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The Strong Domination Alteration Sets in Fuzzy Graphs

Research Article

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Abstract: In this paper, the effect of the removal of a node or an arc of a fuzzy graph on the minimum strong dominating set of the fuzzy graph is studied. The motivation behind this study is the identification of fault tolerance in the topological design of a network. That is, the ability of the network to provide service even when it contains a faulty component or components.

MSC: 05C69.

Keywords: Fuzzy graph, strong arcs, weight of arcs, strong domination, private strong neighbors.

1. Introduction

Fuzzy graphs were introduced by Rosenfeld [13], who has described the fuzzy analogue of several graph theoretic concepts like paths, cycles, trees and connectedness and established some of their properties [13]. Bhutani and Rosenfeld have introduced the concept of strong arcs [2]. The concept of domination in graphs was introduced by Ore and Berge in 1962, the domination number and independent domination number are introduced by Cockayne and Hedetniemi [5]. For the terminology of domination in crisp graphs we refer to [4].

For a node v of a graph $G:(V,E)$, a neighbor of v is a node adjacent to v in G. Also the neighborhood $N(v)$ of v is the set of neighbors of v. The closed neighborhood $N[v]$ is defined as $N[v] = N(v) \cup \{v\}$. A node v in G is said to dominate itself and each of its neighbors, that is v dominates the nodes in $N[v]$. A set S of nodes of G is a dominating set of G if every node of $V(G) - S$ is adjacent to some node in S. A minimum dominating set in G is a dominating set of minimum cardinality. The cardinality of a minimum dominating set is called the domination number of G and is denoted by $\gamma(G)$. Somasundaram studied the effect of removal of a node or an arc in fuzzy graphs using effective arcs [16]. Nagoor Gani and Vijayalakshmi discussed domination critical nodes in fuzzy graphs using strong arcs [12]. In this paper, the effect of the removal of a node or an arc of a fuzzy graph on the minimum strong dominating set of the fuzzy graph is studied.

This paper is organized as follows. Section 2 contains preliminaries and in section 3, we have investigated the effect of removal of a node or an arc of a fuzzy graph on the minimum strong dominating set.

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2. Preliminaries

We summarize briefly some basic definitions in fuzzy graphs which are presented in [1, 2, 6, 8–10, 13–15, 17].

A **fuzzy graph** is denoted by $G : (V, \sigma, \mu)$ where V is a node set, σ and μ are mappings defined as $\sigma : V \rightarrow [0,1]$ and $\mu : V \times V \rightarrow [0,1]$, where σ and μ represent the membership values of a node and an arc respectively. For any fuzzy graph, $\mu(x,y) \leq min\{\sigma(x), \sigma(y)\}$. We consider fuzzy graph G with no loops and assume that V is finite and nonempty, μ is reflexive (i.e.,$\mu(x,x) = \sigma(x)$, for all x) and symmetric (i.e., $\mu(x,y) = \mu(y,x)$, for all (x,y)). In all the examples σ is chosen suitably. Also, we denote the underlying crisp graph by $G^* : (\sigma^*, \mu^*)$ where $\sigma^* = \{u \in V : \sigma(u) > 0\}$ and $\mu^* = \{(u,v) \in V \times V : \mu(u,v) > 0\}$. Throughout we assume that $\sigma^* = V$. The fuzzy graph $H : (\tau, \nu)$ is said to be a **partial fuzzy subgraph** of $G : (\sigma, \mu)$ if $\nu \subseteq \mu$ and $\tau \subseteq \sigma$. In particular we call $H : (\tau, \nu)$ a **fuzzy subgraph** of $G : (\sigma, \mu)$ if $\tau(u) = \sigma(u)$ for all $u \in \tau^*$ and $\nu(u,v) = \mu(u,v)$ for all $(u,v) \in \nu^*$. A fuzzy graph $G : (V, \sigma, \mu)$ is called **trivial** if $|\sigma^*| = 1$. Two nodes u and v in G are said to be **adjacent** if $\mu(u,v) > 0$.

A **path** P of length n in a fuzzy graph $G : (V, \sigma, \mu)$ is a sequence of distinct nodes $u_0, u_1, ..., u_n$ such that $\mu(u_{i-1}, u_i) > 0, i = 1, 2, ..., n$ and the degree of membership of a weakest arc is defined as its strength. If $u_0 = u_n$ and $n \geq 3$ then P is called a **cycle** and P is called a **fuzzy cycle**, if it contains more than one weakest arc. The **strength of a cycle** is the strength of the weakest arc in it. The **strength of connectedness** between two nodes x and y is defined as the maximum of the strengths of all paths between x and y and is denoted by $CONN_G(x,y)$. A fuzzy graph $G : (V, \sigma, \mu)$ is **connected** if for every x, y in $\sigma^*, CONN_G(x,y) > 0$.

An arc (u,v) of a fuzzy graph $G : (V, \sigma, \mu)$ is called an **effective arc** (M-strong arc) if $\mu(u,v) = \sigma(u) \wedge \sigma(v)$. Then u and v are called effective neighbors. The set of all effective neighbors of u is called **effective neighborhood of** u and is denoted by $EN(u)$. A fuzzy graph $G : (V, \sigma, \mu)$ is said to be **complete** if $\mu(u,v) = \sigma(u) \wedge \sigma(v)$, for all $u, v \in \sigma^*$.

The **order** p and **size** q of a fuzzy graph $G : (\sigma, \mu)$ are defined to be $p = \sum_{x \in V} \sigma(x)$ and $q = \sum_{(x,y) \in \mu^*} \mu(x,y)$. Let $G : (V, \sigma, \mu)$ be a fuzzy graph and $S \subseteq V$. Then the **scalar cardinality** of S is defined to be $\sum_{v \in S} \sigma(v)$. Let p denotes the scalar cardinality of V, also called the order of G. The **complement** of a fuzzy graph $G : (V, \sigma, \mu)$, denoted by $\overline{G}$ is defined to be $\overline{G} = (V, \sigma, \overline{\mu})$ where $\overline{\mu}(x,y) = \sigma(x) \wedge \sigma(y) - \mu(x,y)$ for all $x, y \in V$ [18].

An arc of a fuzzy graph $G : (V, \sigma, \mu)$is called **strong** if its weight is at least as great as the strength of connectedness of its end nodes when it is deleted. A fuzzy graph $G : (V, \sigma, \mu)$ is called a **strong fuzzy graph** if each arc in G is a strong arc. Depending on $CONN_G(x,y)$ of an arc (x,y) in G, Mathew and Sunitha [17] defined three different types of arcs. Note that $CONN_{G-(x,y)}(x,y)$ is the the strength of connectedness between x and y in the fuzzy graph obtained from G by deleting the arc (x,y). An arc (x,y) in G is $\alpha-$ **strong** if $\mu(x,y) > CONN_{G-(x,y)}(x,y)$. An arc (x,y) in G is $\beta-$ **strong** if $\mu(x,y) = CONN_{G-(x,y)}(x,y)$. An arc (x,y) in G is $\delta-$ **arc** if $\mu(x,y) < CONN_{G-(x,y)}(x,y)$.

