > 0 \tag{8}$$

Then there exists $\epsilon > 0$, such that for any $k \in N$, there exists $m_k > n_k \ge k$, such that

$$d(x_{m_k}, x_{n_k}) \ge \epsilon \tag{9}$$

Furthermore assume that for each k, m_k is the smallest number greater than n_k for which (9) holds. In view of (6), there exists k_0 such that $k \ge k_0$ implies that $d(x_k, x_{k+1}) \ge \epsilon$. For such k, we have

$$\begin{aligned}
\epsilon &\le d(x_{m_k}, x_{n_k}) \\
&\le d(x_{m_k}, x_{m_{k-1}}) + d(x_{m_{k-1}}, x_{n_k}) \\
&\le d(x_{m_k}, x_{m_{k-1}}) + \epsilon \\
&\le d(x_k, x_{k-1}) + \epsilon.
\end{aligned}$$

This proves $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$. On the other hand

$$\begin{aligned}
d(x_{m_k}, x_{n_k}) &\le d(x_{m_k}, x_{m_{k+1}}) + d(x_{m_{k+1}}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k}) \\
&\le 2d(x_k, x_{k+1}) + \psi(d(x_{m_k}, x_{n_k})).
\end{aligned}$$

Since $\lim_{k\to\infty} d(x_k, x_{k+1}) = 0$, the above inequality yields

$$\epsilon \leq \limsup_{m,n\to\infty} d(x_{m_k}, x_{n_k}) \leq \limsup_{m,n\to\infty} \psi(d(x_{m_k}, x_{n_k})) \leq \psi(\epsilon).$$

It follows that $\epsilon \leq \psi(\epsilon)$, a contradiction. Therefore $\{x_n\}$ is a cauchy sequence. Since $\{x_n\} \subset A$ and A is closed subset of the complete metric space (X, d), we can find $x^* \in A$ such that $x_n \to x^*$. Since the mapping is contractive and continuous, we have $f(x_n) \to f(x^*)$. This implies that $d(x_n, x_{n+1}) \to d(x^*, f(x^*))$. Taking into consideration that the sequence $\{d(x_{n+1}, f(x_n))\}$ is a constant sequence with the value $d(A, B)$, we deduce that $d(x^*, f(x^*)) = d(A, B)$. This means that x^* is a best proximity point of f. This proves the existence of our result. For the uniqueness, suppose that x_1 and x_2 are two best proximity points of f with $x_1 \neq x_2$. This means that $d(x_i, f(x_i)) = d(A, B)$ for $i = 1, 2$. Using the P-property, we have $d(x_1, x_2) = d(f(x_1), f(x_2))$. Using the fact that f is contractive and continuous, we have

$$d(x_1, x_2) = d(f(x_1), f(x_2)) \leq \psi(d(x_1, x_2)) < d(x_1, x_2)$$

which is a contradiction. Therefore $x_1 = x_2$. This completes the proof. In the following result we introduce the concept of generalized weakly contractive mapping and find best proximity point based on the work of Binayak S. Choudhury [6]. □

Definition 3.2 ([3]). *A mapping $f : X \to X$, where (X, d) is a metric space, is said to be weakly contractive if for any $x, y \in X$, then*

$$d(f(x), f(y) \leq d(x, y) - \phi(d(x, y)) \tag{10}$$

where $\phi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$. If one takes $\phi(t) = (1 - k)t$, where $0 < k < t$, a weak contraction reduces to a Banach contraction.

In [4] Alber and Guerre proved that if $f : \Omega \to \Omega$ is a weakly contractive self-map, where Ω is a closed convex subset of a Hilbert space, then f has a unique fixed point in Ω. Later, in [3] Rhodes proved that the existence of a unique fixed point for a weakly contractive self-map could be achieved even in a complete metric space setting.

Definition 3.3 ([12]). *Let A, B be nonempty subsets of a metric space X. A map $f : A \to B$ is said to be weakly contractive mapping if*

$$d(f(x), f(y)) \leq d(x, y) - \psi(d(x, y)), \text{ for all } x, y \in A,$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that ψ is positive on $(0, \infty), \psi(0) = 0$ and $\lim_{n\to\infty} \psi(t) = \infty$. If A is bounded, then the infinity condition can be omitted.

