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Rectilinear Crossing Number of Complete Graph Imbedded Inside the Complete Bipartite Graph of $\Gamma(Z_n)$

Research Article

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Abstract: In this paper we evaluate the Rectilinear crossing number of the zero divisor graph $\Gamma(Z_{2p^2})$ and $\Gamma(Z_{3p^2})$, which can be decomposed into a star graph, complete graph and complete bipartite graph. We introduce the minimum number of Rectilinear crossing which can be obtained by imbedding the complete graph and star graph inside the complete bipartite graph, by framing formula for prime iterations.

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Keywords: Zero divisor graph, Crossing number, Rectilinear crossing number.

1. Introduction

Let **R** be a commutative ring with unity and let Z(R) be its set of zero divisors. The zero divisor graph of **R** denoted by $\Gamma(R)$ is a graph which is undirected with vertices $Z(R)^* = Z(R) - \{0\}$, the set of non-zero divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent iff $xy = 0$. Throughout this paper, we consider the commutative ring by R and the zero divisor graph $\Gamma(R)$ by $\Gamma(Z_n)$.The idea of a zero divisor graph of a commutative ring was introduced by I. Beck [2]. For notation and graph theory terminology are considered as in [1, 2].

The crossing number cr(G) of a graph G is the minimum number of edge crossings among the drawings of G in the plane such that the edges of G are Jordan arcs [3, 7]. The rectilinear crossing number of a graph with minimum number of edge crossing drawn in a plane satisfies the following conditions: (i) edges are line segments. (ii) no three vertices are collinear. (iii) no three edges may intersect at a common vertex. The rectilinear crossing number of G is denoted by $\bar{cr}(G)$.

2. Maximum Rectilinear Crossing Number

We derived the Rectilinear crossing number of some zero divisor graphs [4–6]. In this paper, we evaluate the rectilinear crossing number of $\Gamma(Z_{2p^2})$ in a commutative ring. we will show that this graph involves only the drawing of complete graph and complete bipartite graph only. we find that crossings between these graphs will be minimum only if we immbed complete graph inside the complete bipartite graph. Our task is to minimize the rectilinear crossings can be achieved by maximizing

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from inside, that is the complete graph. So first we prove the theorem for finding maximum rectilinear crossing number for complete graphs. Although we come across many rectilinear drawings for maximum crossings, we give a convenient way so that crossings from complete bipartite graph is minimized from the total crossings.The convenient way is that keeping all the vertices of a complete graph in a cresent shaped manner and arranging majority of the vertices of complete bipartite graphs horizontally below the cresent. This will be evident from the following theorems.

Theorem 2.1. *For any complete graph K_{p-1}, the maximum Rectilinear crossing number is Max $\bar{cr}(K_{p-1}) =^{p-1} C_4$.*

Proof. Since K_{p-1} is a complete graph, all the vertices are adjacent to every vertex in K_{p-1}.Since $p-1$ is even, place equally all the vertices in a cresent shaped manner on either side of the cresent which is symmetrical. The crossings will be as follows:

$$\begin{aligned}\max\ \bar{cr}(K_{p-1}) &= \bar{cr}(2p) + \bar{cr}(3p) + \cdots + \bar{cr}(p(p-3)) \\ &= 1\,[(p-4) + (p-5) + \cdots + 1] + 2\,[(p-5) + (p-4) + \cdots + 1] \\ &\quad + 2\,[(p-6) + (p-5) + \cdots + 1] + \cdots + (p-4)\,[1]\end{aligned}$$

Case(i): $p = 5$

$$\max\ \bar{cr}(K_4) = \bar{cr}(10) = 1(1) = 1 =^4 C_4 =^{5-1} C_4 =^{p-1} C_4$$

Case(ii): $p = 7$

$$\begin{aligned}\max\ \bar{cr}(K_6) &= \bar{cr}(14) + \bar{cr}(21) + \bar{cr}(28) \\ &= 1(3+2+1) + 2(2+1) + 3(1) \\ &= 6+6+3 = 15 =^6 C_4 =^{7-1} C_4 =^{p-1} C_4\end{aligned}$$

