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On Common Fixed Points of Pair of Selfmaps Satisfying Certain Contractive Condition

Research Article

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Abstract: In this paper we study a common fixed point theorem of a pair of mappings satisfying certain contractive condition in Complete Metric Space.

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1. Introduction

The Fixed point Theory has wide applications in Many Branches of Science and Engineering field, S.Banach [1]derived a well known theorem for a contraction mapping in a Complete metric space.Which states that,"A contraction has a Unique fixed point theorem in a Complete metric space." After that many authors proved fixed point theorems for mappings satisfying certain contraction conditions. In 1968 R.Kanan [4] introduced another type of map called as Kanan Map and investigated Unique fixed point theorem in complete metric space. In 1973 B.K. Das and Sattya Gupta [2] Generalized Banach contraction principle in terms of rational expression. Recently V.V. Latpate and Dolhare U.P.[3] generalized fixed point theorem in Generalized metric space. The study of common fixed points of a functions satisfying a certain conditions has been a very interesting research activity since last some years. Kanan R.[4] has proved common fixed point theorem for a pair of self maps, J.Madhusudan Rao [6] proved important common fixed point theorem .Fisher [5] defined Kanan type map and obtain relation between Kanan map and Contraction. In this paper we have proved a common fixed point theorem for a pair of self maps similar to map of B.k. Das and Satya Gupta [2].

2. Some Preliminaries

Definition 2.1. *Let X be a non empty set. A mapping $d : X \times X \to R$ is said to be a metric or a distance function if it satisfies following conditions.*

(1). $d(x, y)$ is non negative.

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(2). $d(x, y) = 0$ *if and only if* x *and* y *coincides i.e.* $x = y$

(3). $d(x, y) = d(y, x)$ *(symmetry)*

(4). $d(x, y) \leq d(x, z) + d(z, y)$ *(triangle inequality)*

Then the function d *is referred to as metric on* X*.And* (X, d) *or simply* X *is said to as metric space.*

Definition 2.2. *A metric space* (x, d) *is said to be a complete metric space if every cauchy sequence in* X *converges to a point of* X*.*

Definition 2.3. *If* (X, d) *be a complete metric space and a function* $F : X \to X$ *is said to be a contraction map if* $d(F(x), F(y)) \leq \beta\ d(x, y)$ *for all* $x, y \epsilon\ X$ *and for* $0 < \beta < 1$*.*

Definition 2.4. *Let* $F : X \to X$*, then* $x \epsilon X$ *is said to be a fixed point of* F *if* $F(x) = x$*.*

Example 2.5. *If* $f(x) = sin(x)$ *, then o is the fixed point of f since f(0)=0.*

Definition 2.6. *Let* X *be a metric space and if* F_1 *and* F_2 *be any two maps .An element* $a \epsilon X$ *is said to be a common fixed point of* F_1 *and*F_2 *if* $F_1(a) = F_2(a)$*.*

Example 2.7. *If* $F_1(x) = Sin(x)$ *and* $F_2(x) = Tan(x)$*. Then o is the common fixed point of* F_1 *and*F_2 *since* $F_1(0) = Sin(0) = 0$ *and* $F_2(0) = Tan(0) = 0$*.*

3. Main Result

Theorem 3.1. *Let* (X, d) *be a complete metric space. A mapping* $F : X \to X$ *be continuous and* F *satisfies the condition*

$$d(F(x), F(y) \leq \frac{\mu\ d(x, F(x))\ d(y, F(y))}{d(x, y)} + \mu\ \lambda d(x, y) \tag{1}$$

for all $x, y \epsilon X$ *,* $x \neq y$ *and for* $\mu\ , \lambda\ \epsilon [0, \frac{1}{2})$ *with* $\mu\ + \lambda < \frac{1}{2}$ *and* $\mu\ \lambda < \frac{1}{2}$*. Then* F *has a Unique fixed point in X.*

Proof. Let us consider $x_0 \epsilon X$ an arbitrary point and Let x_n be a sequence in X s.t. $f^n(x_0) = x_n$ for n be a positive integer. If $x_n = x_{n+1}$ for some n then the result is obvious. Let $x_n \neq x_{n+1}$ for all n. Consider $d(x_{n+1}, x_n) = d(F(x_n), F(x_{n-1}))$

$$\begin{aligned}
d(x_{n+1}, x_n) &\leq \frac{\mu\ d(x_n, F(x_n))\ d(x_{n-1}, F(x_{n-1}))}{d(x_n, x_{n-1})} + \mu\ \lambda d(x_n, x_{n-1}) \\
d(x_{n+1}, x_n) &\leq \left(\frac{\mu\lambda}{1-\mu}\right) d(x_n, x_{n-1}) \\
&\leq \left(\frac{\mu\lambda}{1-\mu}\right)^2 d(x_{n-1}, x_{n-2}) \\
&\leq \dots \\
&\leq \left(\frac{\mu\lambda}{1-\mu}\right)^n d(x_1, x_0) \ \text{ for } m \geq n \text{ ,using the triangle inequality, we have} \\
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
&\leq (\alpha^n + \alpha^n + 1 + \ldots.. + \alpha^m - 1)\ d(x_1, x_0) \ \text{ where } \alpha = \frac{\mu\lambda}{1-\mu} \\
&\leq \frac{\alpha^n}{1-\alpha}\ d(x_1, x_0)
\end{aligned}$$

But $\frac{\alpha^n}{1-\alpha} \to 0$ *as* $m, n \to \infty$.