Thus an arc (x,y) is a strong arc if it is either $\alpha-$ strong or $\beta-$ strong. A path P is called **strong path** if P contains only strong arcs. If $\mu(u,v) > 0$, then u and v are called neighbors. The set of all neighbors of u is denoted by $N(u)$. Also v is called strong neighbor of u if arc (u,v) is strong. The set of all strong neighbors of u is called the **strong neighborhood** of u and is denoted by $N_s(u)$. The **closed strong neighborhood** $N_s[u]$ is defined as $N_s[u] = N_s(u) \cup \{u\}$.

A fuzzy graph $G : (V, \sigma, \mu)$ is said to be **bipartite** [15] if the node set V can be partitioned into two non empty sets V_1 and V_2 such that $\mu(v_1, v_2) = 0$ if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$. Further if $\mu(u, v) = \sigma(u) \bigwedge \sigma(v)$ for all $u \in V_1$ and $v \in V_2$ then G is called a **complete bipartite graph** and is denoted by K_{σ_1,σ_2}, where σ_1 and σ_2 are respectively the restrictions of σ to V_1 and V_2. A node u in a fuzzy graph $G : (V, \sigma, \mu)$ is said to be **isolated** if $\mu(u, v) = 0$ for all $v \neq u$.

A fuzzy subgraph $H : (\tau, \nu)$ **spans** the fuzzy graph $G : (V, \sigma, \mu)$ if $\tau = \sigma$ [8]. A fuzzy graph $G = (V, \sigma, \mu)$ is called a **fuzzy forest** if it has a fuzzy spanning subgraph $F : (\sigma, \nu)$, which is a forest, where for all arcs (x, y) not in F there exists a path from x to y in F whose strength is more than $\mu(x, y)$ [13]. If G is connected we call G a **fuzzy tree**. Note that here F is a tree which contains all nodes of G and is a spanning tree of G. Also note that F is the unique maximum spanning tree (MST) of G [18], where a **maximum spanning tree** of a connected fuzzy graph $G : (V, \sigma, \mu)$ is a fuzzy spanning subgraph $T : (\sigma, \nu)$, such that T is a tree, and for which $\sum_{u \neq v} \nu(u, v)$ is maximum [8].

An arc is called a **fuzzy bridge** of a fuzzy graph $G : (V, \sigma, \mu)$ if its removal reduces the strength of connectedness between some pair of nodes in G [13]. Similarly a **fuzzy cut node** w is a node in G whose removal from G reduces the strength of connectedness between some pair of nodes other than w [13]. A node z in a fuzzy graph $G : (V, \sigma, \mu)$ is called a **fuzzy end node** if it has exactly one strong neighbor in G [3].

Theorem 2.1 ([3]). *A non trivial fuzzy tree $G : (V, \sigma, \mu)$ contains at least two fuzzy end nodes.*

Theorem 2.2 ([3]). *Every node is either a fuzzy cutnode or a fuzzy end node in a non trivial fuzzy tree $G : (V, \sigma, \mu)$ except K_2.*

In a fuzzy tree G, an arc is strong if and only if it is an arc of F where F is the associated unique maximum spanning tree of G [2, 18]. Note that these strong arcs are α-strong and there are no β-strong arcs in a fuzzy tree [17]. Also note that in G, an arc (x, y) is α-strong if and only if (x, y) is a fuzzy bridge of G [17].

Theorem 2.3 ([14]). *The strong arc incident with a fuzzy end node is a fuzzy bridge in any non trivial fuzzy graph $G : (V, \sigma, \mu)$.*

Corollary 2.4 ([14]). *In a non trivial fuzzy tree $G : (V, \sigma, \mu)$ except K_2, the strong neighbor of a fuzzy end node is a fuzzy cut node of G.*

3. Strong Domination Alteration in Fuzzy Graphs

Nagoorgani and Chandrasekaran [9] introduced the concept of domination using strong arcs in fuzzy graphs as follows.

Definition 3.1. [9] *A node v in a fuzzy graph $G : (V, \sigma, \mu)$ is said to strongly dominate itself and each of its strong neighbors, i.e., v strongly dominates the nodes in $N_s[v]$. A set D of nodes of G is a strong dominating set [SD−set] of G if every node of $V(G) - D$ is a strong neighbor of some node in D.*

In 2002, Somasundaram studied the effect of removal of a node or an arc in fuzzy graphs using effective arcs [16]. Later in 2011, Nagoor Gani and Vijayalakshmi discussed domination critical nodes in fuzzy graphs using strong arcs [12]. Manjusha and Sunitha [7] defined strong domination number using membership values (weights) of arcs in fuzzy graphs as follows.

Definition 3.2 ([7]). *The weight of a strong dominating set [SD−set] D in a fuzzy graph $G : (V, \sigma, \mu)$ is defined as $W(D) = \sum_{u \in D} \mu(u, v)$, where $\mu(u, v)$ is the minimum of the membership values(weights) of strong arcs incident on u. The* *strong domination number of G is defined as the minimum weight of strong dominating sets of G and it is denoted by $\gamma_s(G)$ or simply γ_s. A minimum strong dominating set [$MSD-$set] in G is a strong dominating set of minimum weight.*

Motivated by the following application in networks, the effect on the number of nodes in the $MSD-$set (minimum strong dominating set) of a fuzzy graph $G : (V, \sigma, \mu)$ when G is modified by deleting a node or deleting or adding an arc is examined in this paper.

One of the basic needs for the topological design of a network is the identification of fault tolerance, that is the ability of the network to provide service even when it contains a faulty component or components. The behavior of a network in the presence of a fault can be analyzed by determining the effect on fault tolerance criterion after removing an arc (link failure) or a node (processor failure) from its underlying fuzzy graph $G : (V, \sigma, \mu)$. For example, an $MSD-$set in G represents a minimum set of processors that can communicate directly with all other processors in the system. If it is essential for file servers to have this property and that the number of processors designated as file servers be limited, the number of nodes in an $MSD-$set of G is the fault tolerance criterion. In this example, it may be noted that the number of nodes in the $MSD-$set of G does not increase when G is modified by removing a node or an arc. Also networks can be made fault tolerant by providing redundant communication links (adding arcs).

Let $G - v$ (respectively $G - e$) denote the fuzzy graph obtained by removing a node v (respectively arc e) from G. The following acronyms are used to denote the following classes of fuzzy graphs. (C represents changing; N: node; A: arc; R: removal).

Definition 3.3. *Let $G : (V, \sigma, \mu)$ be any fuzzy graph. Then $G \in CNR$ [changing node removal] if $|MSD(G - v)| \neq |MSD(G)| \forall v \in \sigma^*$. Similarly $G \in CAR$ [changing arc removal] if $|MSD(G - e)| \neq |MSD(G)| \forall e \in \mu^*$.*

Next, the effect of the removal of a node of a fuzzy graph $G : (V, \sigma, \mu)$ on an $MSD-$set of G is investigated.