Case(iii): $p = 11$

$$\begin{aligned}\max\ \bar{cr}(K_{10}) &= \bar{cr}(22) + \bar{cr}(33) + \bar{cr}(44) + \cdots + \bar{cr}(88) \\ &= 1(7+6+5+4+3+2+1) + 2(6+5+4+3+2+1) + 3(5+4+3+2+1) \\ &\quad + 4(4+3+2+1) + 5(3+2+1) + 6(2+1) + 7(1) \\ &= 28+42+45+40+30+18+7 \\ &= 210 =^{10} C_4 =^{11-1} C_4 =^{p-1} C_4\end{aligned}$$

Case(iv): $p = 13$

$$\max\ \bar{cr}(K_{12}) = 495 =^{12} C_4 =^{13-1} C_4 =^{p-1} C_r$$

Case(v): $p = 17$

$$\max\ \bar{cr}(K_{16}) = 1820 =^{16} C_4 =^{17-1} C_4 =^{p-1} C_4$$

In general,for any p,

$$\max\ \bar{cr}(K_{p-1}) =^{p-1} C_4.$$

□

3. Rectilinear Crossing Number of $\Gamma(Z_{2p^2})$ and $\Gamma(Z_{3p^2})$

As the graph $\Gamma(Z_{2p^2})$ and $\Gamma(Z_{3p^2})$ consists of star graph, complete graph and complete bipartite graph which lead us to a new approach in finding the minimum number of Rectilinear crossing of these composition which can be embedded in a single graph with minimum number of crossings. We obtain the minimum number of crossings by imbedding the complete graph (outer planar) and star graph within the middle of complete bipartite graph by following the prescribed orderly manner. So the zero divisor graph facilitates, in finding the different types of graphs embedded in different combinations. The same can be obtained for any combinations of graphs if possible.

Theorem 3.1. *For any graph $\Gamma(Z_{2p^2})$,where p is any prime $p > 3$, then*

$$\bar{cr}(\Gamma(Z_{2p^2})) =^{p-1} C_4 + (p - Odd)\left[\frac{(p-1)^2(p-3)}{8} - \frac{^{\frac{p+1}{2}}P_3}{3}\right] + \frac{K_{p-1,p-Odd} + K_{p-1,p+Odd}}{2},$$

where 1, 3, 5, 7, 9, 11, 13, 15, 17, 19,... are the odd numbers chosen for the prime numbers, $p = 5, 5$ and 7, 11, 13, 17, 17 and 19, 23, 23, 29, 29,... respectively by rightly chosen.

Proof. The vertex set of $\Gamma(Z_{2p^2})$, is $V(\Gamma(Z_{2p^2})) = \{2, 4, 6, 8, \ldots, 2(p^2 - 1), p, 2p, 3p, 5p, \ldots, p(2p - 1)\}$. Hence, $|V(\Gamma(Z_{2p^2}))| = p^2 + p - 1$. Let $u = 4$ and $v = p(2p-1)$. Then $uv = 4p(2p-1)$. This implies $2p^2$ does not divides $4p(2p-1)$. Therefore,u and v are non-adjacent vertices. Let S_1 be the set which contains only the vertices with multiples of $2p$, and S_2 be the set which consists multiples of 2 other than the vertices of S_1. Let $x = 10$ and $y = 20$ when $p = 5$. Then $xy = 200$. Hence $2p^2$ divides 200, which implies x and y are adjacent. Similarly, all the vertices in S_1 are adjacent. Therefore we get a complete graph K_{p-1}. The centre point p^2 is adjacent to all the vertices in the set S_1 and S_2, since if $x = p^2, y = 2$ then $xy = 2p^2$, which clearly divides $2p^2$. Therefore, we get a star graph k_{p^2,p^2-1}. Finally the vertices $p, 3p, 5p...p(2p - 1)$ are adjacent to all the vertices in the set S_1.