R.H.S. $\to 0$. x_n be a Cauchy sequence in X and Since X is complete. Converges to some element of X. Say $p \in X$ $x_n \to p$

and since F is continuous. Consider

$$\begin{aligned} F(p) &= F\big(\lim_{n\to\infty} x_n\big) \\ &= \lim_{n\to\infty} F(x_n) = p \end{aligned}$$

p is a fixed point of F in X. For Uniqueness if q be distinct point F from p in X. Then $F(q) = q$. Consider

$$\begin{aligned} d(q,p) &= d(F(q), F(p)) \\ &\leq d(F(q), F(p)) \leq \frac{\mu\ d(q, F(q))\ d(p, F(p))}{d(q,p)} + \mu\ \lambda d(q,p) \\ &= \mu\ \lambda d(q,p) \\ &< d(q,p) \end{aligned}$$

which is a contradiction. p is a fixed point of F in X. □

Theorem 3.2. *Let $F : X \to X$ where X be a complete metric space such that F satisfies (1) and if for some $n\epsilon I^+$ if F^n is uniformly continuous, then F has a unique fixed point.*

Now we prove a unique common fixed theorem of a pair of self mappings defined on a complete metric space (X,d) which satisfies (1).

Theorem 3.3. *Let F_1, F_2 be two mappings which maps $X \times X$ to R, where (X, d) be a complete metric space satisfying the condition*

(1). $d(F(x), F(y) \leq \frac{\mu\ d(x,F(x))\ d(y,F(y))}{d(x,y)} + \mu\ \lambda d(x,y)$ for all $x, y\epsilon X$, $x \neq y$ and for μ , λ $\epsilon[0, \frac{1}{2})$ with $\mu + \lambda < \frac{1}{2}$ and $\mu\ \lambda < \frac{1}{2}$.

(2). there is an $x_0\epsilon X$, the sequence x_n is such that $x_n = F_1(x_{n-1})$, if n is even and $x_n = F_2(x_{n-1})$, if n is odd, and $x_n \neq x_{n+1}$ for all n.

(3). F_1F_2 is continuous on X. Then F_1 F_2 has a unique common fixed point in X.

Proof. consider

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &= d(F_2(x_{2n}), F_1(x_{2n-1})) \\ &\leq \frac{\mu\ d(x_{2n}, F_2(x_{2n}))\ d(x_{2n-1}, F_1(x_{2n-1}))}{d(x_{2n}, x_{2n-1})} + \mu\ \lambda d(x_{2n}, x_{2n-1}), \text{ which gives} \\ d(x_{2n+1}, x_{2n}) &\leq \left(\frac{\mu\lambda}{1-\mu}\right) d(x_{2n}, x_{2n-1}) \\ &\leq \cdots \\ &\leq \left(\frac{\mu\lambda}{1-\mu}\right)^{2n} d(x_0, x_1) \\ d(x_{2n+2}, x_{2n+1}) &\leq \left(\frac{\mu\lambda}{1-\mu}\right)^{2n+1} d(x_0, x_1) \end{aligned}$$

For $m \geq n$, using triangle inequality. Consider

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1})\ d(x_1, x_0) \text{ where } \alpha = \frac{\mu\lambda}{1-\mu} \\ &\leq \frac{\alpha^n}{1-\alpha}\ d(x_1, x_0) \end{aligned}$$

$\frac{\alpha^n}{1-\alpha} \to 0 \ as \ m, n \to 0$. Since X is complete,there exists p in X such that $x_n \to p$. Therefore its subsequence x_{n_k} also converges to p where $n_k = 2k$. Since F_1F_2 be continuous on X. Therefore, we have

$$F_1F_2(p) = F_1F_2(\lim_{k\to\infty} x_{n_k}) = \lim_{k\to\infty}(x_{n_{k+1}}) = p.$$

If possible suppose that $F_2(p) \neq p$, then

$$\begin{aligned} d(F_2(p), p) &= d(F_2(u), F_1F_2(u)) \\ &\leq \frac{\mu d(p, F_2(p)).d(F_2(p), F_1F_2(p))}{d(p, F_2(p))} + \mu\lambda d(p, F_2(p)) \\ &\leq (\mu + \lambda)d(p, F_2(p)) \end{aligned}$$

which is impossible, since $\mu + \lambda \leq 1$. Therefore, we have $F_2(p) = p$. Similarly,

$$d(F_1(p), p) = d(F_1(F_2(p)), p) = d(p, p) = 0$$

as $d(F_1(p), p) = 0$. Therefore $F_1(p) = p$. Therefore p is a common fixed point of F_1 and F_2. For Uniqueness if possible suppose that $q \epsilon X$ is another fixed point of F_1 which is distinct from p. Consider

$$\begin{aligned} d(q, p) &= d(F_1(q), F_2(p)) \\ &\leq \frac{\mu \ d(q, F_1(q)) \ d(p, F_2(p))}{d(q, p)} + \mu \ \lambda d(q, p) \\ d(q, p) &\leq 0 + \mu \ \lambda d(q, p) < d(q, p) \\ \therefore \ d(q, p) &< d(q, p) \end{aligned}$$

Which is not possible. Therefore p is a unique fixed point of F_1. Similarly we can prove that p is also a unique fixed point of F_2. Therefore p is the unique common fixed point of F_1 and F_2. □

4. Conclusion

We have proved unique common fixed point for pair of self maps in Complete metric space.

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