Example 3.4. *Consider the fuzzy graph G in Fig 1.*

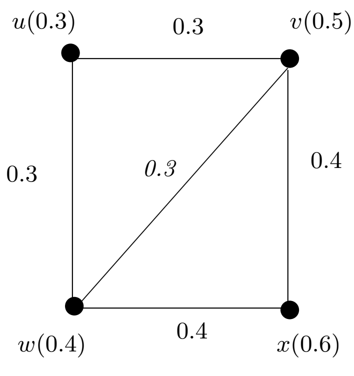


Fig 1: The effect "decreasing" the number of nodes in an $MSD-$set

In the fuzzy graph of Fig 1, strong arcs are $(u, v), (v, x), (x, w)$ and (u, w). The set $D = \{u, v\}$ forms an $MSD-$set and

$$|MSD(G)| = 2 \ \ and \ \ \gamma_s(G) = 0.3 + 0.3 = 0.6.$$

Also

$$|MSD(G - u)| = 1, \ \ \gamma_s(G - u) = 0.4$$

$$|MSD(G - v)| = 1, \ \ \gamma_s(G - v) = 0.3$$

$$|MSD(G - w)| = 1, \ \ \gamma_s(G - w) = 0.3$$

$$|MSD(G - x)| = 1, \ \ \gamma_s(G - x) = 0.3.$$

Thus the removal of any node of G decreases the number of nodes in an $MSD-$set.

Example 3.5. *Consider the fuzzy graph G in Fig 2.*

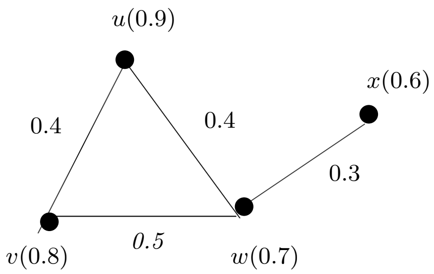


Fig 2: The effect "unaltered" and "increasing" the number of nodes in an $MSD-$set

In the fuzzy graph of Fig 2, all arcs are strong.

$$MSD(G) = \{w\}, \ |MSD(G)| = 1 \ \text{and} \ \gamma_s(G) = 0.3$$

$$|MSD(G-u)| = 1, \ \gamma_s(G-u) = 0.3.$$

$$|MSD(G-v)| = 1, \ \gamma_s(G-v) = 0.3.$$

$$|MSD(G-x)| = 1, \ \gamma_s(G-x) = 0.4.$$

$$|MSD(G-w)| = 2, \ \gamma_s(G-w) = 0.4.$$

Hence the removal of u or v or x does not change the cardinality of an $MSD-$set. But the removal of w increases the cardinality of an $MSD-$set.

The illustrated examples [Example 3.4 and Example 3.5] have shown that the number of nodes in an $MSD-$set of a fuzzy graph $G : (V, \sigma, \mu)$ may increase or decrease or remain unaltered after removing a node from the fuzzy graph. It is beneficial to partition the nodes of G into three crisp subsets according to how their removal affects the cardinality of an $MSD-$set.

$$V_s^o = \{v \in V / |MSD(G-v)| = |MSD(G)|\}$$
$$V_s^+ = \{v \in V / |MSD(G-v)| > |MSD(G)|\}$$
$$V_s^- = \{v \in V / |MSD(G-v)| < |MSD(G)|\}$$

Clearly every isolated node belongs to V_s^-. Also every node of a complete fuzzy graph is in V_s^o. Note that $V = V_s^+ \cup V_s^-$ for a fuzzy graph $G : (V, \sigma, \mu)$ for which the cardinality of an $MSD-$set changes when an arbitrary node is removed (CNR).

Remark 3.6. *Note that removing a node from a fuzzy graph $G : (V, \sigma, \mu)$ can increase the number of nodes in an $MSD-$set by more than one, but can decrease it by at most one.*

Remark 3.7. *Also note that if D is an $MSD-$set of a fuzzy graph $G : (V, \sigma, \mu)$, then removing any node in $V - D$ cannot increase the number of nodes in the $MSD-$set of G. Hence $\left|V_s^+\right| \leq |MSD(G)|$.*

Theorem 3.8. *If $G : (V, \sigma, \mu)$ is a fuzzy tree such that G^* is a tree, then V_s^o is never empty.*

Proof. First note that all arcs in the fuzzy tree $G : (V, \sigma, \mu)$ are strong arcs since G^* is a tree. Clearly the result is true when $G^* = K_2$. Since G is a fuzzy tree, G has at least two fuzzy end nodes [Theorem 2.1] and every node is either a fuzzy cutnode or a fuzzy end node [Theorem 2.2]. Next, let G contains at least 3 nodes. Then G has at least one node v that is adjacent to at least one fuzzy end node and at most one fuzzy cutnode. Then we consider the following two cases.

Case 1: v is adjacent to two (or more) fuzzy end nodes u_1 and u_2. In this case, v is contained in every $MSD-$set of

G, since the only strong neighbor of u_1 and u_2 is v and $|MSD(G-u_1)| = |MSD(G)|$, since as u_1 is a fuzzy end node its deletion makes no change in the $MSD-$set of G.

Case 2: v is adjacent to one fuzzy end node u.

Claim: For any fuzzy tree G, if u is a fuzzy end node then $|MSD(G-u)| \leq |MSD(G)|$, for, let w be a strong neighbor of u and if w is adjacent to at least one fuzzy end node other than u then as in Case 1, w is in the $MSD-$set of G and $|MSD(G-u)| = |MSD(G)|$. Next, if the strong neighbors of w other than u are fuzzy cutnodes of G, then u or w belongs to $MSD(G)$. If $u \in MSD(G)$ then $|MSD(G-u)| = |MSD(G)|$. If $w \in MSD(G)$, then in $G-u$, w may become a fuzzy end node and strong neighbor of w other than u strongly dominates w. Hence $|MSD(G-u)| < |MSD(G)|$. Thus the claim.

Now let $G' = G-v-u$. Then $\left|MSD(G')\right| \leq |MSD(G-u)| \leq |MSD(G)|$, by the claim. Also $\left|MSD(G')\right| \geq |MSD(G)|-1$, since removing a node from a fuzzy graph G can decrease the number of nodes in an $MSD-$set by at most one [Remark 3.3.4]. If $\left|MSD(G')\right| = |MSD(G)|-1$, then $|MSD(G-v)| = |MSD(G)|$, which implies $v \in V_s^o$. If $\left|MSD(G')\right| > |MSD(G)|-1$, then $\left|MSD(G')\right| = |MSD(G)| = |MSD(G-u)|$, which implies $u \in V_s^o$. □

Remark 3.9. *The converse of Theorem 3.8 is not true. That is V_s^o is non- empty does not imply that G is a fuzzy tree which is illustrated in the following example.*

Example 3.10. *In Fig 2 of Example 3.5, $V_s^o = \{u, v, x\}$ which is non- empty. But G is not a fuzzy tree.*

The following theorem gives a characterization of the nodes in V_s^+ in strong fuzzy graphs. Note that $N_s[u]$ is the closed strong neighborhood of u for any node $u \in \sigma^*$.