Similarly if $x = 2p, y = p(2p - 1)$, implies $xy = 2p^2(2p - 1)$ which divides $2p^2$. Therefore we get a complete bipartite graph $K_{p-1,p-1}$. As the graph of k_{p^2,p^2-1} is a star graph, the vertices of S_2 which are adjacent only to the vertex p^2, whose edges between them involves no crossing with the other edges of the graph if p^2 is placed either at the topmost end or at the last vertex in the bottom so the crossings of p^2, with the set S_1 is zero.

Now, the graph $\Gamma(Z_{2p^2})$ involves the drawing of complete graph K_{p-1} and complete bipartite graph $K_{p-1,p-1}$ along with the vertex p^2 only. It can be observed that p^2 is adjacent only to k_{p-1} but non adjacent to S_3 of $K_{p-1,p-1}$. Therefore, we include p^2 in $K_{p-1,p-1}$. Then the modified bipartite graph will be $K_{p+1,p-1}$.

Finally, we arrive at the crossings of only the complete bipartite graph $K_{p+1,p-1}$ and complete graph K_{p-1}. First, we give rectilinear drawing of K_{p-1}. Place all the $(p - 1)$ vertices of S_1 in crescent shaped manner and the edges are drawn such that all the $(p - 1)$ vertices are adjacent to each other. Therefore the drawing of K_{p-1} will lead to maximum upper bound for any rectilinear drawing. Next we place all the $(p + 1)$ vertices of the set S_3 and p^2 vertically from above and below K_{p-1}. The task of splitting the vertices from above and below minimizes from the total crossings.

We observe that the number of vertices in the complete bipartite graph is p which is greater than the number of vertices in complete graph i.e, K_{p-1}. So the inner complete graph K_{p-1} has maximum crossing, but not greater than the crossings from above and below K_{p-1}. Let V_1 be the vertex set from above and V_2 be the vertex set from below the crescent. We infer that the combination of vertices taken in V_1 and V_2 differs for every prime. Let $\bar{cr}\left(\frac{K'_{p-1,p-odd}}{2}\right)$ denote the crossings between the vertices in the set V_1 and $\bar{cr}\left(\frac{K_{p-1,p+odd}}{2}\right)$ denote the crossings between the vertices in the set V_2.

Case(i): Let $p = 5$

The vertex set of $\Gamma(Z_{50})$, is $\{2, 4, 6, 8, ..., 48, 5, 15, ..., 45\}$. Hence, $|V(\Gamma(Z_{50}))| = 29$. Let $S_1 = \{10, 20, 30, 40\}$, $S_2 = \{2, 4, 6, ..., 48\}$ and $S_3 = \{5, 15, 25, 35, 45\}$. Here S_1 is the complete graph of $\Gamma(Z_{50})$, where $p = 5$ is K_4. Therefore by Theorem (1.4.1), $\bar{cr}(K_4) = {}^{5-1}C_4 = {}^4C_4 = 1$. Let $V_1 = \{5, 25\}$ and let $V_2 = \{15, 35, 45\}$. Then,

$$\begin{aligned}\bar{cr}(V_1) &= \bar{cr}\left(\frac{K_{4,4}}{2}\right) = \frac{2.1.2.1}{2} = 2\\ \bar{cr}(V_2) &= \bar{cr}\left(\frac{K_{4,6}}{2}\right) = \frac{2.1.3.2}{2} = 6\\ \bar{cr}(V_1) + \bar{cr}(V_2) &= \bar{cr}\left(\frac{K_{p-1,p-1} + K_{p-1,p+1}}{2}\right)\\ &= \bar{cr}\left(\frac{K'_{p-1,p-odd} + K_{p-1,p+odd}}{2}\right)\\ &= 2 + 6 = 8\end{aligned}$$

Crossings of V_1 over K_4 is,

$$\begin{aligned}\bar{cr}(5) + \bar{cr}(25) &= 2\,[2 + 2(1)]\\ &= (4)(4-2)\\ &= (5-1)\left[\frac{16\times 2}{8}\right] - \frac{6}{3}\\ &= (5-1)\left[\frac{(5-1)^2(5-3)}{8} - \frac{^{\frac{5+1}{2}}P_3}{3}\right]\\ &= (p-1)\left[\frac{(p-1)^2(p-3)}{8} - \frac{^{\frac{p+1}{2}}P_3}{3}\right], \text{ where odd=1.}\end{aligned}$$