Note that $d(f(x), f(y)) \leq d(x, y) - \psi(d(x, y)) < d(x, y)$ if $x, y \in A$ with $x \neq y$. That is f is a contractive map. The notion called the P-property was introduced in [11] and was used to prove a extended version of Banach's contraction principle.

Theorem 3.4 ([12]). *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $f : A \to B$ be a weakly contractive mapping satisfying $f(A_0) \subseteq B_0$. Assume that the pair (A, B) has the p-property. Then there exists a unique $x^* \in A$ such that $d(x^*, f(x^*)) = d(A, B)$.*

Definition 3.5 ([10]). *A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering function if the following properties are satisfied:*

(a) ψ is monotone increasing and continuous

(b) $\psi(t) = 0$ if and only if $t = 0$.

Definition 3.6 ([6]). *Let (X, d) be a metric space, f a self-mapping of X. We shall call f a generalized weakly contractive mapping if for all $x, y \in X$, then*

$$\psi(d(f(x),f(y)) \leq \psi(m(x,y)) - \phi(\max\{d(x,y),d(y,f(y))\})$$

where

$$m(x,y) = \max\{d(x,y),d(x,f(x)),d(y,f(y)),\tfrac{1}{2}[d(x,f(y))+d(y,f(x))]\}$$

and ψ *is an altering distance function also* $\phi : [0,\infty) \to [0,\infty)$ *is a continuous function with* $\phi(t) = 0$ *if and only if* $t = 0$. *A generalized weakly contractive mapping is more general than that satisfying* $d(f(x),f(y)) \leq km(x,y)$ *for some constant* $0 \leq k < 1$ *and is included in those mappings which satisfy*

$$d(f(x),f(y)) < m(x,y).$$

Definition 3.7. *Let* A, B *be nonempty subsets of a metric space* X. *A map* $f : A \to B$ *is said to be a generalized weakly contractive mapping if for all* $x, y \in A$, *then*

$$\psi(d(f(x),f(y)) \leq \psi(m(x,y)) - \phi(\max\{d(x,y),d(y,f(y)) - d(A,B))\})$$

where

$$m(x,y) = \max\{d(x,y),d(x,f(x)) - d(A,B),d(y,f(y)) - d(A,B),\tfrac{1}{2}[d(x,f(y))+d(y,f(x))] - d(A,B)\}.$$

A generalized weakly contractive mapping is more general than that satisfying $d(f(x),f(y)) \leq km(x,y)$ *for some constant* $0 \leq k < 1$ *and is included in those mappings which satisfy*

$$d(f(x),f(y)) < m(x,y).$$

Theorem 3.8. *Let* (A, B) *be a pair of nonempty closed subsets of a complete metric space* (X, d) *such that* A_0 *is nonempty. Let* $f : A \to B$ *be such that* $f(A_0) \subseteq B_0$. *Suppose*

$$\psi(d(f(x),f(y)) \leq \psi(m(x,y)) - \phi(\max\{d(x,y),d(y,f(y)) - d(A,B))\}) \tag{11}$$

Furthermore the pair (A, B) *has the p-property. Then there exists a unique* x^* *in* A *such that* $d(x^*, f(x^*)) = d(A, B)$.

Proof. Choose $x_0 \in A$. Since $f(x_0) \in f(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that $d(x_1, f(x_0)) = d(A, B)$. Analogously regarding the assumption, $f(x_1) \in f(A_0) \subseteq B_0$, we determine $x_2 \in A_0$ such that $d(x_2, f(x_1)) = d(A, B)$. Recursively we obtain a sequence $\{x_n\}$ in A_0 satisfying

$$d(x_{n+1}, f(x_n)) = d(A,B) \quad \text{for any} \quad n \in N \tag{12}$$

Claim: $d(x_n, x_{n+1}) \to 0$.