Theorem 3.11. *Let $G : (V, \sigma, \mu)$ be a strong fuzzy graph. A node $v \in V_s^+$ if and only if the following conditions hold.*

(a). v is not an isolate and is in every $SD-$set of G with cardinality $|MSD(G)|$ and

(b). no subset $D \subseteq V - N_s[v]$ with cardinality $|MSD(G)|$ strongly dominates $G - v$.

Proof. Let $G : (V, \sigma, \mu)$ be a strong fuzzy graph. First suppose that $v \in V_s^+$. Then $|MSD(G-v)| > |MSD(G)|$. If v is an isolated node then $v \in V_s^-$, a contradiction. If there exists an $SD-$set D of G with cardinality $|MSD(G)|$ not containing v then D is also an $SD-$set of $G-v$. Hence $|MSD(G-v)| \leq |MSD(G)|$, a contradiction, since $v \in V_s^+$. Hence v belongs to every $SD-$set of G with cardinality $|MSD(G)|$. If there exists a subset $D \subseteq V - N_s[v]$ with cardinality $|MSD(G)|$ that strongly dominates $G-v$, then $|MSD(G-v)| \leq |MSD(G)|$, a contradiction. Hence all conditions are necessary.

Conversely, suppose that a node $v \in G$ satisfies conditions (a) and (b). To prove that $v \in V_s^+$. Suppose if possible, $v \notin V_s^+$. Then $v \in V_s^o$ or $v \in V_s^-$.

Case 1: $v \in V_s^o$. Then $|MSD(G-v)| = |MSD(G)|$. Hence there exists an $MSD-$set D' of $G-v$ not containing v with cardinality $|MSD(G)|$. By condition (b), no subset $D \subseteq V - N_s[v]$ with cardinality $|MSD(G)|$ strongly dominates $G-v$. Then the set D' must contain a strong neighbor of v. Hence D' is an $SD-$set of G not containing v with cardinality $|MSD(G)|$, which contradicts condition (a).

Case 2: $v \in V_s^-$.

Then $|MSD(G-v)| < |MSD(G)|$. Also since removing a node from a fuzzy graph G can decrease the number of nodes in a $MSD-$set by at most one, $|MSD(G-v)| = |MSD(G)|-1$. Let D' be an $MSD-$set of $G-v$. Take a strong neighbor of v as u. Then $D' \cup \{u\}$ is an $SD-$set of G not containing v with cardinality $|MSD(G)|$, which is a contradiction to condition (a). Thus from Case 1 and Case 2, it follows that $v \in V_s^+$. □

Remark 3.12. *In a general fuzzy graph which is not a strong fuzzy graph, Theorem 3.11 is not true which is illustrated in the following example.*

Example 3.13. *Consider the fuzzy graph $G : (V, \sigma, \mu)$ in Fig 3.*

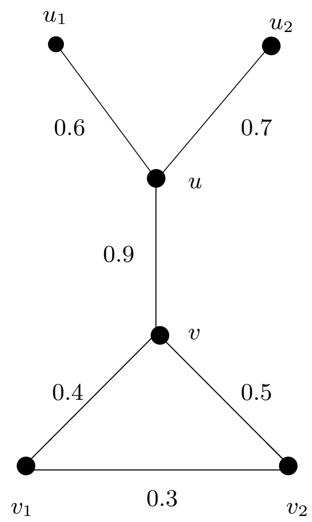
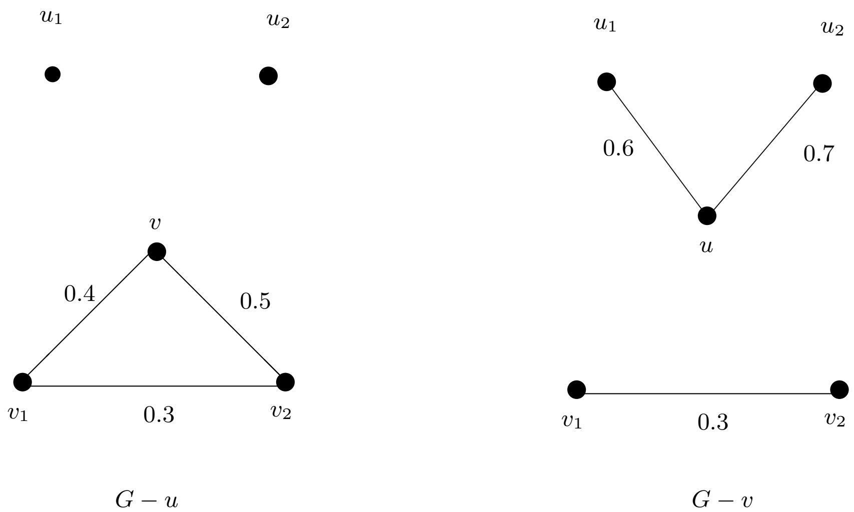


Fig 3: Illustration of Remark 3.12

In the fuzzy graph G of Fig 3, all node weights are taken as 1. All arcs except (v_1, v_2) are strong arcs. Hence G is not a strong fuzzy graph. The $MSD-$set of G is the set $\{u, v\}$, $|MSD(G)| = 2$ and $\gamma_s(G) = 0.6 + 0.4 = 1$. By removing the nodes u and v we get the following graphs $G - u$ and $G - v$ as follows.



For the fuzzy graph $G - u$,

$$MSD(G - u) = \{u_1, u_2, v\}, \ |MSD(G - u)| = 3 \text{ and } \gamma_s(G - u) = 0.4.$$

For the fuzzy graph $G - v$,

$$MSD(G - v) = \{u, v_1\}, \ \{u, v_2\}, \ |MSD(G - v)| = 2 \text{ and } \gamma_s(G - v) = 0.9.$$

Hence $u \in V_s^+$, and $v \in V_s^o$. But for both u and v, conditions (a) and (b) of Theorem 3.11 hold. That is,

(a). u and v are not isolated nodes and are in every $SD-$ set of G with cardinality $|MSD(G)|$.

(b). $V - N_s[u] = \{u_1, u_2, u, v, v_1, v_2\} - \{u_1, u_2, u, v\} = \{v_1, v_2\}$ and
$V - N_s[v] = \{u_1, u_2, u, v, v_1, v_2\} - \{u, v, v_1, v_2\} = \{u_1, u_2\}$.

No subset of $V - N_s[u]$ and $V - N_s[v]$ with cardinality $|MSD(G)|$ strongly dominates $G - u$ and $G - v$.

Theorem 3.14. *In a fuzzy graph $G : (V, \sigma, \mu)$ if a node $v \in V_s^+$ then v is a fuzzy cutnode.*

Proof. Let $G : (V, \sigma, \mu)$ be a fuzzy graph. Suppose $v \in V_s^+$. Then it follows that v is in every $SD-$set of G with cardinality $|MSD(G)|$ by using the proof technique in the first part of Theorem 3.11. To prove that v is a fuzzy cutnode. Suppose if possible, v is not a fuzzy cutnode. Then by definition of a fuzzy cutnode, removal of v does not reduce the strength of connectedness between any pair of nodes x, y both different from v.