Therefore total crossings is,

$$\bar{cr}(K_4) + \bar{cr}\left(\frac{K_{4,4} + K_{4,6}}{2}\right) + (5-1)\left[\frac{(5-1)^2(5-3)}{8} - \frac{^{\frac{5+1}{2}}P_3}{3}\right] = 1 + \frac{4+12}{2} + 4\,[4-2] = 1 + 8 + 8 = 17$$

Case(ii): Let $p = 7$

The vertex set of $\Gamma(Z_{98})$, is $\{2, 4, 6, 8, ..., 96, 7, 21, ..., 91\}$. Hence, $|V(\Gamma(Z_{98}))| = 54$. Let $S_1 = \{14, 28, 42, 56, 70, 84\}$, $S_2 = \{2, 4, 6, ..., 96\}$ and $S_3 = \{7, 21, 35, 49, 63, 77, 91\}$. Here S_1 is the complete graph of $\Gamma(Z_{98})$, where $p = 7$ is K_6. Therefore by theorem (1.4.1), $\bar{cr}(K_6) = {}^{7-1}C_4 = {}^6C_4 = 15$. Let $V_1 = \{7, 49\}$ and let $V_2 = \{21, 35, 63, 77, 91\}$. Then,

$$\begin{aligned}\bar{cr}(V_1) &= \bar{cr}\left(\frac{K_{6,4}}{2}\right) = \frac{3.2.2.1}{2} = 6\\ \bar{cr}(V_2) &= \bar{cr}\left(\frac{K_{6,10}}{2}\right) = \frac{3.2.5.4}{2} = 60\\ \bar{cr}(V_1) + \bar{cr}(V_2) &= \bar{cr}\left(\frac{K_{p-1,p-3} + K_{p-1,p+3}}{2}\right)\\ &= \bar{cr}\left(\frac{K'_{p-1,p-odd} + K_{p-1,p+odd}}{2}\right)\\ &= 6 + 60 = 66\end{aligned}$$

Crossings of V_1 over K_6 is,

$$\begin{aligned}\bar{cr}(7)+\bar{cr}(49) &= 2\left[4+2(3)+3(2)\right]\\ &= (4)(18-8)\\ &= (7-3)\left[\frac{36\times 4}{8}\right]-\frac{24}{3}\\ &= (7-3)\left[\frac{(7-1)^2(7-3)}{8}-\frac{^{\frac{7+1}{2}}P_3}{3}\right]\\ &= (p-1)\left[\frac{(p-1)^2(p-3)}{8}-\frac{^{\frac{p+1}{2}}P_3}{3}\right], \quad \text{where odd=3.}\end{aligned}$$

Therefore total crossings is,

$$\bar{cr}(K_6)+\bar{cr}\left(\frac{K_{6,4}+K_{6,10}}{2}\right)+(7-1)\left[\frac{(7-1)^2(7-3)}{8}-\frac{^{\frac{7+1}{2}}P_3}{3}\right]=15+\frac{12+120}{2}+4\left[18-8\right]=15+40+66=121$$

Case(iii): Let $p=11$

The vertex set of $\Gamma(Z_{242})$, is $\{2,4,6,8,...,240,11,22,...,231\}$. Hence, $|V(\Gamma(Z_{242}))|=131$. Let $S_1=\{22,44,66,...,220\}$, $S_2=\{2,4,6,...,240\}$ and $S_3=\{11,33,55,...,231\}$. Here S_1 is the complete graph of $\Gamma(Z_{242})$, where $p=11$ is K_{12}. Therefore by theorem (1.4.1), $\bar{cr}(K_{12}) =^{11-1} C_4 =^{10} C_4 = 210$. Let $V_1=\{11,33,121\}$ and let $V_2=\{55,77,99,...,231\}$. Then,