If $x_N = x_{N+1}$, then x_N is a best proximity point. By the P-property, we have

$$d(x_{n+1}, x_{n+2}) = d(f(x_n), f(x_{n+1})).$$

Hence we assume that $x_n \neq x_{n+1}$ for all $n \in N$. Since $d(x_{n+1}, f(x_n)) = d(A, B)$, from (11) we have for all $n \in N$

$$\begin{aligned}
\psi(d(x_{n+1}, x_{n+2})) &= \psi(d(f(x_n), f(x_{n+1}))) \\
&\leq \psi(max\{d(x_n, x_{n+1}), d(x_n, f(x_n)) - d(A,B), d(x_{n+1}, f(x_{n+1})) - d(A,B), \\
&\quad \frac{1}{2}[d(x_n, f(x_{n+1})) + d(x_{n+1}, f(x_n))] - d(A,B)\}) - \phi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, f(x_{n+1})) - d(A,B)\}) \\
&\leq \psi(max\{d(x_n, x_{n+1}), d(x_n, f(x_n)) - d(A,B), d(x_{n+1}, f(x_{n+1})) - d(A,B), \\
&\quad \frac{1}{2}(d(x_n, f(x_{n+1}))) - d(A,B)\}) - \phi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, f(x_{n+1})) - d(A,B)\})
\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2}(d(x_n, f(x_{n+1}))) - d(A,B) &\leq \frac{1}{2}(d(x_n, x_{n+1}) + d(x_{n+1}, f(x_{n+1}))) - d(A,B)\\ &\leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, f(x_{n+1})) - d(A,B)\}\\ d(x_n, f(x_n)) - d(A,B) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, f(x_n)) - d(A,B)\\ &= d(x_n, x_{n+1})\end{aligned}$$

It follows that

$$\begin{aligned}\psi(d(f(x_n), f(x_{n+1})) &\leq \psi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, f(x_{n+1})) - d(A,B)\})\\ &\quad - \phi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, f(x_{n+1})) - d(A,B)\})\\ \psi(d(x_{n+1}, x_{n+2})) &\leq \psi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}) - \phi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}) \quad (13)\end{aligned}$$

Suppose that $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$, for some positive integer n. Then from (13), we have

$$\psi(d(x_{n+1}, x_{n+2}) \leq \psi(d(x_{n+1}, x_{n+2})) - \phi(d(x_{n+1}, x_{n+2})),$$

that is $\phi(d(x_{n+1}, x_{n+2})) \leq 0$, which implies that $d(x_{n+1}, x_{n+2}) = 0$, contradicting our assumption. Therefore $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for any $n \in N$ and hence $\{d(x_n, x_{n+1})\}$ is monotone decreasing sequence of non-negative real numbers, hence there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. In view of the facts from (13) for any $n \in N$, we have

$$\psi(d(x_{n+1}, x_{n+2})) \leq \psi(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1})),$$

Taking the limit as $n \to \infty$ in the above inequality and using the continuities of ψ and ϕ we have $\psi(r) \leq \psi(r) - \phi(r)$ which implies $\phi(r) = 0$. Hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \quad (14)$$

Next we show that $\{x_n\}$ is a cauchy sequence. If otherwise there exists an $\epsilon > 0$ for which we can find two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that for all positive integers $k, n_k > m_k > k$,

$$d(x_{m_k}, x_{n_k}) \geq \epsilon \text{ and } d(x_{m_k}, x_{n_k-1}) < \epsilon.$$

Now $\epsilon \leq d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k})$, that is $\epsilon \leq d(x_{m_k}, x_{n_k}) < \epsilon + d(x_{n_k-1}, x_{n_k})$. Taking the limit as $k \to \infty$ in the above inequality and using (14) we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon \quad (15)$$

Again $d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k})$. Taking the limit as $k \to \infty$ in the above inequalities and using (14) and (15) we have

$$\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) = \epsilon \quad (16)$$

Again

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) \text{ and } d(x_{m_k}, x_{n_k+1}) \leq d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k+1})$$

Letting $k \to \infty$ in the above inequalities and using (14) and (15), we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k+1}) = \epsilon \tag{17}$$