Claim: v has at least two strong neighbors. For, suppose v has only one strong neighbor say u. Let D be an $SD-$set of G containing v with cardinality $|MSD(G)|$. Then we consider two cases.

Case 1: $u \in D$. Then $D - v$ is an $SD-$set of $G - v$. Hence $|MSD(G - v)| < |MSD(G)|$, which is a contradiction to the assumption that $v \in V_s^+$.

Case 2: $u \notin D$. Then $(D - v) \cup \{u\}$ is an $SD-$set of $G - v$ with cardinality $|MSD(G)|$. Hence $|MSD(G - v)| \leq |MSD(G)|$, which is a contradiction to the assumption that $v \in V_s^+$. Hence in either case [Case 1 and Case 2] we get a contradiction. Hence the claim.

Since $v \in V_s^+$ and v is in every $SD-$set of G with cardinality $|MSD(G)|$, at least one of the strong neighbors say x of v is strong dominated as well as connected by only v and the deletion of v makes x as an isolated node of $G - v$ [If such a node x does not exist then $G - v$ is a fuzzy subgraph of G and hence at most $|MSD(G)|$ nodes are needed to strongly dominate $G - v$, which is a contradiction since $v \in V_s^+$]. Consider a strong neighbor say y $(\neq x)$ of v. Then the path say $P : x - v - y$ connecting x and y is a strong path and strength of $P= S(P) = min\{\mu(x, v), \mu(v, y)\} > 0$. Since $CONN_G(x, y)$ is the maximum of the strengths of all paths connecting x and y it follows that $CONN_G(x, y) > 0$.

Now, since the deletion of v makes x as an isolated node, the strength of connectedness between x and any node in $N_s(v)$ other than x becomes zero in $G - v$. Hence $CONN_{G-v}(x, y) = 0$. That is the removal of v reduces the strength of connectedness between the nodes x and y from a nonzero value to zero, which contradicts the assumption that v is not a fuzzy cutnode. Hence it follows that v is a fuzzy cutnode of G. □

The following example shows that a fuzzy cutnode v need not belong to V_s^+.

Example 3.15. *Consider the fuzzy graph G in Fig 4.*

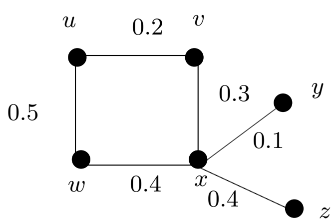


Fig 4: A fuzzy cutnode not in V_s^+

In the fuzzy graph of Fig 4, all node weights are taken as 1. Here all arcs except (u, v) are strong arcs. The nodes x and w are fuzzy cutnodes. One of the $MSD -$ set is $\{w, x\}$. Hence $|MSD(G)| = 2$. But $|MSD(G - w)| = 2$. Hence $w \notin V_s^+$.

Theorem 3.16. *In any strong fuzzy graph $G : (V, \sigma, \mu)$ with at least 3 nodes, if a fuzzy cutnode v is in every $SD-$set of G with cardinality $|MSD(G)|$ then $v \in V_s^+$.*

Proof. Let $G : (V, \sigma, \mu)$ be any strong fuzzy graph with at least 3 nodes. Let v be a fuzzy cutnode of G which is in every $SD-$set of G with cardinality $|MSD(G)|$. Suppose if possible, $v \notin V_s^+$. Since v is in every $SD-$set of G with cardinality $|MSD(G)|$, condition (a) of Theorem 3.11 holds and since $v \notin V_s^+$, condition (b) should get violated. Hence there must exist at least one subset $D \subseteq V - N_s[v]$ with cardinality $|MSD(G)|$, and D strongly dominates $G - v$. Now by definition of a fuzzy cutnode, there are two nodes say u and w such that v is on every strongest as well as strong path (since G is a strong fuzzy graph) joining u and w. Since v is in every $SD-$set of G with cardinality $|MSD(G)|$, either u or w or both are strongly dominated only by the node v. Suppose u is strongly dominated only by v. Then the deletion of v makes u as an isolated node in $G - v$. Since $D \subseteq V - N_s[v]$ with cardinality $|MSD(G)|$ and D strongly dominates $G - v$, u must belong to D, which is a contradiction, since $u \in N_s(v)$ and D cannot contain any node in $N_s[v]$ by the condition $D \subseteq V - N_s[v]$. The proofs follow similarly when w or both u and w are strongly dominated only by v. Hence it follows that $v \in V_s^+$. ☐

Remark 3.17. *In a general fuzzy graph which is not a strong fuzzy graph, Theorem 3.16 is not true which is illustrated in the following example.*

Example 3.18. *In the fuzzy graph of Fig 3 in Example 3.13, v is a fuzzy cutnode since deletion of v reduces the strength of connectedness between the nodes v_1 and v_2 from 0.4 to 0.3. Also v is in every $SD-$set of G with cardinality $|MSD(G)|$. But $v \notin V_s^+$.*

Remark 3.19. *The major difference between strong domination number of fuzzy graphs and that of crisp graphs is that in any crisp graph G, the number of nodes in a minimum dominating set of G is strictly less than that of any other dominating set. But in fuzzy graphs, the strong domination number of fuzzy graph is defined based on weight of strong arcs and not based on node weights. Hence in fuzzy graphs, there are strong dominating sets with same number of nodes as minimum strong dominating set. That is, there are fuzzy graphs G with $|SD(G)| = |MSD(G)|$.*

Theorem 3.20. *If a node v in a non trivial fuzzy graph $G : (V, \sigma, \mu)$ is in V_s^+, then v is in every $SD-$set of G with cardinality $|MSD(G)|$.*

Proof. Let $G : (V, \sigma, \mu)$ be a non trivial fuzzy graph. Suppose $v \in V_s^+$. Then

$$|MSD(G - v)| > |MSD(G)| \tag{1}$$

To prove that v is in every $SD-$set of G with cardinality $|MSD(G)|$. If possible, let D be an $SD-$set of G not containing v with cardinality $|MSD(G)|$. Then D is also an $SD-$set $G - v$. Hence

$$|MSD(G - v)| \leq |D| = |MSD(G)|.$$

which is a contradiction to (1). Hence v is in every $SD-$set of G with cardinality $|MSD(G)|$. ☐

Remark 3.21. *The converse of Theorem 3.20 is not true which is illustrated in the following example.*

Example 3.22. *In the fuzzy graph of Fig 3 in Example 3.13, G is a non trivial fuzzy graph. The node v is in every $SD-$set of G with cardinality $|MSD(G)|$. But $v \notin V_s^+$.*

The following definitions of strong independent set and private strong neighbor in fuzzy graphs are due to Nagoorgani, Chandrasekharan and Vadivel [9, 11]. The authors used strong arcs to define these concepts as follows.