$$\begin{aligned}\bar{cr}(V_1) &= \bar{cr}\left(\frac{K_{10,6}}{2}\right)=\frac{5.4.3.2}{2}=60\\ \bar{cr}(V_2) &= \bar{cr}\left(\frac{K_{10,16}}{2}\right)=\frac{5.4.8.7}{2}=560\\ \bar{cr}(V_1)+\bar{cr}(V_2) &= \bar{cr}\left(\frac{K_{p-1,p-5}+K_{p-1,p+5}}{2}\right)\\ &= \bar{cr}\left(\frac{K'_{p-1,p-odd}+K_{p-1,p+odd}}{2}\right)\\ &= 60+560=620\end{aligned}$$

Crossings of V_1 over K_6 is,

$$\begin{aligned}\bar{cr}(11)+\bar{cr}(33)+\bar{cr}(121) &= 6\left[8+2(7)+3(6)+4(5)\right]\\ &= (6)(100-40)\\ &= (11-5)\left[\frac{100\times 8}{8}\right]-\frac{120}{3}\\ &= (11-5)\left[\frac{(11-1)^2(11-3)}{8}-\frac{^{\frac{11+1}{2}}P_3}{3}\right]\\ &= (p-5)\left[\frac{(p-1)^2(p-3)}{8}-\frac{^{\frac{p+1}{2}}P_3}{3}\right], \quad \text{where odd=5.}\end{aligned}$$

Therefore total crossings is,

$$\bar{cr}(K_{10})+\bar{cr}\left(\frac{K_{10,5}+K10,16}{2}\right)+(11-5)\left[\frac{(11-1)^2(11-3)}{8}-\frac{^{\frac{11+1}{2}}P_3}{3}\right]=210+\frac{120+1120}{2}+6\left[100-40\right]=210+620+360=1190$$

In general, for any prime $p>3$, then $\bar{cr}(\Gamma(Z_{2p^2})) =^{p-1} C_4 + (p-Odd)\left[\frac{(p-1)^2(p-3)}{8}-\frac{^{\frac{p+1}{2}}P_3}{3}\right] + \bar{cr}\left(\frac{K_{p-1,p-Odd}+K_{p-1,p+Odd}}{2}\right)$ where Odd represents the odd numbers, $1,3,5,7,9,11,13,15,17,19,...$ for the corresponding prime, $p=5,5$ and $7,11,13,17,17$ and $19,23,23,29,29,...$ by rightly choosen. □

Theorem 3.2. *For any graph $\Gamma(Z_{3p^2})$,where p is any prime, then*

$$\bar{cr}(\Gamma(Z_{3p^2})) = {}^{p+1}C_4 + 2\left[p-(2+n)\right]\left[1(s-2)+2(s-3)+...+S_0(S-S_1)\right] + \bar{cr}\left(\frac{K_{p-1},2(p+n)}{2}+\frac{K_{p-1},2(p-(2+n))}{2}\right),$$

and for $p>3$ where $S=p+1, S_0=\frac{p-3}{2}, S_1=\frac{p-1}{2}$ and $n=1,2,3,4,5,...$ for the corresponding primes $p=5,7,11,13,...$

Proof. The vertex set of $\Gamma(Z_{3p^2})$, is $V(\Gamma(Z_{3p^2}))=\{3,6,9,......3(p^2-1),p,2p,3p,.....p(3p-1)\}$. Hence, $|V(\Gamma(Z_{2p^2}))|=p^2+2p-1$. The vertices of $\Gamma(Z_{3p^2})$ be partitioned into V_1,V_2,V_3, and V_4 where $V_1=\{p^2,2p^2\}$,$V_2=\{3p,6p,9p,...3p(p-1)\}$,$V_3=\{p,2p,4p,...p(3p-1)\}$ and $V_4=\{3,6,9,...3(p^2-1)\}$.