Similarly

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k+1}) = \epsilon \tag{18}$$

For $x = x_{m_k}, y = y_{m_k}$, we have

$$\begin{aligned} d(x_{m_k}, f(x_{m_k})) - d(A,B) &\leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, f(x_{m_k})) - d(A,B) \\ &= d(x_{m_k}, x_{m_k+1}) \end{aligned}$$

similarly $d(x_{n_k}, f(x_{n_k})) - d(A,B) = d(x_{n_k}, x_{n_k+1})$. Also $d(x_{m_k}, f(x_{n_k})) - d(A,B) = d(x_{m_k}, x_{n_k+1})$ and $d(x_{n_k}, f(x_{m_k})) - d(A,B) = d(x_{n_k}, x_{m_k+1})$. From (11) we have

$$\begin{aligned} \psi(d(x_{m_k+1}, x_{n_k+1})) &= \psi(d(f(x_{m_k}), f(x_{n_k}))) \\ &\leq \psi(\max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, f(x_{m_k})) - d(A,B), d(x_{n_k}, f(x_{n_k})) - d(A,B), \\ &\quad \frac{1}{2}[d(x_{m_k}, f(x_{n_k})) + d(x_{n_k}, f(x_{m_k})) - d(A,B)]\}) - \phi(\max\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, f(x_{n_k})) - d(A,B)\}) \\ &\leq \psi(\max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_k+1}), d(x_{n_k}, x_{n_k+1}), \\ &\quad \frac{1}{2}[d(x_{m_k}, x_{n_k+1}) + d(x_{n_k}, x_{m_k+1})]\}) - \phi(\max\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, x_{n_k+1})\}) \end{aligned}$$

It follows that

$$\begin{aligned} \psi(d(f(x_{m_k}), f(x_{n_k}))) &\leq \psi(\max\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, f(x_{n_k+1})), \ \frac{1}{2}[d(x_{m_k}, x_{n_k+1}) + d(x_{n_k}, x_{m_k+1})]\}) \\ &\quad - \phi(\max\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, f(x_{n_k+1}))\}) \\ \psi(d(x_{m_k+1}, x_{n_k+1}) &\leq \psi(\max\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, x_{n_k+1})\}) - \phi(\max\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, x_{n_k+1})\}) \end{aligned}$$

From (14), (15), (17), (18) and Letting $k \to \infty$ in the above inequalities and using the continuities of ψ and ϕ, we have $\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon)$ which is contradiction by virtue of property of ϕ. Hence $\{x_n\}$ is a cauchy sequence. Since $\{x_n\} \subset A$ and A is a closed subset of the complete metric space (X,d), there exists x^* in A such that $x_n \to x^*$. Putting $x = x_n$ and $y = x^*$ in (11) and since

$$d(x_n, f(x^*)) \leq d(x_n, x^*) + d(x^*, f(x_n)) \text{ and } d(x^*, f(x_n)) \leq d(x^*, f(x^*)) + d(f(x^*), f(x_n))$$

we have

$$\begin{aligned} \psi(d(x_{n+1}, f(x^*)) - d(A,B) &\leq \psi(d(f(x_n), f(x^*))) \\ &\leq \psi(\max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, f(x^*)) - d(A,B), \\ &\quad \frac{1}{2}[d(x_n, f(x^*)) + d(x^*, f(x_n))] - d(A,B)\}) - \phi(\max\{d(x_n, x^*), d(x^*, f(x^*)) - d(A,B)\}) \end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality and using the continuities of ψ and ϕ, we have

$$\psi(d(x^*, f(x^*)) - d(A,B)) \leq \psi(d(x^*, f(x^*)) - d(A,B)) - \phi(d(x^*, f(x^*)) - d(A,B).$$

Which implies that $d(x^*, f(x^*)) = d(A, B)$. Hence x^* is a best proximity point of f. For the uniqueness. Let p and q be two best proximity points of f and suppose that $p \neq q$. Then putting $x = p$ and $y = q$ in (11) we obtain

$$\psi(d(f(p), f(q))) \leq \psi(\max\{d(p,q), d(p, f(p)) - d(A,B), d(q, f(q)) - d(A,B), \frac{1}{2}[d(p, f(q)) + d(q, f(p))] - d(A,B)\}) - \phi(\max\{d(p,q), d(q, f(q)) - d(A,B)\})$$

That is

$$\psi(d(p,q)) \leq \psi(d(p,q)) - \phi(d(p,q))$$

which is contradiction by virtue of a property of ϕ. Therefore $p = q$. This completes the proof. □

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