Definition 3.23 ([9]). *Two nodes in a fuzzy graph $G : (V, \sigma, \mu)$ are said to be strongly independent if there is no strong arc between them. A set of nodes in G is strong independent if any two nodes in the set are strongly independent.*

Definition 3.24 ([11]). *Let $G : (V, \sigma, \mu)$ be any fuzzy graph. Let $D \subseteq V$ be a set of nodes and let $u \in D$. Then a node v is called a private strong neighbor of u (with respect to D) if $N_s[v] \cap D = \{u\}$. Also the private strong neighbor set of u with respect to D is defined to be $psn[u, D] = \{v : N_s[v] \cap D = \{u\}\}$ or $psn[u, D] = N_s[u] - N_s[D - u]$.*

Note that $u \in psn[u, D]$ if D is a strong independent set of nodes, in which case u is said to be its own private strong neighbor. The above defined concepts are illustrated in the following example.

Example 3.25. *Consider the fuzzy graph G in Fig 5.*

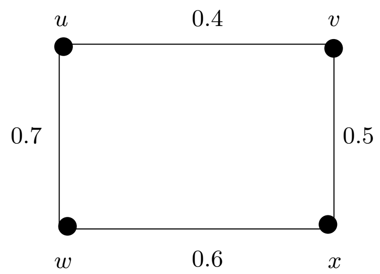


Fig 5: Illustration of private strong neighbor set

In this fuzzy graph, all node weights are taken as 1. All arcs except (u, v) are strong arcs. Consider the set $D = \{u, v\}$. Then D is a strong independent set of G, since no strong arc between the nodes u and v. Now in Fig 3.6,

$N_s[u] = \{u, w\}$, $N_s[v] = \{v, x\}$, $N_s[w] = \{u, w, x\}$, $N_s[x] = \{x, w, v\}$.

$N_s[u] \cap D = \{u, w\} \cap \{u, v\} = \{u\}$,

$N_s[v] \cap D = \{v, x\} \cap \{u, v\} = \{v\}$,

$N_s[w] \cap D = \{u, w, x\} \cap \{u, v\} = \{u\}$,

$N_s[x] \cap D = \{x, w, v\} \cap \{u, v\} = \{v\}$.

Hence u and w are private strong neighbors of u with respect to D. Also v and x are private strong neighbors of v with respect to D. The private strong neighbor set of u with respect to D is

$psn[u, D] = \{v : N_s[v] \cap D = \{u\}\} = N_s[u] - N_s[D - u] = N_s[u] - N_s[v] = \{u, w\} - \{v, x\} = \{u, w\}$.

Similarly $psn[v, D] = \{v, x\}$. Also note that $u \in psn[u, D]$ and $v \in psn[v, D]$. Hence u and v are said to be its own private strong neighbors.

The next theorem gives a characterization of nodes in V_s^- in strong fuzzy graphs.

Theorem 3.26. *Let $G : (V, \sigma, \mu)$ be a strong fuzzy graph. Then a node $v \in V_s^-$ if and only if $psn[v, D] = \{v\}$, for some strong dominating set D containing v with cardinality $|MSD(G)|$.*

Proof. Let $G : (V, \sigma, \mu)$ be a strong fuzzy graph. Let $v \in V_s^-$ and S be an $MSD-$set of $G - v$. Then $D = S \cup \{v\}$ is an $SD-$set of G with cardinality $|MSD(G)|$. By Definition 3.24,

$$psn[v, D] = N_s[v] - N_s[D - v] = N_s[v] - N_s[S]. \tag{2}$$

We have to prove that $psn[v, D] = \{v\}$. Suppose if possible, $psn[v, D]$ contains a node say u other than v. Then u is a strong neighbor of v. Also $u \notin N_s[S]$ by (2). That is, u is not in S and u is not a strong neighbor of any node in S, which is a contradiction, since S is a strong dominating set of $G - v$. Hence our assumption is wrong and $psn[v, D] = \{v\}$.

Conversely suppose that $psn[v, D] = \{v\}$ for some strong dominating set D containing v with cardinality $|MSD(G)|$. Obviously $D - v$ strongly dominates $G - v$. Hence $v \in V_s^-$. □

Remark 3.27. *In a general fuzzy graph except strong fuzzy graphs, Theorem 3.26 is not true which is illustrated in the following example.*

Example 3.28. *Consider the fuzzy graph G in Fig 6.*

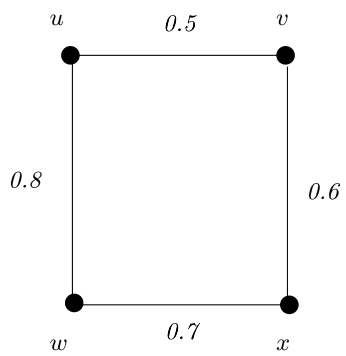


Fig 6: Illustration of Remark 3.27

In the fuzzy graph of Fig 6, all node weights are taken as 1. All arcs except (u, v) are strong arcs. The $MSD-$ sets are $D_1 = \{v, w\}$ and $D_2 = \{w, x\}$ with $\gamma_s(G) = 0.7 + 0.6 = 1.3$. Also $D_3 = \{u, x\}$ and $D_4 = \{u, v\}$ are $SD-$sets with cardinality $|MSD(G)|$. Here, $u, v, w, x \in V_s^-$. But $psn[x, D_2] = N_s[x] - N_s[w]$. That is $psn[x, D_2] = \{v, w, x\} - \{u, w, x\} = \{v\}$. Also $psn[x, D_3] = \{v, w, x\} - \{u, w\} = \{v, x\}$. Hence $x \notin psn[x, D_2]$, and $psn[x, D_3] \neq \{x\}$, which is a contradiction, since $x \in V_s^-$.

Theorem 3.29. *For any strong fuzzy graph $G : (V, \sigma, \mu)$ with at least 2 nodes,*

(a). If $v \in V_s^+$, then for every $SD-$set D of G with cardinality $|MSD(G)|$, $v \in D$ and $psn[v, D]$ contains at least two non adjacent nodes.

(b). If $x \in V_s^+$ and $y \in V_s^-$ then x and y are not adjacent.

(c). $|V_s^o| \geq 2\left|V_s^+\right|$.

(d). $|MSD(G)| \neq |MSD(G - v)|$ $\forall v \in V$ if and only if $V = V_s^-$.

(e). If $v \in V_s^-$ and v is not an isolate in G, then there exists an $SD-$set D of G with cardinality $|MSD(G)|$ such that $v \notin D$.

Proof.

(a). Let $G : (V, \sigma, \mu)$ be any strong fuzzy graph with at least 2 nodes. Suppose $v \in V_s^+$. Then from Theorem 3.11, v is not an isolated node and is in every $SD-$set D of G with cardinality $|MSD(G)|$. If $psn[v, D] = \{v\}$ then by Theorem 3.26, $v \in V_s^-$, a contradiction, since $v \in V_s^+$. Next, we have to prove that $psn[v, D]$ contains at least two non adjacent nodes. Suppose on the contrary that the strong private neighbors of v induce a fuzzy subgraph H such that H^* is a complete graph. Then $(D - v) \cup \{u\}$ for any $u \in psn[v, D]$ is an $SD-$set of G with cardinality $|MSD(G)|$ not containing v, which is a contradiction, since $v \in V_s^+$ and hence v should belong to every $SD-$set of G with cardinality $|MSD(G)|$ by Theorem 3.11. Hence the result follows.