Next, consider the set V_2. Let $u=3p, v=6p$. Then $uv=18p^2$ divides $3p^2$. Similarly all the vertices in V_2 are adjacent among itself. Therefore the vertex set V_2 forms the complete graph K_{p-1}. Next consider the set V_1. Let $u=p^2, v=2p^2$. Then $uv=2p^4$ does not divides $3p^2$. But if, $u=p^2$ in V_1 and $v=3p$ in V_2. Then $uv=3p^3$ divides $3p^2$. Similarly, all the the vertices in V_1 are adjacent to every vertex in V_2. Therefore V_1 and V_2 together forms a complete graph except an edge $(p^2,2p^2)$ in V_1. Let us denote the complete graph without an edge as K'_{p+1}. Therefore, $K'_{p+1}=K_{p+1}-(p^2,2p^2)$.Then edge set of K'_{p+1} will be $E\left(K'_{p+1}\right)=\{(p^2,2p^2),...(p^2,3p(p-1)),(2p^2,3p),...(2p^2,3p(p-1))\}-(p^2,2p^2)$.

Now consider the vertex set V_3. Let $x=4p, y=2p$. Then $xy=8p^2$ doesnot divide $3p^2$. Therefore x and y are non-adjacent. Similarly every vertices of V_3 are non-adjacent among themselves. Let $r=p$ in V_3 and $s=3p$ in V_2. Then $rs=3p^2$ divides $3p^2$. Therefore V_2 and V_3 together forms a complete bipartite graph $K_{p-1,2(p-1)}$. With the same verification, we note that V_1 and V_4 together forms a complete bipartite graph $K_{2,p(p-1)}$. Now let us find the rectilinear crossing number of $\Gamma(Z_{3p^2})$. The minimum number of rectilinear crossing number is obtained by imbedding complete graph K'_{p+1} inside the complete bipartite graph $K_{p-1,2(p-1)}$. The vertices of K'_{p+1} contains the vertex set V_1 and V_2. Therefore the vertex set of K'_{p+1} is $V(K'_{p+1})=\{p^2,2p^2,3p,...3p(p-1)\}$.

The drawing of D involves first in drawing of K'_{p+1}. The vertices of K'_{p+1} are arranged in a crescent shaped such that the vertices of V_1 comes in the middle. That is $V(K'_{p+1})=\{3p,...,p^2,2p^2,...,3p(p-1)\}$. Now the vertices of V_4 are kept vertically between p^2 and $2p^2$, such that a complete bipartite graph $K_{2,p(p-1)}$ is formed. Clearly the rectilinear crossing number is zero for $K_{2,p(p-1)}$. Finally, we are left only with the crossings between complete graph K'_{p+1} and complete bipartite graph $K_{p-1,2(p-1)}$. Since only $p-1$ vertices in the crescent are adjacent to $2(p-1)$ vertices of V_3, the vertices are arranged on either side of the cresent vertically, which doesnot disturbs the $K_{2,p(p-1)}$ crossings. Let $p+n$ be the number of vertices placed below the cresent and the remaining vertices are placed above the cresent. We observe that $n=1,2,3,...$ for the subsequent primes $p=5,7,11,13,...$ which will be revealed from the following cases. Now let us find the total rectilinear crossing number of $\Gamma(Z_{3p^2})$ by the method of induction.

Case(i): When $p=5$,

The vertex set of $\Gamma(Z_{75})$, is $\{3,6,9,......72,5,10,15,...,70\}$. Hence, $|V(\Gamma(Z_{75}))|=34$. Here $V_1=\{25,50\}$, $V_2=\{15,30,45,60\}$, $V_3=\{5,10,20,35,40,55,65,70\}$ and $V_4=\{3,6,..72\}$. The complete graph of $\Gamma(Z_{75})$ where $p=5$ is K_6 has the vertices $\{15,30,45,60,25,50\}$. Since we place the vertices of V_1 in the middle then, $K'_6=\{15,30,25,50,45,60\}$. Therefore the rectilinear crossing number of $K_6={}^6C_4=15$. As $n=1$, 6 vertices are placed below the cresent. Then $\bar{cr}\left(\frac{K_{4,12}}{2}\right)=\frac{2.1.6.5}{2}=30$. Similarly, the remaining 2 vertices of V_3 are placed above the crescent. Then $\bar{cr}\left(\frac{K_{4,4}}{2}\right)=\frac{2.1.2.1}{2}=2$. Therefore the total rectilinear crossing number of complete bipartite graph is 32. The rectilinear crossing number over $K_6^{'}$ is

$$= \bar{cr}(5) \text{ over } K_6^{'} + \bar{cr}(10) \text{ over } K_6^{'}$$
$$= 16$$
$$= 2(2)(4)$$
$$= 2\,[5-(2+1)]\,[1(6-2)]$$
$$= 2\,[p-(2+n)]\,[1(s-2)+2(s-3)+...+S_0(S-S_1)]$$

where $S=6, S_0=1, S_1=2, n=1$. Therefore $\bar{cr}(\Gamma(Z_{3p^2}))$ is

$$= 63$$
$$= 15+16+32$$
$$=^6 C_4 + 2\,[5-(2+1)]\,[1(6-2)] + \bar{cr}\left(\frac{K_{4,12}}{2}\right) + \bar{cr}\left(\frac{K_{4,4}}{2}\right)$$
$$=^{p+1} C_4 + 2\,[p-(2+n)]\,[1(s-2)+2(s-3)+...+S_0(S-S_1)] + \bar{cr}\left(\frac{K_{p-1}, 2(p+n)}{2}\right) + \bar{cr}\left(\frac{K_{p-1}, 2(p-(2+n))}{2}\right),$$

where $S=p+1, S_0=\frac{p-3}{2}, S_1=\frac{p-1}{2}$

Case(ii): When $p=7$

The vertex set of $\Gamma(Z_{147})$, is $\{3,6,9,......144,7,14,...,140\}$. Hence, $|V(\Gamma(Z_{147}))| = 62$. Here $V_1 = \{49,98\}$,$V_2 = \{21,42,63,84,105,126\}$, $V_3 = \{7,14,...,140\}$ and $V_4 = \{3,6,..144\}$. The complete graph of $\Gamma(Z_{147})$ where $p=7$ is K_8 has the vertices $\{21,42,63,84,105,126,49,98\}$. Since we place the vertices of V_1 in the middle then $K_8^{'} = \{21,42,63,49,98,84,105,126\}$. Therefore the rectilinear crossing number of $K_8 =^8 C_4 = 70$. As $n=2$, 9 vertices are placed below the cresent. Then $\bar{cr}\left(\frac{K_{6,18}}{2}\right) = \frac{3.2.9.8}{2} = 216$. Similarly, the remaining 3 vertices of V_3 are placed above the crescent. Then $\bar{cr}\left(\frac{K_{6,6}}{2}\right) = \frac{3.2.3.2}{2} = 18$. Therefore the total rectilinear crossing number of complete bipartite graph is 234. The rectilinear crossing number over $K_8^{'}$ is

$$= \bar{cr}(7) \text{ over } K_8^{'} + \bar{cr}(14) \text{ over } K_8^{'} + \bar{cr}(28) \text{ over } K_8^{'}$$
$$= 96$$
$$= 2(3)(6+2.5)$$
$$= 2\,[7-(2+2)]\,[1(8-2)+2(8-3)]$$
$$= 2\,[p-(2+n)]\,[1(s-2)+2(s-3)+...+S_0(S-S_1)]$$

where $S=8, S_0=2, S_1=3, n=2$. Therefore $\bar{cr}(\Gamma(Z_{3p^2}))$ is

$$= 400 = 70+96+234$$
$$=^8 C_4 + 2\,[7-(2+2)]\,[1(8-2)+2(8-3)] + \bar{cr}\left(\frac{K_{6,18}}{2}\right) + \bar{cr}\left(\frac{K_{6,6}}{2}\right)$$
$$=^{p+1} C_4 + 2\,[p-(2+n)]\,[1(s-2)+2(s-3)+...+S_0(S-S_1)] + \bar{cr}\left(\frac{K_{p-1}, 2(p+n)}{2}\right) + \bar{cr}\left(\frac{K_{p-1}, 2(p-(2+n))}{2}\right),$$

where $S=p+1$, $S_0=\frac{p-3}{2}$, $S_1=\frac{p-1}{2}$.