(b). Assume that $x \in V_s^+$ and $y \in V_s^-$. To prove that x and y are not adjacent. Suppose on the contrary that x and y are adjacent (i.e. $\mu(x,y) > 0$). Since removing a node from a fuzzy graph $G : (V, \sigma, \mu)$ can decrease the number of nodes in a $MSD-$set by at most one by Remark 3.6 and since $y \in V_s^-$, $\exists$ an $SD-$set $D^{'}$ (say) of $G - y$ with cardinality $|MSD(G)| - 1$. Then we consider the following two cases.

Case 1: $D^{'}$ contains x.

Then $D^{'}$ strongly dominates G and hence $D^{'}$ is an $SD-$set of G. But $\left|D^{'}\right| = |MSD(G)| - 1 < |MSD(G)|$ which is a contradiction, since $|MSD(G)|$ nodes are necessary to strongly dominate G.

Case 2: $D^{'}$ does not contain x.

Then $D^{'} \cup \{y\}$ is an $SD-$set of G with cardinality $|MSD(G)|$ not containing x, which is a contradiction since $x \in V_s^+$ and hence x should belong to every $SD-$set of G with cardinality $|MSD(G)|$. Hence if $x \in V_s^+$ and $y \in V_s^-$, then x and y are not adjacent.

(c). To prove that $|V_s^o| \geq 2\left|V_s^+\right|$.

For each $v \in V_s^+$, (a) describes that for every $SD-$set D of G with cardinality $|MSD(G)|$, $v \in D$ and $psn[v, D]$ contains at least two non adjacent nodes say x and y. Hence these private strong neighbors x and y of v are in $V - D$, since $psn[v, D] = N_s[v] - N_s[D - v]$. Thus these private strong neighbors x and y of v are not in V_s^+, since any node in V_s^+ is a node of D. Also from (b), since $v \in V_s^+$, note that v has no private strong neighbor in V_s^-, and thus these private strong neighbors x and y must be in V_s^o. Hence the result follows.

(d). Clearly if $V = V_s^-$, then $|MSD(G)| \neq |MSD(G - v)|$, $\forall v \in V$. Conversely suppose that $|MSD(G)| \neq |MSD(G - v)|$, $\forall v \in V$. Then

$$V = V_s^+ \cup V_s^-. \tag{3}$$

If any $v \in V_s^+$, then by (c), V_s^o is non empty, which contradicts (1). Thus $V_s^+ = \phi$ and $V = V_s^-$.

(e). Suppose $v \in V_s^-$ and v is not an isolate in G. To prove that $\exists$ an $SD-$set D of G with cardinality $|MSD(G)|$ such that $v \notin D$. Suppose on the contrary that every $SD-$set of G with cardinality $|MSD(G)|$ contains v. Since $v \in V_s^-$, by Theorem 3.26, $psn[v, D] = \{v\}$, for some $SD-$set D containing v with cardinality $|MSD(G)|$. Since $psn[v, D] = N_s[v] - N_s[D - v]$ by Def.3.24, $psn[v, D] = \{v\}$ implies that either $D - v$ contains all strong neighbors of v or the nodes of $D - v$ are strong neighbors of the nodes in $N_s(v)$. Then in both of these cases, we get a contradiction as follows.

Case 1: $D - v$ contains all strong neighbors of v.

Then $D - v$ itself is an $SD-$set of G with cardinality less than $|MSD(G)|$, which is a contradiction.

Case 2: The nodes of $D - v$ are strong neighbors of the nodes in $N_s(v)$.

Then $(D - v) \cup \{u\}$ for any strong neighbor u of v is an $SD-$set of G with cardinality $|MSD(G)|$ not containing v, which is a contradiction to our assumption.

□

Corollary 3.30. *A strong fuzzy graph $G : (V, \sigma, \mu) \in CNR$ if and only if for each node $v \in V$, $psn[v, D] = \{v\}$ for some strong dominating set D containing v with cardinality $|MSD(G)|$.*

Proof. A strong fuzzy graph $G : (V, \sigma, \mu) \in CNR$ if and only if $|MSD(G)| \neq |MSD(G - v)|$ $\forall v \in V$, by Definition 3.3. That is if and only if $V = V_s^-$, by part (d) of Theorem 3.29. That is if and only if for each node $v \in V$, $psn[v, D] = \{v\}$ for some strong dominating set D containing v with cardinality $|MSD(G)|$, by Theorem 3.26.

Next, we discuss the effect of the removal of an arc in the $MSD-$set of a fuzzy graph $G:(V,\sigma,\mu)$. Let us partition the arcs of G into two crisp subsets according to how their removal affects the cardinality of a $MSD-$set as follows.

$$E_s^o = \{(u,v)\in E/\,|MSD(G-(u,v))| = |MSD(G)|\}$$
$$E_s^+ = \{(u,v)\in E/\,|MSD(G-(u,v))| > |MSD(G)|\}$$

□

Remark 3.31. *Clearly all δ and δ^*- arcs of a fuzzy graph $G:(V,\sigma,\mu)$ are in E_s^o. Also note that the removal of any arc of G does not decrease the number of nodes in a $MSD-$set of G and can increase it by at most one. Thus a fuzzy graph G, for which the number of nodes in a $MSD-$set changes when an arbitrary arc is removed, has the property that $|MSD(G-e)| = |MSD(G)|+1\ \forall$ arcs $e\in\mu^*$. The fuzzy graphs in CAR [Definition 3.3] are called MSD^+ critical fuzzy graphs.*

Proposition 3.32. *A non trivial complete fuzzy graph $G:(V,\sigma,\mu)$ is in CAR if and only if G is K_2.*

Proof. Let $G:(V,\sigma,\mu)$ be a non trivial complete fuzzy graph. If G is K_2, then clearly G is in CAR. Conversely assume that G is a non trivial complete fuzzy graph and $G\in CAR$. Then by Remark 3.31,

$$|MSD(G-e)| = |MSD(G)|+1\ \forall\ e\in\mu^*. \tag{4}$$

To prove that $G=K_2$. Suppose if possible $G\neq K_2$. Hence $|V|=n\geq 3$. We have $|MSD(G)|=1$. Hence by (4),

$$|MSD(G-e)| = 2\ \ \forall\ e\in\mu^* \tag{5}$$

Let $e=(x,y)$ be any arbitrary arc of G. Note that each node in G is adjacent with the the remaining $n-1$ nodes of G and every arc in G and $G-e$ is an effective arc. Since each of x and y is adjacent to $n-1$ nodes in G, each of x and y is adjacent to $n-2$ nodes in $G-e$. Let $D=\{u_1,u_2,\cdots,u_{n-2}\}$ be the $n-2$ nodes of G except x and y. Note that each node in D is adjacent to the remaining $n-1$ nodes of $G-e$ by an effective arc. Hence $D^{'}=\{u_i\}$ for each $u_i\in D$ is an $MSD-$set of $G-e$. Hence $|MSD(G-e)|=1$, which contradicts (5). Now since $e=(x,y)$ is arbitrary, this is true for every arc of G. Hence the proposition follows. □

Proposition 3.33. *If $G:(V,\sigma,\mu)$ is a fuzzy cycle, then $E_s^o=\mu^*$. That is E_s^+ is empty.*

Proof. In a fuzzy cycle $G:(V,\sigma,\mu)$ on n nodes, every arc is strong. Also, the number of nodes in any $SD-$set of G and G^* is same because each arc in both graphs is strong. Now, the strong domination number of G^* is $\lceil\frac{n}{3}\rceil$. Hence the minimum number of nodes in any $SD-$set of G is $\lceil\frac{n}{3}\rceil$. Now the deletion of any arc e in G makes $G-e$ as a path P_n on n nodes and every arc in P_n is strong. Hence the minimum number of nodes in any $SD-$set of $G-e=P_n$ is $\lceil\frac{n}{3}\rceil$. Hence the number of nodes in any $MSD-$set of G and $G-e$ is same. That is, $|MSD(G-e)|=|MSD(G)|$ for any $e\in\mu^*$. Hence $E_s^o=\mu^*$. □

Proposition 3.34. *If $G:(V,\sigma,\mu)$ is a fuzzy tree such that G^* contains at least one cycle, then E_s^o is never empty.*

Proof. Let $G:(V,\sigma,\mu)$, be a fuzzy tree such that G^* contains at least one cycle. Let C be a cycle in G. Then the weakest arc of C is a $\delta-$arc say e, since G is a fuzzy tree. Then for the arc e, $|MSD(G-e)|=|MSD(G)|$. Hence $e\in E_s^o$ and E_s^o is non-empty. □

Next theorem shows that the only fuzzy graphs in CAR are fuzzy galaxy. The notion of fuzzy galaxy is introduced as follows.

Definition 3.35. *A fuzzy star is a bipartite fuzzy graph $G : (V, \sigma, \mu)$ with $|V_1| = 1$ or $|V_2| = 1$. A fuzzy galaxy is a fuzzy forest in which each component is a fuzzy star.*

Theorem 3.36. *A fuzzy graph $G : (V, \sigma, \mu)$ is in CAR if and only if G is a fuzzy galaxy.*

Proof. If G is a fuzzy galaxy then each component of G is a fuzzy star [Definition 3.35]. Hence deletion of any arc in G increases the cardinality of any $MSD-$set of G. Hence $G \in CAR$. Conversely, let $G \in CAR$. Then

$$|MSD(G-e)| \neq |MSD(G)|\,, \ \ \forall \ \ e \in \mu^*. \tag{6}$$

First note that all arcs in G are strong, for, if G contains a $\delta-$ arc (say) e, then $e \in E_s^o$ [Remark 3.31], which contradicts (6). To prove that G is a fuzzy galaxy. Suppose on the contrary that G is not a fuzzy galaxy. Then at least one of the component of G contains a cycle say C. Then C contains more than one weakest arc, for, if C has only one weakest arc then it is a $\delta-$arc and it belongs to E_s^o which contradicts (6). Hence C is a fuzzy cycle. Let D be an $MSD-$set of G. Then all arcs in C with both ends in D or both ends in $V-D$ are in E_s^o since D will still strongly dominate $V-D$ if such an arc is removed. This is a contradiction, since by our assumption, $G \in CAR$ implies that all arcs of G should be in E_s^+. Hence it follows that G is a fuzzy galaxy. This completes the proof. □

4. Conclusion

In this paper, the effect of the removal of a node or an arc on the $MSD-$set of a fuzzy graph is investigated.

References

[1] K.R.Bhutani, *On automorphisms of fuzzy graphs*, Pattern Recognition Letters, 9(1989), 159-162.

[2] K.R.Bhutani and A.Rosenfeld, *Strong arcs in fuzzy graphs*, Information Sciences, 152(2003), 319-322.

[3] K.R.Bhutani and A.Rosenfeld, *Fuzzy end nodes in fuzzy graphs*, Information Sciences, 152(2003), 323-326.

[4] Gray Chartrand and Ping Zang, *Introduction to Graph Theory*, Tata McGraw-Hill Edition, (2006).

[5] T.W.Haynes, S.T.Hedetniemi and P.J.Slater, *Fundamentals of domination in graphs*, (Marcel Dekker, Inc., New York), (1998).

[6] O.T.Manjusha and M.S.Sunitha, *Notes on Domination in Fuzzy Graphs*, Journal of intelligent and Fuzzy Systems, 27(2014), 3205-3212.

[7] O.T.Manjusha and M.S.Sunitha, *Strong Domination in Fuzzy Graphs*, Fuzzy Inf. Eng., 7(2015), 369-377.

[8] J.N.Mordeson and P.S.Nair, *Fuzzy Graphs and Fuzzy Hypergraphs*, Physica-Verlag, (2000).

[9] A.Nagoorgani and V.T.Chandrasekaran, *Domination in Fuzzy graph*, Adv.in Fuzzy Sets and Systems, I(1)(2006), 17-26.

[10] A.Nagoorgani and P.Vijayalakshmi, *Insensitive Arc in Domination of fuzzy graph*, Int.J. Contemp. Math. Sciences, 6(26)(2011), 1303-1309.

[11] A.Nagoor Gani and P.Vadivel, *A Study on Domination, Independent Domination and Irredundance in Fuzzy Graph*, Applied Mathematical Sciences, 5(47)(2011), 2317-2325.

[12] A.Nagoorgani and P.Vijayalakshmi, *Domination critical nodes in fuzzy graph*, Int. J. of Math. Sci. Engg. Appls., 5(I)(2011), 295-301.

[13] A.Rosenfeld, *Fuzzy Graphs, In Fuzzy sets and their application to Cognitive and decision processes*, Academic Press, (1975), 77-95.

[14] Sameena Kalathodi and M.S.Sunitha, *Distance in Fuzzy Graphs*, LAP LAMBERT Academic Publishing, (2012).

[15] A.Somasundaram and S.Somasundaram, *Domination in fuzzy graphs-I*, Pattern Recognition Letters, 19(1998), 787-791.

[16] A.Somasundaram, *Domination in fuzzy graphs-II*, Journal of Fuzzy Mathematics, 13(2)(2005), 281-288.

[17] Sunil Mathew and M.S.Sunitha, *Fuzzy graph; basic concepts and applications*, LAP LAMBERT Academic Publishing, (2012).

[18] M.S.Sunitha and Sunil Mathew, *Fuzzy Graph Theory: A Survey*, Annals of Pure and Applied Mathematics, 4(1)(2013), 92-110.