Case(iii): When $p=11$,

The vertex set of $\Gamma(Z_{363})$, is $\{3, 6, 9, ...360, 11, 22, ..., 352\}$. Hence, $|V(\Gamma(Z_{363}))| = 142$. Here $V_1 = \{121, 242\}, V_2 = \{33, 66, 99, 132, 165, 198, 231, 264, 297, 330\}$, $V_3 = \{11, 22, 44, 55, 77, ...352\}$ and $V_4 = \{3, 6, ..360\}$. The complete graph of $\Gamma(Z_{363})$ where $p = 11$ is K_{12} has the vertices $\{33, 66, 99, 132, 165, 198, 231, 264, 297, 330\}$. Since we place the vertices of V_1 in the middle then $K'_{12} = \{33, 66, 99, 132, 165, 121, 242, 198, 231, 264, 297, 330\}$. Therefore the rectilinear crossing number of $K_{12} =^{12} C_4 = 495$. As $n = 3$, 14 vertices are placed below the cresent. Then $\bar{cr}\left(\frac{K_{10,28}}{2}\right) = \frac{5.4.14.13}{2} = 1820$. Similarly, the remaining 6 vertices of V_3 are placed above the crescent. Then $\bar{cr}\left(\frac{K_{10,12}}{2}\right) = \frac{5.4.6.5}{2} = 300$. Therefore the total rectilinear crossing number of complete bipartite graph is 2120. The rectilinear crossing number over K'_{12} is

$$= \bar{cr}(11) \quad \text{over} \quad K'_{12} + \bar{cr}(22) \quad \text{over} \quad K'_{12} + \bar{cr}(44) \quad \text{over} \quad K'_{12} + \bar{cr}(55) \quad \text{over} \quad K'_{12} + \bar{cr}(77) \quad \text{over} \quad K'_{12} + \bar{cr}(88) \quad \text{over} \quad K'_{12}$$
$$= 960$$
$$= 2(6)(10 + 2.9 + 3.8 + 4.7)$$
$$= 2\,[11 - (2 + 3)]\,[1(12 - 2) + 2(12 - 3) + 3(12 - 4) + 4(12 - 5)]$$
$$= 2\,[p - (2 + n)]\,[1(s - 2) + 2(s - 3) + ... + S_0(S - S_1)]$$

where $S = 12, S_0 = 2, S_1 = 3, n = 3$. Therefore $\bar{cr}(\Gamma(Z_{3p^2}))$ is

$$= 3575 = 495 + 960 + 2120$$
$$=^{12} C_4 + 2\,[11 - (2 + 3)]\,[1(12 - 2) + 2(12 - 3)] + \bar{cr}\left(\frac{K_{10,28}}{2}\right) + \bar{cr}\left(\frac{K_{10,12}}{2}\right)$$
$$=^{p+1} C_4 + 2\,[p - (2 + n)]\,[1(s - 2) + 2(s - 3) + ... + S_0(S - S_1)] + \bar{cr}\left(\frac{K_{p-1}, 2(p + n)}{2}\right) + \bar{cr}\left(\frac{K_{p-1}, 2(p - (2 + n))}{2}\right),$$

where $S = p + 1, S_0 = \frac{p-3}{2}, S_1 = \frac{p-1}{2}$. In general, for any prime $p > 3$,

$$\bar{cr}(\Gamma(Z_{3p^2})) =^{p+1} C_4 + 2\,[p - (2 + n)]\,[1(s - 2) + 2(s - 3) + ... + S_0(S - S_1)] + \bar{cr}\left(\frac{K_{p-1}, 2(p + n)}{2} + \frac{K_{p-1}, 2(p - (2 + n))}{2}\right),$$

where $S = p + 1, S_0 = \frac{p-3}{2}, S_1 = \frac{p-1}{2}$ and $n = 1, 2, 3, 4, 5...$ for the corresponding primes $p = 5, 7, 11, 13, ...$ □

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